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Combinatorics Properties of Order-preserving Full Contraction Transformation Semigroup by Their Equivalence Classes

G. R. Ibrahim[1,*] and S. O. Makanjuola[2]

1 Department of Statistics and Mathematical Sciences, Kwara state University Malete, Kwara State, Nigeria.

2 Department of Mathematics, University of Ilorin, Ilorin, Nigeria.

Abstract: In this paper the cardinalities of equivalence classes of starred Green's relations in order-preserving full contraction transformation semigroup of a finite set(OCT_n) and the elements in each equivalence classes were investigated. For each class tables were formed, elements were arrange based on their kernel and image sets, patterns of arrangement observed and formulae were deduced in each case through the combinatorial principles.

Keywords: Order-preserving, Contraction mapping, Starred Green's Relation.

1. Introduction

Algebraic and Combinatorial properties of transformation semigroup have been studied over a long period and interesting results have emerged for example [4]. A semigroup is simply a set S which is closed under an associative binary operation usually denoted by $(xy)\,z = x\,(yz)$ for all $x, y, z \in S$. The semigroup S is called a monoid if it has identity, that is, if it contains an element 1 with the property that $x1 = 1x = x$ for all $x \in S$. Transformation of X is a function of X to itself. Let the finite set $1, 2, 3, ..., n$ be X_n, then a mapping $\alpha : A \to B$, where A and B are subsets of X_n is called a partial transformation of X_n. If $A = X_n$, the mapping α is called a full transformation of X_n. If for $x, y \in X_n$ such that $x \neq y \Rightarrow x\alpha \neq y\alpha$, The mapping α is called partial one-one transformation of X_n. The set of all full, partial and partial one-one transformations of X_n form semigroup under composition of mappings are respectively denoted by T_n, P_n and I_n.These semigroups are often referred to as full transformation, partial transformation and symmetric inverse semigroup respectively. This is supported by the fact that every finite inverse semigroup can be embedded in a symmetric inverse semigroup. Regular semigroups were introduced by [6] and was the paper in which popular Green's relations were introduced. It was his study of regular semigroups which led Green to define relations. Let S be a semigroup $a, b \in S$. If a and b generate the same left principal ideal,that is $S^1a = S^1b$, then we say that a and b are $\mathcal{L}$- equivalent and write $a\mathcal{L}\ b$ or $(a, b) \in \mathcal{L}$.If a and b generate the same right principal ideal that is $aS^1 = bS^1$, then we say that a and b are $\mathcal{R}$- equivalent and write $a\mathcal{R}\ b$ or $(a, b) \in \mathcal{R}$. If a and b generate the same principal ideal that is S^1aS^1, then we say that a and b are $\mathcal{J}$ equivalent and write $a\mathcal{J}\ b$ or $(a, b) \in \mathcal{J}$, Let $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, $\mathcal{D} = \mathcal{L} \cup \mathcal{R}$, then $\mathcal{H}, \mathcal{D}$ are equivalences on S too. It is well known that $\mathcal{J} = \mathcal{D}$ in any finite semigroup.These five

* *E-mail: risqot.ibrahim@kwasu.edu.ng*

equivalences are usually called Green's equivalence relations. They were first introduced by [6] and have a great importance in the study of the algebraic structure of semigroups. Two functions are $\mathcal{L}$-related if and only if they have the same image, $\mathcal{R}$-related if and only if for x and y in the set X_n; such functions are in the same table row(kernel). Similarly, two functions are $\mathcal{D}$-related if and only if their image sets have the same size. Each $\mathcal{D}$-class in a semigroup S is a union of $\mathcal{L}$-class and $\mathcal{R}$ and their intersection of is either empty or an $\mathcal{H}$-class. Hence, it is convenient to visualize $\mathcal{D}$-class as what [2] called an egg box , in which row represents an $\mathcal{R}$-class, each column represents $\mathcal{L}$-class and each cell represents $\mathcal{H}$-class. Note that, it is possible for the eggbox to contain a single row or single column of cells. even to contain only one cell. Similarly, it may be an infinite eggbox. Recently, [5] characterised the Green's relations and starred Green's relations in CT_n, CI_n and also starred Green's relation in OCT_n.

Table 1. **Structure of $\mathcal{D}$-class**

2. Main Results

Theorem 2.1. *Let $S = OCT_n$, the number of*

(a). $\mathcal{D}^*$-class of height r is 1;

(b). $\mathcal{L}^*$ -classes of height r is $n - r + 1$;

(c). $\mathcal{R}^*$-classes of height r is $\begin{pmatrix} n-1 \\ r-1 \end{pmatrix}$; and

(d). $\mathcal{H}^*$ -classes of height r is $(n-r+1)\begin{pmatrix} n-1 \\ r-1 \end{pmatrix}$.

Proof.

(a). From the characterisation of starred Green's relations [5] that if $|im\,(\alpha)\,| = |im\,(\beta)\,|$ then $(\alpha, \beta) \in \mathcal{D}^*\,(OCT_n)$. Hence, $\mathcal{D}^*$-class of height r is equal to 1.

(b). Since for each $\alpha \in OCT_n$, $im\,(\alpha)$ is convex subset of X_n, by characterisation of $\mathcal{L}^*$-relation in OCT_n, the number of $\mathcal{L}^*$-class of height r corresponds to the number of ways of selecting r consecutively terms out of n. This is given by $n - r + 1$.

(c). From the order-preservedness of OCT_n each block of the $ker\,(\alpha)$, $\alpha \in OCT_n$ must be convex in X_n. Thus, by characterisation of $\mathcal{R}$-relation,the number of $\mathcal{R}$-classes of height r in OCT_n is equal to the number of partitions of X_n into r convex classes. This is not difficult to see that, this number equal to the number of ways of inserting $r-1$ identical strokes into $n-1$ places since inserting a stroke in space is the same as selecting that place. Hence, the number of $\mathcal{R}$-classes of height r in OCT_n is the number of ways of selecting $r-1$ places out of $n-1$ places, that is, $\begin{pmatrix} n-1 \\ r-1 \end{pmatrix}$

(d). By characterisation of the $\mathcal{H}$-relation in OCT_n, the number of $\mathcal{H}$-classes of height r is equal to the product of number of the columns and number of the rows of height r. That is, $(n-r+1)\begin{pmatrix} n-1 \\ r-1 \end{pmatrix}$ □

Theorem 2.2. *Let $\mathcal{D}^*$ be a $\mathcal{D}^*$-class of height r in OCT_n. Then*

(1). The number of elements in each $\mathcal{L}^*$ -class within $\mathcal{D}^*$-class is $\begin{pmatrix} n-1 \\ r-1 \end{pmatrix}$*;*

(2). The number of elements in each $\mathcal{R}^*$ -class within $\mathcal{D}^*$-class is $n-r+1$; and

(3). The number of elements in each $\mathcal{H}^*$ -class within $\mathcal{D}^*$-class is 1

Proof. The proof of (1)and (2) follows from the proof of Theorem 2.1. For (3), since $\mathcal{H}^*$-class is the intersection of $\mathcal{L}^*$-class and $\mathcal{R}^*$-class and each cell contained only one element. Hence, the number of element in each $\mathcal{H}^*$ -class is equal to 1. □

Corollary 2.3. *Let $S = OCT_n$, the total number of*

(a). $\mathcal{D}^*$-classes in OCT_n is n;

(b). $\mathcal{L}^*$ -classes in OCT_n is $\frac{n(n+1)}{2}$;

(c). $\mathcal{R}^*$-classes in OCT_n is 2^{n-1}; and

(d). $\mathcal{H}^*$-classes in OCT_n is $(n+1)\,2^{n-2}$

Proof.

(a). Since for each $r = 1, 2, \ldots, n$ there is 1 $\mathcal{D}^*$-class, an $\alpha \in OCT_n$ with $|im\alpha| = r$, then we have n $\mathcal{D}^*$-class.

(b). The total number of $\mathcal{L}^*$-classes in OCT_n is

$$\sum_{r=1}^{n}(n-r+1) = \sum_{r=1}^{n} n - \sum_{r=1}^{n} r + \sum_{r=1}^{n} 1 = n^2 - \frac{n(n+1)}{2} + n = \frac{2n^2 - n^2 - n + 2n}{2} = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$$

(c). The total number of $\mathcal{R}$-classes in OCT_n is equal to

$$\sum_{r=1}^{n}\begin{pmatrix} n-1 \\ r-1 \end{pmatrix} = \sum_{r=0}^{n-1}\begin{pmatrix} n-1 \\ r \end{pmatrix} = 2^{n-1}$$

(d). Since $\mathcal{H}$-classes are trivial in OCT_n, then the total number of $\mathcal{H}$-classes in OCT_n is equal to the order of OCT_n, That is, $|OCT_n|$, and this has been found by Adeshola (2013) to be $(n+1)\,2^{n-2}$. Hence,

$$\sum_{r=1}^{n}\begin{pmatrix} n-1 \\ r-1 \end{pmatrix}(n-r+1) = |OCT_n| = (n+1)\,2^{n-2}$$

□

Example 2.4. *Let $\alpha \in OCT_4$ and $|im\alpha| = r$.*

For $r = 1$

n/r	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$
1234	$\begin{pmatrix}\{1234\}\\ 1\end{pmatrix}$	$\begin{pmatrix}\{1234\}\\ 2\end{pmatrix}$	$\begin{pmatrix}\{1234\}\\ 3\end{pmatrix}$	$\begin{pmatrix}\{1234\}\\ 4\end{pmatrix}$

For $r = 2$

n/r	$\{12\}$	$\{23\}$	$\{34\}$
$1\mid 234$	$\begin{pmatrix}1 & \{234\}\\ 1 & 2\end{pmatrix}$	$\begin{pmatrix}1 & \{234\}\\ 2 & 3\end{pmatrix}$	$\begin{pmatrix}1 & \{234\}\\ 3 & 4\end{pmatrix}$
$12\mid 34$	$\begin{pmatrix}\{12\} & \{34\}\\ 1 & 2\end{pmatrix}$	$\begin{pmatrix}\{12\} & \{34\}\\ 2 & 3\end{pmatrix}$	$\begin{pmatrix}\{12\} & \{34\}\\ 3 & 4\end{pmatrix}$
$123\mid 4$	$\begin{pmatrix}\{123\} & 4\\ 1 & 2\end{pmatrix}$	$\begin{pmatrix}\{123\} & 4\\ 2 & 3\end{pmatrix}$	$\begin{pmatrix}\{123\} & 4\\ 3 & 4\end{pmatrix}$

For $r = 3$

n/r	$\{123\}$	$\{234\}$
$1\mid 2\mid 34$	$\begin{pmatrix}1 & 2 & \{34\}\\ 1 & 2 & 3\end{pmatrix}$	$\begin{pmatrix}1 & 2 & \{34\}\\ 2 & 3 & 4\end{pmatrix}$
$1\mid 23\mid 4$	$\begin{pmatrix}1 & \{23\} & 4\\ 1 & 2 & 3\end{pmatrix}$	$\begin{pmatrix}1 & \{23\} & 4\\ 2 & 3 & 4\end{pmatrix}$
$12\mid 3\mid 4$	$\begin{pmatrix}\{12\} & 3 & 4\\ 1 & 2 & 3\end{pmatrix}$	$\begin{pmatrix}\{12\} & 3 & 4\\ 2 & 3 & 4\end{pmatrix}$

For $r = 4$

n/r	$\{1234\}$
1234	$\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 2 & 3 & 4\end{pmatrix}$

Number of $\mathcal{L}^*$*-classes in* OCT_4 *is 10; number of* $\mathcal{R}^*$*-classes in* OCT_4 *is 8; number of* $\mathcal{D}^*$*-classes in* OCT_4 *is 4; number of* $\mathcal{H}^*$*-classes in* OCT_4 *is 20.*

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