International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **11(2)**(2023), 79–87
Available Online: http://ijmaa.in

The Generalized Repetitious Number Puzzle

John Rafael M. Antalan[1,*], Richard P. Tagle[1]

[1] *Department of Mathematics and Physics, College of Science, Central Luzon State University (3120), Science City of Muñoz, Nueva Ecija, Philippines*

Abstract

In this paper, we use simple divisibility property of integers to provide a generalization to the "Repetitious NumberPuzzle" found in *The Second Scientific American Book of Mathematical Puzzles and Diversions* by Martin Gardner. We also show that the solution to the generalized repetitious number puzzle provides a particular recreational application of the sequences A000533 and A261544 in *The On-line Encyclopedia of Integer Sequences* (OEIS).

Keywords: Repetitious number puzzle; generalized repetitious number puzzle; integer sequence.
2020 Mathematics Subject Classification: 00A08, 11A05.

1. Introduction

In [1], Martin Gardner presented the puzzle below:

> "**The Repetitious Number**. An unusual parlor trick is performed as follows. Ask spectator A to jot down any three-digit number, and then to repeat the digits in the same order to make a six-digit number (e.g., 394 394). With your back turned so that you cannot see the number, ask A to pass the sheet of paper to spectator B, who is requested to divide the number by 7.
>
> Don't worry about the remainder, you tell him, because there won't be any. B is surprised to discover that you are right (e.g., 394 394 divided by 7 is 56 342). Without telling you the result, he passes it on to spectator C, who is told to divide it by 11. Once again you state that there will be no remainder, and this also proves correct (56 342 divided by 11 is 5 122).
>
> With your back still turned, and no knowledge whatever of the figures obtained by these computations, you direct a fourth spectator D, to divide the last result by 13. Again the division comes out even (5 122 divided by 13 is 394). This final result is written on a slip of

*Corresponding author (jrantalan@clsu.edu.ph)

paper which is folded and handed to you. Without opening it you pass it on to spectator A.

Open this, you tell him, and you will find your original three-digit number.

Prove that the trick cannot fail to work regardless of the digits chosen by the first spectator."

The puzzle was originally written by Yakov Perelman [2]. In this paper, we provide a generalization to this puzzle and show that the solution to the generalization provides a particular recreational application of the sequences A000533 and A261544 in the OEIS.

2. Preliminaries

2.1 The OEIS Sequence A000533 and A261544

The OEIS sequence A000533 [3] is the sequence defined by

$$a(0) = 1$$
$$a(n) = 10^n + 1, \quad n \geq 1.$$

Its first 15 terms are:

1, 11, 101, 1001, 10001, 100001, 1000001, 10000001, 100000001, 1000000001, 10000000001, 100000000001, 1000000000001, 10000000000001, 100000000000001.

On the otherhand, the sequence A261544 [4] is the sequence defined by

$$b(n) = \sum_{k=0}^{n} 1000^k.$$

Its first 10 terms are:

1, 1001, 1001001, 1001001001, 1001001001001, 1001001001001001, 1001001001001001001, 1001001001001001001001, 1001001001001001001001001, 1001001001001001001001001001.

2.2 Some Definitions, Notations and Preliminary Results

Definition 2.1. *Let* $n = d_1d_2\ldots d_kd_1d_2\ldots d_k\ \ldots\ d_1d_2\ldots d_k$ *be a positive repetitive integer. We say that the positive integer* $g = d_1d_2\ldots d_k$ *is a* **generator** *of* n *if* g *is a positive integer such that replicating* g *a finite number of times generates* n.

Definition 2.2. *Let* $g = d_1d_2\ldots d_k$ *be a generator of* n. *Then the* **length** *of* g *denoted by* $l(g)$ *is the number of digits in* g.

Definition 2.3. *Let* $g = d_1d_2\ldots d_k$ *be a generator of* n. *The* **replication number** *of* g *denoted by* $r(g)$ *is the number of replication performed in* g *in order to generate* n.

Example 2.4. *To illustrate the concepts being discussed, we consider some examples.*

(i) Consider the positive repetitive integer $n_1 = 394\ 394$ *in the Repetitious Number Puzzle. The positive integer* $g_1 = 394$ *is a generator for* n_1 *with length* $l(g_1) = 3$ *and replication number* $r(g_1) = 2$.

(ii) The positive repetitive integer $n_2 = 111\ 111$ *is generated by* $g_2 = 1$ *with length* $l(g_2) = 1$ *and replication number* $r(g_2) = 6$. *The integers 11, 111, and 111 111 are the other generators of* n_2.

(iii) The positive integer $n_3 = 223\ 344$ *generates itself with length* 6 *and replication number* 1.

Note that as shown in Example 2.4, a generator is not unique. Moreover, for any positive integer n, we have n generates itself. Finally, if n is not repetitive, then its generator is unique. We now state a very important divisibility property whose proof is available in any standard number theory text such as [5] and [6].

Lemma 2.5. *Let* $a, d_1, d_2, \ldots, d_n$ *be integers. If* $a|d_j$ *for* $j = 1, 2, \ldots, n$ *then*

$$a|(d_1x_1 + d_2x_2 + \ldots + d_nx_n)$$

for all integers $x_1, x_2, \ldots, x_n$.

By taking $x_1 = x_2 = \ldots = x_n = 1$, we have a corollary that will be used extensively in the proof of main results.

Corollary 2.6. *Let* $a, d_1, d_2, \ldots, d_n$ *be integers. If* $a|d_k$ *for* $k = 1, 2, \ldots, n$ *then*

$$a|(d_1 + d_2 + \ldots + d_n).$$

We end this subsection by noting that if g generates n with replication number r, we write $n = g_r$.

2.3 Solution of the Repetitious Number Puzzle

The solution discussed in this subsection is due to the solution presented by Gardner [1].

Any three digit number takes the form $d_1d_2d_3$ where d_1, d_2 and d_3 are non-negative integers with bounds

$$0 < d_1 \leq 9$$
$$0 \leq d_2 \leq 9$$
$$0 \leq d_3 \leq 9.$$

Repeating the digits in the same order yields the six-digit integer $d_1d_2d_3d_1d_2d_3$. This integer can be factored into $1\ 001 \times d_1d_2d_3$ as shown in the following computation

$$
\begin{array}{rcccccc}
 & & & 1 & 0 & 0 & 1 \\
\times & & & & d_1 & d_2 & d_3 \\
\hline
 & & & d_3 & 0 & 0 & d_3 \\
 & & d_2 & 0 & 0 & d_2 & \\
+ & d_1 & 0 & 0 & d_1 & & \\
\hline
d_1 & d_2 & d_3 & d_1 & d_2 & d_3. &
\end{array}
$$

Thus, $d_1d_2d_3$ and 1 001 divides $d_1d_2d_3d_1d_2d_3$ and that $d_1d_2d_3d_1d_2d_3 = 1\ 001 \times d_1d_2d_3$. By the Fundamental Theorem of Arithmetic, 1 001 can be expressed as a product of primes $7, 11$ and 13. Hence $7, 11$ and 13 divides $d_1d_2d_3d_1d_2d_3$ and that

$$d_1d_2d_3d_1d_2d_3 = 7 \times 11 \times 13 \times d_1d_2d_3.$$

Using the Division Algorithm, dividing the integer $d_1d_2d_3d_1d_2d_3$ by 7 gives the integer $11 \times 13 \times d_1d_2d_3$ with remainder 0. Using the Division Algorithm again, dividing the integer $11 \times 13 \times d_1d_2d_3$ by 11 gives the integer $13 \times d_1d_2d_3$ with remainder 0. A final application of Division Algorithm on dividing the integer $13 \times d_1d_2d_3$ by 13 gives the integer $d_1d_2d_3$ with remainder 0.
Hence, dividing the six-digit repetitive number $d_1d_2d_3d_1d_2d_3$ in succession by the integers $7, 11$ and 13 returns the repetitive number into its generator $d_1d_2d_3$. This solves the puzzle.
Before we end this subsection, we note that the order of dividing the integer $d_1d_2d_3d_1d_2d_3$ by the integers $7, 11$ and 13 do not matter in the puzzle. For $d_1d_2d_3d_1d_2d_3$ can be written as

$$7 \times (11 \times 13 \times d_1d_2d_3), 7 \times (13 \times 11 \times d_1d_2d_3)\ 11 \times (7 \times 13 \times d_1d_2d_3), 11 \times (13 \times 7 \times d_1d_2d_3),$$
$$13 \times (7 \times 11 \times d_1d_2d_3) \text{ and } 13 \times (11 \times 7 \times d_1d_2d_3).$$

With all the needed definitions and results being stated, we are now ready to present our main results.

3. Results

3.1 Extension on the Length of Generator

In this subsection, we answer the question: Suppose, in the *repetitious number puzzle*, spectator A was asked to write down any $k-$digit positive integer. To what sequence of prime numbers does the resulting $2k-$digit number be divided in order to return to the original $k-$digit number? The solution to the problem is given in the result below.

Theorem 3.1. *Let $n = (d_1d_2 \ldots d_k)_2$ be a repetitive number generated by $g = d_1d_2 \ldots d_k$ of length k. Then the prime factors of $a(k)$ solves the extended repetitious number puzzle.*

Proof. Given a repetitive number $n = (d_1d_2 \ldots d_k)_2$, we express it as a sum of two positive integers both divisible by $g = d_1d_2 \ldots d_k$. In particular n can be expressed as the sum

$$\begin{array}{r} d_1\, d_2 \ldots d_k\ 0\ \ 0\ \ldots\ 0 \\ +\qquad\qquad\quad d_1\, d_2 \ldots\, d_k \\ \hline d_1\, d_2 \ldots d_k\, d_1\, d_2 \ldots d_k. \end{array}$$

Note that since $g \mid g$ and $g \mid d_1d_2 \ldots d_k \underbrace{00\ldots0}_{\text{k-zeros}}$, by Corollary 2.6 we have

$$g \mid (g + d_1d_2 \ldots d_k \underbrace{00\ldots0}_{\text{k-zeros}}).$$

But $g + d_1d_2 \ldots d_k \underbrace{00\ldots0}_{\text{k-zeros}} = n$. So, $g \mid n$.

After factoring out the common factor g in both summands we have

$$\begin{aligned} n &= g \times (1 + 1\underbrace{00\ldots0}_{\text{k-zeros}}) \\ &= g \times (1\ \underbrace{00\ldots0}_{k-1\text{-zeros}}\ 1) \\ &= g \times a(k). \end{aligned}$$

By the Fundamental Theorem of Arithmetic, $a(k)$ is either a prime or a product of primes. If $a(k)$ is prime, then the finite sequence of divisors to be divided to n to become g is $a(k)$ itself. If $a(k)$ is non-prime then the finite sequence of divisors to be divided to n to become g is the finite sequence whose terms are the prime divisors of $a(k)$. ☐

Example 3.2. *Suppose that spectator A wrote the number* $g = 451\ 220\ 125$. *Duplicating* g *gives the number* $n = 451\ 220\ 125\ 451\ 220\ 125$. *Dividing* n *by the numbers* $7, 11, 13, 19$ *and* $52\ 579$, *which are the prime divisors of* $a(9) = 1\ 000\ 000\ 001$ *gives the original number* $g = 451\ 220\ 125$.

3.2 Extension on the Number of Replication

In this subsection, we answer the question: Suppose that in the *repetitious number puzzle* spectator A was asked to write down any $3-$digit positive integer and replicate it $r-$times. To what sequence of divisors does the resulting $3r-$digit number be divided in order to produce the original $3-$digit number? The solution to the problem is given in the next result.

Theorem 3.3. *Let* $n = (d_1d_2d_3)_r$ *be a repetitive number generated by* $g = d_1d_2d_3$ *of length* 3. *Then the prime factors of* $b(r-1)$ *solves the extended repetitious number puzzle.*

Proof. Given a repetitive number $n = (d_1d_2d_3)_r$, we express it as a sum of r positive integers both divisible by $g = d_1d_2d_3$. In particular n can be expressed as the sum

$$n = d_1d_2d_3(0)_{3(r-1)} + d_1d_2d_3(0)_{3(r-2)} + \ldots + d_1d_2d_3(0)_{3(r-r)}.$$

Since $g \mid g$ and $g \mid d_1d_2d_3(0)_{3j}$, for $j = 1, 2, \ldots r-1$, by Corollary 2.6, we have

$$g \mid d_1d_2d_3(0)_{3(r-1)} + d_1d_2d_3(0)_{3(r-2)} + \ldots + d_1d_2d_3(0)_{3(r-r)}.$$

So, $g \mid n$.

Factoring out the common factor g in all of the summands we have

$$\begin{aligned} n &= g \times \left(1(0)_{3(r-1)} + 1(0)_{3(r-2)} + \ldots + 1(0)_{3(r-r)}\right) \\ &= g \times b(r-1). \end{aligned}$$

By the Fundamental Theorem of Arithmetic, $b(r-1)$ is either a prime or a product of primes. However, (except for the zeroth term) the terms of the sequence A261544 are all composite [7]. So the finite sequence of divisors to be divided to n in order to become g is the finite sequence whose terms are the prime divisors of $b(r-1)$. □

Example 3.4. *Suppose that spectator A wrote the number $g = 721$. Replicating g 4-times gives the number $n = 721\ 721\ 721\ 721$. Dividing n by the numbers $7, 11, 13, 101, 9\ 901$, which are the prime divisors of $b(3) = 1\ 001\ 001\ 001$, gives the original number $g = 721$.*

3.3 The Generalized Repetitious Number Puzzle

Finally, in this subsection, we answer the generalized repetitious number puzzle. In the generalized repetitious number puzzle, we allow spectator A to write down any $k-$digit number and replicate it $r-$times to generate the integer $n = g_r$ with $l(g) = k$. The answer to the generalization is given in the final result of this paper.

Theorem 3.5. *Let $n = (d_1d_2 \ldots d_k)_r$ be a repetitive number generated by $g = d_1d_2 \ldots d_k$ of length k. Then the sequence of prime factors of the integer*

$$\left(1(0)_{k-1}\right)_{r-1}1$$

is a finite sequence such that n upon division by all the sequence terms becomes g.

Proof. Given a repetitive number $n = (d_1d_2 \ldots d_k)_r$, we express it as a sum of r positive integers both divisible by $g = d_1d_2 \ldots d_k$. In particular n can be expressed as the sum

$$n = d_1d_2 \ldots d_k(0)_{k(r-1)} + d_1d_2 \ldots d_k(0)_{k(r-2)} + \ldots + d_1d_2 \ldots d_k(0)_{k(r-r)}.$$

Since $g \mid g$ and $g \mid d_1d_2 \ldots d_k(0)_{kj}$, for $j = 1, 2, \ldots r-1$, by Corollary 2.6, we have

$$g \mid d_1d_2 \ldots d_k(0)_{k(r-1)} + d_1d_2 \ldots d_k(0)_{k(r-2)} + \ldots + d_1d_2 \ldots d_k(0)_{k(r-r)}.$$

So, $g \mid n$.

Factoring out the common factor g in all of the summands we have

$$n = g \times \left(1(0)_{k(r-1)} + 1(0)_{k(r-2)} + \ldots + 1(0)_{k(r-r)}\right)$$

$$= g \times \left(1(0)_{k-1}\right)_{r-1}1.$$

By the Fundamental Theorem of Arithmetic, $\left(1(0)_{k-1}\right)_{r-1}1$ is either a prime or a product of primes. If $\left(1(0)_{k-1}\right)_{r-1}1$ is prime, then the finite sequence of divisors to be divided to n to become g is $\left(1(0)_{k-1}\right)_{r-1}1$ itself. If $\left(1(0)_{k-1}\right)_{r-1}1$ is non-prime then the finite sequence of divisors to be divided to n to become g is the finite sequence whose terms are the prime divisors of $\left(1(0)_{k-1}\right)_{r-1}1$. □

Theorem 3.5 proves the validity of a Grade 7 teacher's clever way in verifying if his students correctly performed a series of division.

Example 3.6. A Relay Involving Division of Large Numbers. *Sir DELTA is a grade 7 mathematics teacher in the Philippines. To test the proficiency of his students on performing division of large numbers, he grouped his students such that each group is consist of 10 members.*

He then instructed the first student which we name S1 to write down in a 1/4 sheet of paper any $4-$*digit positive integer (say 2 019) and replicate it* $8-$ *times to get a* $32-$*digit number (*20 192 019 201 920 192 019 201 920 192 019*). Then he asked S1 to give the paper containing the* $32-$*digit number to S2. S2 then was asked to divide the* $32-$*digit number by* 17 *and write down the answer (1 187 765 835 407 070 118 776 583 540 707) in another 1/4 sheet of paper. After S2 was done writing the answer in a 1/4 sheet of paper, Sir Delta asked S2 to give the paper to S3.*

Denote by A_n *the answer of student n. Suppose that the process continues with the following given*

S3 performs $A_2 \div 73$

S4 performs $A_3 \div 137$

S5 performs $A_4 \div 353$

S6 performs $A_5 \div 449$

S7 performs $A_6 \div 641$

S8 performs $A_7 \div 1\,409$

S9 performs $A_8 \div 69\,857$

S10 performs $A_9 \div 5\,882\,353$.

Sir DELTA then asked S10 to give his/her answer to him.

If Sir DELTA wants to determine whether his students performed their assigned division problem correctly or not, show that it is enough for him to ask S1: "Is this your $4-$*digit number?"*

Given below are the correct answers for the assigned sequence of divisions generated using Wolfram Alpha [8].

$$20\,192\,019\,201\,920\,192\,019\,201\,920\,192\,019 \div 17 = 1\,187\,765\,835\,407\,070\,118\,776\,583\,540\,707$$

$$1\,187\,765\,835\,407\,070\,118\,776\,583\,540\,707 \div 73 = 16\,270\,764\,868\,590\,001\,627\,076\,486\,859$$

$$16\,270\,764\,868\,590\,001\,627\,076\,486\,859 \div 137 = 118\,764\,707\,070\,000\,011\,876\,470\,707$$

$$118\ 764\ 707\ 070\ 000\ 011\ 876\ 470\ 707 \div 353 = 336\ 443\ 929\ 376\ 770\ 571\ 888\ 019$$

$$336\ 443\ 929\ 376\ 770\ 571\ 888\ 019 \div 449 = 749\ 318\ 328\ 233\ 342\ 030\ 931$$

$$749\ 318\ 328\ 233\ 342\ 030\ 931 \div 641 = 1\ 168\ 983\ 351\ 378\ 068\ 691$$

$$1\ 168\ 983\ 351\ 378\ 068\ 691 \div 1\ 409 = 829\ 654\ 614\ 178\ 899$$

$$829\ 654\ 614\ 178\ 899 \div 69\ 857 = 11\ 876\ 470\ 707$$

$$11\ 876\ 470\ 707 \div 5\ 882\ 353 = 2\ 019.$$

4. Concluding Remarks

In this paper, we were able to generalize the repetitious number puzzle. The generalization allows spectator A to write down any k-digit number and replicate it r number of times resulting to a new number n. In order for the resulting number to return to the original k-digit number, we must divide n by the prime factors of the number $\left(1(0)_{k-1}\right)_{r-1}1$. We call the number $\left(1(0)_{k-1}\right)_{r-1}1$ the (l,r) **co-divisor** of the k-digit number i.e. the generator. The name (l,r) co-divisor number is based from the idea that the number $\left(1(0)_{k-1}\right)_{r-1}1$ is dependent to the length (l) of the generator and its replication number (r).

The concept of (l,r) co-divisor number allows us to view the sequence A000533 and the sequence A261544 in the OEIS as a particular member of a family of sequence which we call (l,r) **co-divisor sequences**. In particular, if we let $s(k,r) = \left(1(0)_{k-1}\right)_{r-1}1$ we have

$$s(k,2) = a(k), k = 1,2,3,\ldots$$

where $a(k)$ is the k^{th} term of the sequence A000533. We also have

$$s(3,r) = b(r-1), r = 1,2,3,\ldots.$$

where $b(r-1)$ is the $(r-1)^{st}$ term of the sequence A261544.

Finally, we recommend further studies on the (l,r) co-divisor number and sequences.

Acknowledgements

The creation of this article would not be possible without the suggestion of the authors colleague Mr. Melchor A. Cupatan. The first author would also like to express his gratitude to Ms. Josephine Joy Tolentino-Antalan of Philippine Science High School Central Luzon Campus for her valuable comments that led to the improvement of the manuscript. The authors also thank the Central Luzon State University for their support and encouragement throughout the conduct of this research. Finally, the authors would like to thank the various referees for their valuable comments and suggestions that helped improve the content of the paper.

References

[1] M. Gardner, *The Second Scientific American Book of Mathematical Puzzles and Diversions*, The University of Chicago Press, Chicago, (1987).

[2] Y. Perelman, *Figures for Fun: Stories and Conundrums*, Foreign Language Publishing House, Moscow, (1957).

[3] N. J. A. Sloane, Sequence A000533 in The On-Line Encyclopedia of Integer Sequences.

[4] I. Gutkovskiy, Sequence A261544 in The On-Line Encyclopedia of Integer Sequences, (2015).

[5] D. M. Burton, *Elementary Number Theory Seventh Edition*, Mc Graw-Hill, New York, (2011).

[6] K. H. Rosen, *Elementary Number Theory and its Applications Sixth Edition*, Addison-Wesley, Boston, (2011).

[7] M. Gardiner, *The Mathematical Olympiad Handbook: An Introduction to Problem Solving Based on the First 32 British Mathematical Olympiads 1965-1996*, Oxford University Press, England, (1997).

[8] Wolfram Alpha, Wolfram Alpha LLC, (2020).