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α-Triangular Fuzzy Matrix and its Application

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Abstract: In this paper, a-triangular matrix is introduced with the help of α-cut of each its elements and the arithmetic operations on α-triangular matrices are defined. The determinant, adjoint, trace of it are also defined and with the help of it some properties are established. The correlation coefficient is defined between two triangular fuzzy numbers in terms of its a-cut and using this idea in α-triangular matrix it became feasible to solve a real life problem using analytic hierarchy process (AHP) for $\alpha \in [0, 1]$.

Keywords: α-triangular matrix, determinant, adjoint, trace, correlation coefficient.

1. Introduction

In the field of science and engineering there are ample uses of matrix theory. But the old matrix theory cannot solve the uncertainties in such areas. Real life problem which is fuzzy in nature are solved by fuzzy matrix theory that is developed by many researchers. Fuzzy matrices deals with both membership and non-membership values. The work on intuitionistic fuzzy matrix was already done by Khan, Shyamal and Pal [10] where membership and non membership values are points. Khan and Pal [6] has presented a new kind of matrix interval-valued intuitionistic fuzzy matrix (IVIFM) based on membership and non - membership value as intervals. Here we introduce α-triangular matrix by considering confidence level intervals, that is the a - cut of each entry of matrix. So it can be considered as a closed interval in terms of a. Arithmetic operations on this kind of matrices are defined. The determinant, adjoint, trace of the matrix is defined with the help of it some properties are established. The correlation coefficient is defined and using this real life problems can be solved using AHP for various values of a ranging from 0 to 1.

2. Preliminaries

Definition 2.1 (Fuzzy Matrix)**.** *A matrix $A = A = [a_{ij}]$ of order $m \times n$, whose components are in the unit interval [0,1] is called a fuzzy matrix.*

Definition 2.2 (α-cut)**.** *For $\alpha \in [0, 1]$, the set ${}^{\alpha}A = \{x \in X | \mu_A(x) \geq \alpha\}$ is said to be the α- cut of a fuzzy set A.*

Definition 2.3 (Triangular number fuzzy matrices)**.** *Let $A = (a_{ij})$ be a triangular number fuzzy matrix of order $m \times n$ where, the elements a_{ij} are triangular fuzzy numbers for every $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$.*

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Definition 2.4. *The α-triangular number fuzzy matrix is defined as* $A_\alpha = (a_{ij}^{\alpha})$ *where* $a_{11} = \left(a_{11}^{(1)}, a_{11}^{(2)}, a_{11}^{(3)}\right)$, *where* $a_{ij}^{\alpha} = \left(a_{ij}^{1(\alpha)}, a_{ij}^{2(\alpha)}\right)$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ *and* $a_{ij}^{1(\alpha)} = a_{ij}^{(1)} + \alpha\left(a_{ij}^{(2)} - a_{ij}^{(1)}\right)$; $a_{ij}^{2(\alpha)} = a_{ij}^{(3)} - \alpha\left(a_{ij}^{(3)} - a_{ij}^{(2)}\right)$.

Definition 2.5 (Triangular fuzzy Number)**.** *A triangular fuzzy number on* $\mathbb{R}$ *is a fuzzy number* A *which has a membership function*

$$\mu_A(x) = \begin{cases} 0 & x < a_1,\ a_3 \le x \\ \frac{x-a_1}{a_2-a_1} & a_1 \le x < a_2 \\ \frac{a_3-x}{a_3-a_2} & a_2 \le x < a_3 \\ 0 & x > a_3 \end{cases}$$

where $a_i \in \mathbb{R}$, $i = 1, 2, 3$.

Definition 2.6. *Let* A *and* B *be a triangular fuzzy matrix of order* $m \times n$, A_α *and* B_α *be the* α*-triangular fuzzy matrix, then*

$$\begin{aligned}
(A+B)_\alpha &= A_\alpha(+)B_\alpha \\
&= [a_{ij}^{\alpha}](+)[b_{ij}^{\alpha}] \\
&= [a_{ij}^{1(\alpha)}, a_{ij}^{2(\alpha)}](+)[b_{ij}^{1(\alpha)}, b_{ij}^{2(\alpha)}] \\
&= [a_{ij}^{1(\alpha)} + b_{ij}^{1(\alpha)}, a_{ij}^{2(\alpha)} + b_{ij}^{2(\alpha)}] \\
(A-B)_\alpha &= A_\alpha(-)B_\alpha \\
&= [a_{ij}^{\alpha}](-)[b_{ij}^{\alpha}] \\
&= [a_{ij}^{1(\alpha)}, a_{ij}^{2(\alpha)}](-)[b_{ij}^{1(\alpha)}, b_{ij}^{2(\alpha)}] \\
&= [a_{ij}^{1(\alpha)} - b_{ij}^{2(\alpha)}, a_{ij}^{2(\alpha)} - b_{ij}^{1(\alpha)}] \\
(A \cdot B)_\alpha &= A_\alpha(\cdot)B_\alpha \\
&= [a_{ij}^{\alpha}](\cdot)[b_{ij}^{\alpha}] \\
&= [a_{ij}^{1(\alpha)}, a_{ij}^{2(\alpha)}](\cdot)[b_{ij}^{1(\alpha)}, b_{ij}^{2(\alpha)}] \\
&= \left[\min\left(a_{ij}^{1(\alpha)} \cdot b_{ij}^{1(\alpha)}, a_{ij}^{1(\alpha)} \cdot b_{ij}^{2(\alpha)}, a_{ij}^{2(\alpha)} \cdot b_{ij}^{1(\alpha)}, a_{ij}^{2(\alpha)} \cdot b_{ij}^{2(\alpha)}\right), \max\left(a_{ij}^{1(\alpha)} \cdot b_{ij}^{1(\alpha)}, a_{ij}^{1(\alpha)} \cdot b_{ij}^{2(\alpha)}, a_{ij}^{2(\alpha)} \cdot b_{ij}^{1(\alpha)}, a_{ij}^{2(\alpha)} \cdot b_{ij}^{2(\alpha)}\right)\right]
\end{aligned}$$

Example 2.7. *Let us verify* $[A+B]_\alpha = [A_\alpha] + [B_\alpha]$ *by considering the following numerical example.*

$$\begin{pmatrix} (1,2,3) & (3,4,5) \\ (5,6,7) & (7,8,9) \end{pmatrix} + \begin{pmatrix} (3,4,5) & (1,2,3) \\ (7,8,9) & (5,6,7) \end{pmatrix} = \begin{pmatrix} (4,6,8,) & (4,6,8) \\ (12,14,16) & (12,14,16) \end{pmatrix}$$
$$= \begin{pmatrix} [2\alpha+4, 8-2\alpha] & [2\alpha+4, 8-2\alpha] \\ [2\alpha+12, 16-2\alpha] & [2\alpha+12, 16-2\alpha] \end{pmatrix}$$

$$\begin{pmatrix} (1,2,3) & (3,4,5) \\ (5,6,7) & (7,8,9) \end{pmatrix} + \begin{pmatrix} (3,4,5) & (1,2,3) \\ (7,8,9) & (5,6,7) \end{pmatrix} = \begin{pmatrix} [\alpha+1, 3-\alpha] & [\alpha+3, 5-\alpha] \\ [\alpha+5, 7-\alpha] & [\alpha+7, 9-\alpha] \end{pmatrix} + \begin{pmatrix} [\alpha+3, 5-\alpha] & [\alpha+1, 3-\alpha] \\ [\alpha+7, 9-\alpha] & [\alpha+5, 7-\alpha] \end{pmatrix}$$
$$= \begin{pmatrix} [2\alpha+4, 8-2\alpha] & [2\alpha+4, 8-2\alpha] \\ [2\alpha+12, 16-2\alpha] & [2\alpha+12, 16-2\alpha] \end{pmatrix}$$

Example 2.8. *Let us verify* $[A-B]_\alpha = [A_\alpha] - [B_\alpha]$ *by considering the following numerical example.*

$$\begin{pmatrix} (1,2,3) & (3,4,5) \\ (5,6,7) & (7,8,9) \end{pmatrix} - \begin{pmatrix} (3,4,5) & (1,2,3) \\ (7,8,9) & (5,6,7) \end{pmatrix} = \begin{pmatrix} (-4,-2,0) & (0,2,4) \\ (-4,-2,0) & (0,2,4) \end{pmatrix}$$

$$= \begin{pmatrix} [2\alpha-4,-2\alpha] & [2\alpha,4-2\alpha] \\ [2\alpha-4,-2\alpha] & [2\alpha,4-2\alpha] \end{pmatrix}$$

$$\begin{pmatrix} (1,2,3) & (3,4,5) \\ (5,6,7) & (7,8,9) \end{pmatrix} - \begin{pmatrix} (3,4,5) & (1,2,3) \\ (7,8,9) & (5,6,7) \end{pmatrix} = \begin{pmatrix} [\alpha+1,3-\alpha] & [\alpha+3,5-\alpha] \\ [\alpha+5,7-\alpha] & [\alpha+7,9-\alpha] \end{pmatrix} - \begin{pmatrix} [\alpha+3,5-\alpha] & [\alpha+1,3-\alpha] \\ [\alpha+7,9-\alpha] & [\alpha+5,7-\alpha] \end{pmatrix}$$

$$= \begin{pmatrix} [2\alpha-4,-2\alpha] & [2\alpha,4-2\alpha] \\ [2\alpha-4,-2\alpha] & [2\alpha,4-2\alpha] \end{pmatrix}$$

Example 2.9. *Let us verify $(A \cdot B)_\alpha = A_\alpha(\cdot)B_\alpha$ by considering the following numerical example.*

$$\begin{pmatrix} (1,2,3) & (3,4,5) \\ (5,6,7) & (7,8,9) \end{pmatrix} \cdot \begin{pmatrix} (3,4,5) & (1,2,3) \\ (7,8,9) & (5,6,7) \end{pmatrix} = \begin{pmatrix} [\alpha+1,3-\alpha] & [\alpha+3,5-\alpha] \\ [\alpha+5,7-\alpha] & [\alpha+7,9-\alpha] \end{pmatrix} \cdot \begin{pmatrix} [\alpha+3,5-\alpha] & [\alpha+1,3-\alpha] \\ [\alpha+7,9-\alpha] & [\alpha+5,7-\alpha] \end{pmatrix}$$

$$\begin{aligned}(a_{11}^{(\alpha)} \cdot b_{11}^{(\alpha)}) &= [\alpha+1,3-\alpha]\,[\alpha+3,5-\alpha] \\ &= [\min\{(\alpha+1)(\alpha+3),(\alpha+1)(5-\alpha),(3-\alpha)(\alpha+3),(3-\alpha)(5-\alpha)\}, \\ &\quad \max\{(\alpha+1)(\alpha+3),(\alpha+1)(5-\alpha),(3-\alpha)(\alpha+3),(3-\alpha)(5-\alpha)\}] \\ &= \left[\min\left\{(\alpha^2+4\alpha+3),(-\alpha^2+4\alpha+5),(-\alpha^2+9),(\alpha^2-8\alpha+15)\right\}, \right. \\ &\quad \left. \max\left\{(\alpha^2+4\alpha+3),(-\alpha^2+4\alpha+5),(-\alpha^2+9),(\alpha^2-8\alpha+15)\right\}\right]\end{aligned}$$

when $\alpha = 0$,

$$= [\min(3,5,9,15), \max(3,5,9,15)] = [3,15]$$

when $\alpha = \frac{1}{2}$,

$$= \left[\min\left(\frac{21}{4},\frac{27}{4},\frac{35}{4},\frac{45}{4}\right), \max\left(\frac{21}{4},\frac{27}{4},\frac{35}{4},\frac{45}{4}\right)\right] = [5.25, 11.25]$$

when $\alpha = 1$,

$$= [\min(8,8,8,8), \max(8,8,8,8)] = [8,8]$$

For approximation, let us take $\alpha^2 = \alpha$, *then*

$$(a_{11}^{(\alpha)} \cdot b_{11}^{(\alpha)}) = [\alpha^2+4\alpha+3, \alpha^2-8\alpha+15] = [5\alpha+3, -7\alpha+15]$$

Similarly we obtain,

$$\begin{aligned}(a_{12}^{(\alpha)} \cdot b_{21}^{(\alpha)}) &= [\alpha^2+10\alpha+21, \alpha^2-14\alpha+45] = [11\alpha+21, -13\alpha+45] \\ (a_{11}^{(\alpha)} \cdot b_{12}^{(\alpha)}) &= [\alpha^2+2\alpha+1, \alpha^2-6\alpha+9] = [3\alpha+1, -5\alpha+9] \\ (a_{12}^{(\alpha)} \cdot b_{22}^{(\alpha)}) &= [\alpha^2+8\alpha+15, \alpha^2-12\alpha+35] = [9\alpha+15, -11\alpha+35] \\ (a_{21}^{(\alpha)} \cdot b_{11}^{(\alpha)}) &= [\alpha^2+8\alpha+15, \alpha^2-12\alpha+35] = [9\alpha+15, -11\alpha+35] \\ (a_{22}^{(\alpha)} \cdot b_{21}^{(\alpha)}) &= [\alpha^2+14\alpha+49, \alpha^2-18\alpha+81] = [15\alpha+49, -17\alpha+81]\end{aligned}$$

$$(a_{21}^{(\alpha)} \cdot b_{12}^{(\alpha)}) = [\alpha^2 + 6\alpha + 5, \alpha^2 - 10\alpha + 21] = [7\alpha + 5, -9\alpha + 21]$$

$$(a_{22}^{(\alpha)} \cdot b_{22}^{(\alpha)}) = [\alpha^2 + 12\alpha + 35, \alpha^2 - 16\alpha + 63] = [13\alpha + 35, -15\alpha + 63]$$

Again, $\begin{pmatrix} (1,2,3) & (3,4,5) \\ (5,6,7) & (7,8,9) \end{pmatrix} \cdot \begin{pmatrix} (3,4,5) & (1,2,3) \\ (7,8,9) & (5,6,7) \end{pmatrix}$

$$= \begin{pmatrix} [5\alpha+3, -7\alpha+15] + [11\alpha+21, -13\alpha+45] & [3\alpha+1, -5\alpha+9] + [9\alpha+15, -11\alpha+35] \\ [9\alpha+15, -11\alpha+35] + [15\alpha+49, -17\alpha+81] & [7\alpha+5, -9\alpha+21] + [13\alpha+35, -15\alpha+63] \end{pmatrix}$$

$$= \begin{pmatrix} [16\alpha+24, -20\alpha+60] & [12\alpha+16, -16\alpha+44] \\ [24\alpha+64, -28\alpha+116] & [20\alpha+40, -24\alpha+84] \end{pmatrix}$$

$$\begin{pmatrix} (1,2,3) & (3,4,5) \\ (5,6,7) & (7,8,9) \end{pmatrix} \cdot \begin{pmatrix} (3,4,5) & (1,2,3) \\ (7,8,9) & (5,6,7) \end{pmatrix}$$

We see that,

$$\begin{aligned}
(a_{11} \cdot b_{11}) &= (1,2,3) \cdot (3,4,5) \\
&= (\min\{1\cdot 3, 1\cdot 5, 3\cdot 3, 3\cdot 5\}, 2\cdot 4, \max\{1\cdot 3, 1\cdot 5, 3\cdot 3, 3\cdot 5\}) \\
&= (\min\{3,5,9,15\}, 8, \max\{3,5,9,15\}) = (3,8,15) = [5\alpha+3, -7\alpha+15] \\
(a_{11} \cdot b_{11}) &= [5\alpha+3, -7\alpha+15](a_{12} \cdot b_{21}) = [11\alpha+21, -13\alpha+45] \\
(a_{11} \cdot b_{12}) &= [3\alpha+1, -5\alpha+9](a_{12} \cdot b_{22}) = [9\alpha+15, -11\alpha+35] \\
(a_{21} \cdot b_{11}) &= [9\alpha+15, -11\alpha+35](a_{22} \cdot b_{21}) = [15\alpha+49, -17\alpha+81] \\
(a_{21} \cdot b_{12}) &= [7\alpha+5, -9\alpha+21](a_{22} \cdot b_{22}) = [13\alpha+35, -15\alpha+63]
\end{aligned}$$

Hence, $\begin{pmatrix} (1,2,3) & (3,4,5) \\ (5,6,7) & (7,8,9) \end{pmatrix} \cdot \begin{pmatrix} (3,4,5) & (1,2,3) \\ (7,8,9) & (5,6,7) \end{pmatrix}$

$$= \begin{pmatrix} [5\alpha+3, -7\alpha+15] + [11\alpha+21, -13\alpha+45] & [3\alpha+1, -5\alpha+9] + [9\alpha+15, -11\alpha+35] \\ [9\alpha+15, -11\alpha+35] + [15\alpha+49, -17\alpha+81] & [7\alpha+5, -9\alpha+21] + [13\alpha+35, -15\alpha+63] \end{pmatrix}$$

$$= \begin{pmatrix} [16\alpha+24, -20\alpha+60] & [12\alpha+16, -16\alpha+44] \\ [24\alpha+64, -28\alpha+116] & [20\alpha+40, -24\alpha+84] \end{pmatrix}$$

Definition 2.10. *Determinant of an α-triangular fuzzy matrix of order $n \times n$ is denoted by $|A_\alpha|$ or $det(A)$ and is defined as,*

$$|A_\alpha| = \sum_{\sigma \in S_n} \prod_{i=1}^{n} \langle [a_{i\sigma(i)}^{1(\alpha)}, a_{i\sigma(i)}^{2(\alpha)}] \rangle$$

S_n denotes the symmetric group of all permutations of the symbols $\{1, 2, \ldots, n\}$.

Example 2.11. *Let us find the determinant of a 3×3, α-triangular matrix as follows:*

$$\begin{vmatrix} (1,2,3) & (3,4,5) & (7,8,9) \\ (4,5,6) & (5,6,7) & (2,3,4) \\ (6,7,8) & (7,8,9) & (1,2,3) \end{vmatrix} = \begin{vmatrix} [\alpha+1, 3-\alpha] & [\alpha+3, 5-\alpha] & [\alpha+7, 9-\alpha] \\ [\alpha+4, 6-\alpha] & [\alpha+5, 7-\alpha] & [\alpha+2, 4-\alpha] \\ [\alpha+6, 8-\alpha] & [\alpha+7, 9-\alpha] & [\alpha+1, 3-\alpha] \end{vmatrix}$$

$$= ([\alpha+1,3-\alpha]\{[\alpha+5,7-\alpha][\alpha+1,3-\alpha]+[\alpha+2,4-\alpha][\alpha+7,9-\alpha]\}$$
$$+[\alpha+3,5-\alpha]\{[\alpha+4,6-\alpha][\alpha+1,3-\alpha]+[\alpha+2,4-\alpha][\alpha+6,8-\alpha]\}$$
$$+[\alpha+7,9-\alpha]\{[\alpha+4,6-\alpha][\alpha+7,9-\alpha]+[\alpha+5,7-\alpha][\alpha+6,8-\alpha]\}) \text{ (by putting } \alpha^2{=}\alpha)$$
$$= ([\alpha+1,3-\alpha]\{[7\alpha+5,-9\alpha+21]+[10\alpha+14,-12\alpha+36]\}$$
$$+[\alpha+3,5-\alpha]\{[6\alpha+4,-8\alpha+18]+[9\alpha+12,-11\alpha+32]\}$$
$$+[\alpha+7,9-\alpha]\{[12\alpha+28,-14\alpha+54]+[12\alpha+30,-14\alpha+56]\})$$
$$= ([\alpha+1,3-\alpha][17\alpha+19,-21\alpha+57]+[\alpha+3,5-\alpha][15\alpha+16,-19\alpha+50]$$
$$+[\alpha+7,9-\alpha][24\alpha+58,-28\alpha+110])$$
$$= [53\alpha+19,-91\alpha+171]+[76\alpha+48,-136\alpha+250]+[250\alpha+406,-334\alpha+990]$$
$$= [379\alpha+473,-559\alpha+1411]$$

Definition 2.12. *The adjoint of α-triangular fuzzy matrix of order $n \times n$, which is denoted by adj A, is defined by $[A_{ji}]$, where A_{ji} is the determinant of the α-triangular fuzzy matrix of order $(n-1)\times(n-1)$ formed by suppressing row j and column i of the α-triangular fuzzy matrix. In other word, $[A_{ji}]$ can be written in the form $\sum_{\sigma \in S_{n_i n_j}} \prod_{t \in n_j} \left\langle [a^{1(\alpha)}_{t\sigma(t)}, a^{2(\alpha)}_{t\sigma(t)}] \right\rangle$, where $n_j = \{1,2,...,n\}n\{j\}$ and $S_{n_i n_j}$ is the set of all permutation of set n_j over the set n_i.*

Definition 2.13. *The transpose of an α-triangular fuzzy matrix $A_\alpha = [a^{1(\alpha)}_{ij}, a^{2(\alpha)}_{ij}]_{m\times n}$ is defined as $[A_\alpha]^T = [a^{1(\alpha)}_{ji}, a^{2(\alpha)}_{ji}]_{n\times m}$.*

Definition 2.14. *Let A_α be an α-triangular fuzzy matrix of order $n \times n$, the trace of A_α is defined as*

$$\begin{aligned} Tr(A_\alpha) &= Tr(a^\alpha_{ii}) \\ &= (a^\alpha_{11} + a^\alpha_{22} + ... + a^\alpha_{nn}) \\ &= \left(\left[a^{1(\alpha)}_{11}, a^{2(\alpha)}_{11}\right] + \left[a^{1(\alpha)}_{22}, a^{2(\alpha)}_{22}\right] + ... + \left[a^{1(\alpha)}_{nn}, a^{2(\alpha)}_{nn}\right]\right) \\ &= \left[\left(a^{1(\alpha)}_{11} + a^{1(\alpha)}_{22} + ... + a^{1(\alpha)}_{nn}\right), \left(a^{2(\alpha)}_{22} + a^{2(\alpha)}_{11} + ... + a^{2(\alpha)}_{nn}\right)\right] \\ &= \left[\sum_{i=1}^{n} a^{1(\alpha)}_{ii}, \sum_{i=1}^{n} a^{2(\alpha)}_{ii}\right], \quad \text{i} = 1,2,...,\text{n} \end{aligned}$$

where a^α_{ii} be an $(i,i)^{th}$ term of an α-triangular fuzzy matrix.

Theorem 2.15. *For A_α and B_α be an α-triangular fuzzy matrix of order $n \times n$ and k a scalar*

(1). $Tr(A+B)_\alpha = Tr(A_\alpha) + Tr(B_\alpha)$.

(2). $Tr(kA)_\alpha = kTr(A_\alpha)$.

(3). $Tr(A^T)_\alpha = Tr(A_\alpha)$.

Proof.

(1). $Tr(A+B)_\alpha = Tr(A_\alpha + B_\alpha)$

$$
\begin{aligned}
&= Tr\left[(a_{ij}^{\alpha}) + (b_{ij}^{\alpha})\right] \\
&= Tr\left[(a_{ij}^{1(\alpha)} + b_{ij}^{1(\alpha)}), (a_{ij}^{2(\alpha)} + b_{ij}^{2(\alpha)})\right] \\
&= \left[(a_{11}^{1(\alpha)} + b_{11}^{1(\alpha)}), (a_{11}^{2(\alpha)} + b_{11}^{2(\alpha)})\right] + \left[(a_{22}^{1(\alpha)} + b_{22}^{1(\alpha)}), (a_{22}^{2(\alpha)} + b_{22}^{2(\alpha)})\right] + ... \\
&+ \left[(a_{nn}^{1(\alpha)} + b_{nn}^{1(\alpha)}), (a_{nn}^{2(\alpha)} + b_{nn}^{2(\alpha)})\right] \\
&= \left[\sum_{i=1}^{n}(a_{ii}^{1(\alpha)} + b_{ii}^{1(\alpha)}), \sum_{i=1}^{n}(a_{ii}^{2(\alpha)} + b_{ii}^{2(\alpha)})\right] \\
&= \left[\sum_{i=1}^{n} a_{ii}^{1(\alpha)}, \sum_{i=1}^{n} a_{ii}^{2(\alpha)}\right] + \left[\sum_{i=1}^{n} b_{ii}^{1(\alpha)}, \sum_{i=1}^{n} b_{ii}^{2(\alpha)}\right] \\
&= Tr(A_\alpha) + Tr(B_\alpha)
\end{aligned}
$$

(2). $Tr(kA)_\alpha = Tr(ka_{ij}^{\alpha})$

$$
\begin{aligned}
&= Tr\left[ka_{ij}^{1(\alpha)}, ka_{ij}^{2(\alpha)}\right] \\
&= \sum_{i=1}^{n}\left[ka_{ii}^{1(\alpha)}, ka_{ii}^{2(\alpha)}\right] \\
&= k\sum_{i=1}^{n}\left[a_{ii}^{1(\alpha)}, a_{ii}^{2(\alpha)}\right] \\
&= k\left[Tr(a_{ij}^{\alpha})\right] \\
&= k\left[Tr(A_\alpha)\right]
\end{aligned}
$$

(3). $Tr(A^T)_\alpha = Tr\left[(a_{ij}^{\alpha})^T\right]$

$$
\begin{aligned}
&= Tr\left[(a_{ji}^{\alpha})\right] \\
&= Tr\left[a_{ji}^{1(\alpha)}, a_{ji}^{2(\alpha)}\right] \\
&= \sum_{i=1}^{n}\left[a_{ii}^{1(\alpha)}, a_{ii}^{2(\alpha)}\right] \\
&= Tr\left[a_{ij}^{1(\alpha)}, a_{ij}^{2(\alpha)}\right] \\
&= Tr\left[a_{ij}^{\alpha}\right] \\
&= Tr\left[A_\alpha\right]
\end{aligned}
$$

□

Theorem 2.16. *Let A_α and B_α be an α-triangular fuzzy matrix of order $m \times n$*

$$\left[(AB)_\alpha\right]^T = (B_\alpha)^T \cdot (A_\alpha)^T$$

Proof. Let $A_\alpha = (a_{ij}^{\alpha})$, $B_\alpha = (b_{ij}^{\alpha})$ then $(A_\alpha)^T = (a_{ji}^{\alpha}), (B_\alpha)^T = (b_{ji}^{\alpha})$, where a_{ij}^{α} and b_{ij}^{α} are a triangular fuzzy number.

$$
\begin{aligned}
[(AB)_\alpha]^T &= [(AB)^T]_\alpha = [B^T A^T]_\alpha = B_\alpha^T A_\alpha^T \\
(A_\alpha)^T &= \left(A^T\right)_\alpha \\
\left(A^T\right)_\alpha &= (a_{ij}^{\alpha})^T = \left(\left[a_{ij}^{1(\alpha)}, a_{ij}^{2(\alpha)}\right]\right)^T = \left(\left[a_{ji}^{1(\alpha)}, a_{ji}^{2(\alpha)}\right]\right) \\
\left(A^T\right) &= (a_{ji}) \\
\left(A^T\right)_\alpha &= (a_{ji})_\alpha = \left[a_{ji}^{1(\alpha)}, a_{ji}^{2(\alpha)}\right]
\end{aligned}
$$

□

Proposition 2.17. *If A is a constant α-triangular fuzzy matrix and B is an α-triangular fuzzy matrix of the same order then AB is a constant α-triangular fuzzy matrix.*

Proof. Consider an α-triangular constant fuzzy matrix A_α, where $A_\alpha = \left[a_{ij}^{1(\alpha)}, a_{ij}^{2(\alpha)}\right]$, $a_{ij}^{1(\alpha)}$ and $a_{ij}^{2(\alpha)}$ are the same for all i and j. Let $B_\alpha = \left[b_{ij}^{1(\alpha)}, b_{ij}^{2(\alpha)}\right]$ be a α-triangular fuzzy matrix. By the definition of $A_\alpha \cdot B_\alpha$ we have

$$\begin{aligned}A_\alpha \cdot B_\alpha &= \left(\left[a_{ij}^{1(\alpha)}, a_{ij}^{2(\alpha)}\right] \cdot \left[b_{ij}^{1(\alpha)}, b_{ij}^{2(\alpha)}\right]\right)\\ &= \left[\min\left(a_{ij}^{1(\alpha)} \cdot b_{ij}^{1(\alpha)}, a_{ij}^{1(\alpha)} \cdot b_{ij}^{2(\alpha)}, a_{ij}^{2(\alpha)} \cdot b_{ij}^{1(\alpha)}, a_{ij}^{2(\alpha)} \cdot b_{ij}^{2(\alpha)}\right), \max\left(a_{ij}^{1(\alpha)} \cdot b_{ij}^{1(\alpha)}, a_{ij}^{1(\alpha)} \cdot b_{ij}^{2(\alpha)}, a_{ij}^{2(\alpha)} \cdot b_{ij}^{1(\alpha)}, a_{ij}^{2(\alpha)} \cdot b_{ij}^{2(\alpha)}\right)\right]\end{aligned}$$

The elements of $A_\alpha \cdot B_\alpha$are i - independent. Therefore $A_\alpha \cdot B_\alpha$ is constant. □

Proposition 2.18. *If any two rows (or columns) of a square α-triangular fuzzy matrix are interchanged then determinant of that α-triangular fuzzy matrix remain unchanged.*

Proof. Let A_α be a α-triangular fuzzy matrix then,

$$A_\alpha = \begin{pmatrix} \left[a_{11}^{1(\alpha)}, a_{11}^{2(\alpha)}\right] & \left[a_{12}^{1(\alpha)}, a_{12}^{2(\alpha)}\right] & \left[a_{13}^{1(\alpha)}, a_{13}^{2(\alpha)}\right] \\ \left[a_{21}^{1(\alpha)}, a_{21}^{2(\alpha)}\right] & \left[a_{22}^{1(\alpha)}, a_{22}^{2(\alpha)}\right] & \left[a_{23}^{1(\alpha)}, a_{23}^{2(\alpha)}\right] \\ \left[a_{31}^{1(\alpha)}, a_{31}^{2(\alpha)}\right] & \left[a_{32}^{1(\alpha)}, a_{32}^{2(\alpha)}\right] & \left[a_{33}^{1(\alpha)}, a_{33}^{2(\alpha)}\right] \end{pmatrix}$$

$$|A_\alpha| = \begin{array}{l} \left(\left[a_{11}^{1(\alpha)}, a_{11}^{2(\alpha)}\right]\left\{\left[a_{22}^{1(\alpha)}, a_{22}^{2(\alpha)}\right]\left[a_{33}^{1(\alpha)}, a_{33}^{2(\alpha)}\right]\right\} + \left[a_{11}^{1(\alpha)}, a_{11}^{2(\alpha)}\right]\left\{\left[a_{23}^{1(\alpha)}, a_{23}^{2(\alpha)}\right]\left[a_{32}^{1(\alpha)}, a_{32}^{2(\alpha)}\right]\right\}\right. \\ + \left[a_{12}^{1(\alpha)}, a_{12}^{2(\alpha)}\right]\left\{\left[a_{21}^{1(\alpha)}, a_{21}^{2(\alpha)}\right]\left[a_{33}^{1(\alpha)}, a_{33}^{2(\alpha)}\right]\right\} + \left[a_{12}^{1(\alpha)}, a_{12}^{2(\alpha)}\right]\left\{\left[a_{23}^{1(\alpha)}, a_{23}^{2(\alpha)}\right]\left[a_{31}^{1(\alpha)}, a_{31}^{2(\alpha)}\right]\right\} \\ \left. + \left[a_{13}^{1(\alpha)}, a_{13}^{2(\alpha)}\right]\left\{\left[a_{21}^{1(\alpha)}, a_{21}^{2(\alpha)}\right]\left[a_{32}^{1(\alpha)}, a_{32}^{2(\alpha)}\right]\right\} + \left[a_{13}^{1(\alpha)}, a_{13}^{2(\alpha)}\right]\left\{\left[a_{22}^{1(\alpha)}, a_{22}^{2(\alpha)}\right]\left[a_{31}^{1(\alpha)}, a_{31}^{2(\alpha)}\right]\right\}\right) \end{array}$$

by interchanging any two rows (or columns) of A_α, we get

$$|A_\alpha^*| = \begin{pmatrix} \left[a_{11}^{1(\alpha)}, a_{11}^{2(\alpha)}\right] & \left[a_{12}^{1(\alpha)}, a_{12}^{2(\alpha)}\right] & \left[a_{13}^{1(\alpha)}, a_{13}^{2(\alpha)}\right] \\ \left[a_{31}^{1(\alpha)}, a_{31}^{2(\alpha)}\right] & \left[a_{32}^{1(\alpha)}, a_{32}^{2(\alpha)}\right] & \left[a_{33}^{1(\alpha)}, a_{33}^{2(\alpha)}\right] \\ \left[a_{21}^{1(\alpha)}, a_{21}^{2(\alpha)}\right] & \left[a_{22}^{1(\alpha)}, a_{22}^{2(\alpha)}\right] & \left[a_{23}^{1(\alpha)}, a_{23}^{2(\alpha)}\right] \end{pmatrix}$$

Where A_α^* represents the new matrix obtained after interchanging

$$\begin{aligned}|A_\alpha^*| = &\left(\left[a_{11}^{1(\alpha)}, a_{11}^{2(\alpha)}\right]\left\{\left[a_{32}^{1(\alpha)}, a_{32}^{2(\alpha)}\right]\left[a_{23}^{1(\alpha)}, a_{23}^{2(\alpha)}\right]\right\} + \left[a_{11}^{1(\alpha)}, a_{11}^{2(\alpha)}\right]\left\{\left[a_{33}^{1(\alpha)}, a_{33}^{2(\alpha)}\right]\left[a_{22}^{1(\alpha)}, a_{22}^{2(\alpha)}\right]\right\}\right. \\ &+ \left[a_{12}^{1(\alpha)}, a_{12}^{2(\alpha)}\right]\left\{\left[a_{31}^{1(\alpha)}, a_{31}^{2(\alpha)}\right]\left[a_{23}^{1(\alpha)}, a_{23}^{2(\alpha)}\right]\right\} + \left[a_{12}^{1(\alpha)}, a_{12}^{2(\alpha)}\right]\left\{\left[a_{33}^{1(\alpha)}, a_{33}^{2(\alpha)}\right]\left[a_{21}^{1(\alpha)}, a_{21}^{2(\alpha)}\right]\right\} \\ &\left. + \left[a_{13}^{1(\alpha)}, a_{13}^{2(\alpha)}\right]\left\{\left[a_{31}^{1(\alpha)}, a_{31}^{2(\alpha)}\right]\left[a_{22}^{1(\alpha)}, a_{22}^{2(\alpha)}\right]\right\} + \left[a_{13}^{1(\alpha)}, a_{13}^{2(\alpha)}\right]\left\{\left[a_{32}^{1(\alpha)}, a_{32}^{2(\alpha)}\right]\left[a_{21}^{1(\alpha)}, a_{21}^{2(\alpha)}\right]\right\}\right)\end{aligned}$$

Therefore, $|A_\alpha| = |A_\alpha^*|$. □

Proposition 2.19. *Let A_α be a square constant α-triangular fuzzy matrix, then we have $(adj\ A_\alpha)^T$ is a constant.*

Proof. Consider $A_\alpha = \begin{pmatrix} [\alpha+1, 3-\alpha] & [\alpha+1, 3-\alpha] \\ [\alpha+1, 3-\alpha] & [\alpha+1, 3-\alpha] \end{pmatrix}$ then $adj\ A_\alpha = \begin{pmatrix} [\alpha+1, 3-\alpha] & [\alpha+1, 3-\alpha] \\ [\alpha+1, 3-\alpha] & [\alpha+1, 3-\alpha] \end{pmatrix}$, we get $(adj\ A_\alpha)^T = \begin{pmatrix} [\alpha+1, 3-\alpha] & [\alpha+1, 3-\alpha] \\ [\alpha+1, 3-\alpha] & [\alpha+1, 3-\alpha] \end{pmatrix}$ is a constant α-triangular fuzzy matrix. □

Proposition 2.20. *Let A_α be a square constant α-triangular fuzzy matrix, then $c = A(adj\ A)$ is constant and $c_{ij} = |A|$.*

Proof. Since A_α is a constant, we can see that $A_{jk} = A_{ik}$ and so $|A_{jk}| = |A_{ik}|$ for every $i,j \in \{1,2,3,...,n\}$ so $\left[c_{ij}^{1(\alpha)}, c_{ij}^{2(\alpha)}\right] = \left[a_{ij}^{1(\alpha)}, a_{ij}^{2(\alpha)}\right] \cdot \sum_{\sigma \in S_{n_i n_j}} \prod_{t \in n_j} \left\langle \left[a_{t\sigma(t)}^{1(\alpha)}, a_{t\sigma(t)}^{2(\alpha)}\right]\right\rangle$ is a constant α-triangular fuzzy matrix. □

Proposition 2.21. *For an $n \times n$, α-triangular fuzzy matrix A_α, if A_α is symmetric, then adj A_α is symmetric.*

Proof. Let $B_\alpha = adj\ A_\alpha$, then

$$\begin{aligned} \left[b_{ij}^{1(\alpha)}, b_{ij}^{2(\alpha)}\right] &= \sum_{\sigma \in S_{n_j n_i}} \prod_{t \in n_j} \left\langle a_{t\sigma(t)}^{1(\alpha)}, a_{t\sigma(t)}^{2(\alpha)} \right\rangle \\ &= \sum_{\sigma \in S_{n_j n_i}} \prod_{t \in n_i} \left\langle a_{t(t)\sigma}^{1(\alpha)}, a_{t(t)\sigma}^{2(\alpha)} \right\rangle \\ &= \left[b_{ij}^{1(\alpha)}, b_{ij}^{2(\alpha)}\right] \quad \text{(since } A_\alpha \text{ is symmetric)} \end{aligned}$$

□

3. The Correlation Coefficient Between Two Triangular Fuzzy Numbers

Correlation analysis is a relationship between two variables, with a central focus on the strength of that relationship. Thus, the correlation measure is defined as follows

Definition 3.1. *Let $A = \left[a^{1(\alpha)}, a^{2(\alpha)}\right]$ and $B = \left[b^{1(\alpha)}, b^{2(\alpha)}\right]$ be two closed intervals, where $a^{1(\alpha)} \leq a^{2(\alpha)}, b^{1(\alpha)} \leq b^{2(\alpha)}$.*

$$\rho(A,B) = \frac{a^{1(\alpha)}b^{1(\alpha)} + a^{2(\alpha)}b^{2(\alpha)}}{\sqrt{\left((a^{1(\alpha)})^2 + (a^{2(\alpha)})^2\right)\left((b^{1(\alpha)})^2 + (b^{2(\alpha)})^2\right)}}$$

is called the correlation coefficient between A and B.

Remark 3.2. *For all $A = \left[a^{1(\alpha)}, a^{2(\alpha)}\right]$, $B = \left[b^{1(\alpha)}, b^{2(\alpha)}\right]$*

(1). $\rho(A,B) = \rho(B,A)$

(2). If $A = B$, then $\rho(A,B) = 1$

(3). If $A = cB$ for some $c > 0$, then $\rho(A,B) = 1$

(4). $|\rho(A,B)| \leq 1$

Now let $A = \left[a^{1(\alpha)}, a^{2(\alpha)}\right]$, $B = \left[b^{1(\alpha)}, b^{2(\alpha)}\right]$ be two fuzzy triangular numbers, then for any $\alpha \in [0,1]$, we define

$$\rho(A,B) = \frac{a^{1(\alpha)}b^{1(\alpha)} + a^{2(\alpha)}b^{2(\alpha)}}{\sqrt{\left((a^{1(\alpha)})^2 + (a^{2(\alpha)})^2\right)\left((b^{1(\alpha)})^2 + (b^{2(\alpha)})^2\right)}}$$

This correlation coefficient analyses the relationship between two fuzzy numbers which is done with the help of the strength of interdependence between them. This correlates the fuzzy concepts like very small, beautiful, fat etc. These correlation coefficient also exhibits the positive and negative relationship between the two fuzzy numbers using this definition we can obtain the correlation coefficient matrices for different values of $\alpha \in [0,1]$triangular fuzzy numbers $\tilde{A} = (a_1, a_2, a_3)$ can be used to indicate the relative strength of each pair of elements in the fuzzy analytic hierarchy process (AHP). Such matrices are used to obtain a favourite judgment in the fuzzy AHP.

Example 3.3. *Consider the two level AHP, whose structure of hierarchy can be drawn as follows*
Let us study the Social Risk problem in Nagapattinam District by considering the three major places Tharangambadi, Vedaranyam and Nagapattinam. The selected factors that affect the society are

1. *Environmental Risk and*
2. *Socio Economic Risk*

The effective factors selected under environmental risk are

i. *Damages due to natural calamities*

ii. *Damages due to chemical industries*

The selective factors affected by the above effective factors are

a. *Land*

b. *Air*

c. *Water*

The factor socio economic risk is classified into two

i. *Damages due to Accident*

ii. *Damages due to socio economic status*

The effective factors of these classifications may be

a. *Minor*

b. *Major*

c. *Death*

The hierarchical structure which depicts the factors and scenario relationship is basis of our AHP calculation shown in figure 1.

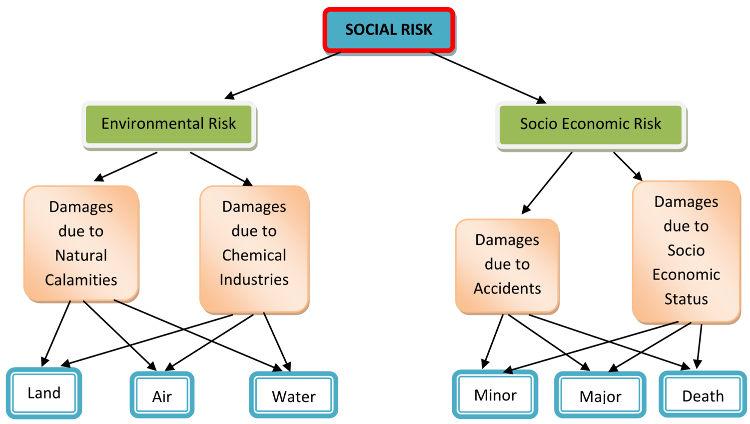

Figure 1.

Now consider population, pollution and accident of three places namely Tharangambadi, Vedaranyam and Nagapattinam which is represented as triangular fuzzy numbers.

Population		
Tharangambadi	Vedaranyam	Nagapattinam
(5,6,7)	(6,8,9)	(12,15,19)
Pollution		
Land	Air	Water
(6,8,11)	(7,10,13)	(9,12,14)
Accident		
Minor	Major	Death
(7,8,9)	(2,5,6)	(3,6,8)

Table 1.

On considering these triangular numbers, we get the correlation coefficients matrix for the level $\alpha = 0$ as follows.

Population			
	Tharangampadi(5,6,7)	Vedaranyam(6,8,9)	Nagapattinam(12,15,19)
Tharangampadi (5,6,7)	1	0.9995	0.9984
Vedaranyam (6,8,9)	0.9995	1	0.9997
Nagapattinam(12,15,19)	0.9984	0.9997	1

Table 2.

Population			
	Land(6,8,11)	Air(7,10,13)	Water(9,12,14)
Land (6,8,11)	1	0.9999	0.9974
Air (7,10,13)	0.9999	1	0.997
Water(9,12,14)	0.9974	0.997	1

Table 3.

Accident			
	Minor(7,8,9)	Major(2,5,6)	Death(3,6,8)
Minor(7,8,9)	1	0.9429	0.9547
Major(2,5,6)	0.9429	1	0.9993
Death(3,6,8)	0.9547	0.9993	1

Table 4.

Paired comparison matrix with priority vector :

Population				**Priority Vector**
	Tharangampadi(5,6,7)	Vedaranyam(6,8,9)	Nagapattinam(12,15,19)	
Tharangampadi (5,6,7)	1	0.9995	0.9984	0.3332
Vedaranyam (6,8,9)	0.9995	1	0.9997	0.3334
Nagapattinam(12,15,19)	0.9984	0.9997	1	0.3333

Table 5.

			Pollution	**Priority Vector**
	Land(6,8,11)	Air(7,10,13)	Water(9,12,14)	
Land (6,8,11)	1	0.9999	0.9974	0.3334
Air (7,10,13)	0.9999	1	0.997	0.3333
Water(9,12,14)	0.9974	0.997	1	0.3331

Table 6.

Accident				**Priority Vector**
	Minor(7,8,9)	Major(2,5,6)	Death(3,6,8)	
Minor(7,8,9)	1	0.9429	0.9547	0.3296
Major(2,5,6)	0.9429	1	0.9993	0.3345
Death(3,6,8)	0.9547	0.9993	1	0.3359

Table 7.

The weight of population matrix must be adjusted, then

$$\text{Adjusted weight for population matrix } = \frac{0.3334}{0.3334 + 0.3333} = 0.5$$

$$\text{Adjusted weight for population matrix } = \frac{0.3333}{0.3334 + 0.3333} = 0.4999$$

Then we compute the overall composite weight of each alternatives choice based on the weight of level 1 and level 2. The overall weight is just normalization of linear combination of multiplication between weight and priority vector.

Overall composite weight of the alternative for population and pollution

$$X = (0.5)(0.3332) + (0.4999)(0.3334) = 0.33326$$

$$Y = (0.5)(0.3334) + (0.4999)(0.3333) = 0.33332$$

$$Z = (0.5)(0.3333) + (0.4999)(0.3331) = 0.33317$$

Overall composite weight of the alternative for population and accident

$$X = (0.5)(0.3332) + (0.4999)(0.3296) = 0.3314$$

$$Y = (0.5)(0.3334) + (0.4999)(0.3345) = 0.3339$$

$$Z = (0.5)(0.3333) + (0.4999)(0.3359) = 0.3346$$

	Population	**Pollution**	**Composite weight**
Land	0.3332	0.3334	0.33326
Air	0.3334	0.3333	0.33332
Water	0.3333	0.3331	0.33317

Table 8.

	Population	**Accident**	**Composite weight**
Minor	0.3332	0.3296	0.3314
Major	0.3334	0.3345	0.3339
Death	0.3333	0.3359	0.3346

Table 9.

By applying the results of the above approach calculated in Table 8 we get the result that "Water Pollution" is the most effective factor that is the highest one that affects the society in Nagapattinam district. From Table 9 we get the result that the selected factor "Major" affects the society in highest rate in Nagapattinam district.

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