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An Approximate Analytical Solution of Boussinesq's Equation for Infiltration Phenomenon in Unsaturated Porous Medium

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Abstract: The present paper discusses the infiltration phenomenon for unsaturated porous medium. The mathematical formulation leads to a one dimensional nonlinear partial differential equation in the form of Boussinesq's equation. An approximate analytical solution of Boussinesq's equation for infiltration phenomenon have been obtained. The homotopy analysis method applied to solve Boussinesq's equation with appropriate boundary conditions. The Mathematica BVPh package for homotopy analysis method has been used to interpret numerically and graphically of solution.

MSC: 76S05, 65Nxx, 35Q35.

Keywords: Infiltration, Porous medium, Boussinesq's equation, Homotopy analysis method.

1. Introduction

The groundwater flow has great importance in various fields of science and engineering like as fluid mechanics, hydrology, environment engineering, water resource engineering, soil science [1, 6, 21–23]. Here, we examine problem related to fluid flow in fluid mechanics. It deals with the filtration of an incompressible fluid (water) through unsaturated porous medium. In particular, the present problem in groundwater infiltration. Infiltration is the process in which precipitation or water on ground surface enters into the subsurface soils and moves into rocks through cracks and pore spaces. The infiltration process can continue only if storage had been available for additional water at the soil surface. The available volume for additional water in the soil depends on the porosity of the soil and the rate at which previously infiltrated water can move away from the surface through the soil. The maximum rate that water can enter a soil in a given condition is the infiltration capacity. If the arrival of the water at the soil surface is less than the infiltration capacity, it is sometimes analyzed using hydrology transport models, mathematical models that consider infiltration, runoff and channel flow to predict river flow rates and stream water quality. Once water has infiltrated the soil it may stay in the soil for a long time until it gradually gets evaporated. If there is a lot of vegetative cover (green plants) the infiltrated water can also be absorbed by plant roots and later transpired. Infiltration occurs in the upper layers of the ground but may also continue further downwards into the water table. The rate of infiltration depends on the different factors like as storage capacity, the depth of the water table,

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water content of the soil, the amount of precipitation, soil texture and structure, the amount of vegetative cover over the area as well as the levels of evapotranspiration in that region.

Infiltration phenomenon has been discussed by many researchers with different point of views. Philip [21] described the development of infiltration equation and its numerical solution. Srivastava and Yeh [24] described one dimensional, vertical infiltration toward the water table in homogeneous and layered soils by analytical solutions. Witelski [27] extended the applicability of the Boltzmann similarity solution by introducing a time-shift constant to describe the long time behavior for absorption into slightly wet soil layers. Wojnar [28] have discussed Boussinesq equation for flow in an aquifer with time dependent porosity. Borana et al. [2] have obtained numerical solution of Boussinesq equation arising in one dimensional infiltration phenomenon by using finite difference method. Patel et al. [8] have discussed a solution of Boussinesq's equation for infiltration phenomenon in unsaturated porous media by homotopy analysis method. Chavan and Panchal [4] have discussed the solution of porous medium equation arising in fluid flow through porous media by homotopy perturbation method using Elzaki transform. Parikh [15] have discussed the numerical solution of Boussinesq's equation in groundwater infiltration phenomenon by differential quadrature method. Desai [7] obtained similarity solution of nonlinear Boussinesq's equation arising in infiltration of incompressible fluid flow.

Vázquez [26] reported that the mathematical model of the present problem was developed first by Boussinesq [3] and is related to the original motivation of Darcy [5]. The mathematical formulation leads to one dimensional nonlinear partial differential equation in the form of Boussinesq's equation. The main aim of the present work is to obtained approximate analytical solution of Boussinesq's equation for infiltration phenomenon. Homotopy analysis method [10] is adopted to obtain solution of Boussinesq's equation with appropriate boundary conditions. The solution represents the height of the free surface of the water mound in unsaturated homogeneous soil. The c_0-curve is used to discuss the convergence of the homotopy series solution. We choose proper value of c_0 with the help of c_0-curve.

2. Mathematical Formulation

Consider the maximum height $OA = h_{max}$ of the groundwater reservoir with impermeable bottom and surrounding of this reservoir is unsaturated homogeneous soil. Infiltration phenomenon is well demonstrated in the figure 1 that shows a vertical cross section of the reservoir surrounded by unsaturated porous medium. The height of the free surface is zero, when $OB = x = L$, the dotted arc below the curve is saturated by infiltrated groundwater and above the curve is the dry region of unsaturated soil. The bottom is assumed impervious bed, so water cannot flow in a downward direction. Infiltration is the process by which the groundwater of the reservoir has entered into the unsaturated soil through vertical permeable wall. The infiltrated groundwater will enter in unsaturated soil then the infiltrated groundwater will develop a curve between saturated soil and unsaturated soil, which is called a water table or water mound. The purpose of the study of infiltration is to examine the effective height of the free surface as a measure of initial storage capacity of a porous stratum.

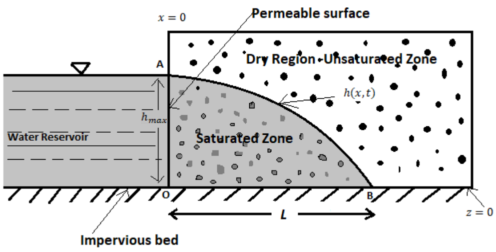

Figure 1. **The infiltration phenomenon**

For mathematical model of the problem, we consider the following assumptions:

(1). the porous stratum has maximum height h_{max} and lies on top of a horizontal impervious bed, which we label as $z = 0$,

(2). we ignore the transversal variable y, and

(3). the water mass which infiltrates the soil occupies a region described as $\Omega = \{(x, z) \in R^2 : 0 \leq z \leq h(x, t)\}$.

We assume that there is no region of partial saturation. This is an evolution model. Clearly, $0 \leq h(x, t) \leq h_{max}$ and the free boundary function $h(x, t)$ is also an unknown of the problem. In this situation, we arrive at a system of three equations with unknowns the two velocity components u, w and the pressure p in a variable domain: one equation of mass conservation for an incompressible fluid and two equations for the conservation of momentum of the Navier-Stokes type. The resulting system is too complicated and can be simplified for the practical computation after introducing a suitable assumption, the hypothesis of almost horizontal flow, i. e., we assume that the flow has an almost horizontal speed $\mathbf{u} \sim (u, 0)$, so that h has small gradients. Thus the vertical component of the momentum equation is of the form

$$\rho\left(\frac{dw}{dt} + \mathbf{u}\cdot\nabla w\right) = -\frac{\partial p}{\partial z} - \rho g. \tag{1}$$

We neglect the inertial term (the left-hand side) of (1) and integrate it w.r.t. z gives

$$p + \rho g z = constant. \tag{2}$$

That means the pressure is determined by means of the hydrostatic approximation. Now we calculate the constant on the free surface $z = h(x, t)$. If we impose continuity of the pressure across the interface, we have $p = 0$ (assuming constant atmospheric pressure in the air that fills the pores of the dry region $z > h(x, t)$). Then we get

$$\rho g h = constant.$$

Thus,

$$p = \rho g(h - z). \tag{3}$$

We go now to the mass conservation law which will give us the equation. We proceed as follows: we take a section $S = (a, x) \times (0, C)$, where C is the free surface $z = h(x, t)$. Then,

$$P\frac{\partial}{\partial t}\int_a^x\int_0^h dz dx = -\int_{\partial S} \mathbf{u}\cdot\mathbf{n}\; dl. \tag{4}$$

So,

$$P\frac{\partial}{\partial t}\int_a^x h dx = -\int_{\partial S} \mathbf{u}\cdot\mathbf{n}\; dl. \tag{5}$$

where P is the porosity of the medium, i. e., the fraction of volume available for the flow circulation, and $\mathbf{u}$ is the velocity, which obeys Darcy's law in the form that includes gravity effects

$$\mathbf{u} = -\frac{K}{\delta}\nabla(p + \rho g z) = -\frac{K}{\delta}\nabla(\rho g h) \tag{6}$$

where K is the permeability of porous medium, δ is the viscosity of water. On the right-hand lateral surface we have $\mathbf{u}\cdot\mathbf{n}\approx(u,0)\cdot(1,0)=u=-\frac{\rho g K}{\delta}\frac{\partial h}{\partial x}$. Differentiating (5) w.r.t. x, we get

$$P\frac{\partial h}{\partial t}=\frac{\rho g K}{\delta}\frac{\partial}{\partial x}\int_0^h \frac{\partial h}{\partial x}dz. \tag{7}$$

Thus we get Boussinesq's equation for infiltration phenomenon

$$\frac{\partial h}{\partial t}=\frac{\rho g K}{2P\delta}\frac{\partial^2(h^2)}{\partial x^2}. \tag{8}$$

This is the porous medium equation. It is a fundamental equation in groundwater infiltration. Using dimensionless variables

$$\mathcal{H}=\frac{h}{L}, X=\frac{x}{L}, T=\frac{\rho g K t}{P\delta L}.$$

(8) gives us

$$\frac{\partial \mathcal{H}}{\partial T}=\mathcal{H}\frac{\partial^2\mathcal{H}}{\partial X^2}+\left\{\frac{\partial \mathcal{H}}{\partial X}\right\}^2. \tag{9}$$

This equation is solved with appropriate boundary conditions by homotopy analysis method. The solution of this equation represents the height of the water mound $\mathcal{H}(X,T)$ at a length X and time T. Assume set of boundary conditions are given by

$$\mathcal{H}(0,T)=h_{max} \text{ and } \mathcal{H}(1,T)=0. \tag{10}$$

3. Homotopy Analysis Method

Homotopy analysis method [10] is used to solve nonlinear equations. This method have been successfully applied to solve various types of nonlinear ordinary as well as partial differential equations [9–13, 16–20, 25]. We apply this method to solve equation (9) with boundary conditions (10). Let us consider the nonlinear partial differential equation

$$\mathcal{N}[\phi(X,T;q)]=0 \tag{11}$$

where $\mathcal{N}$ is a nonlinear operator of the form

$$\mathcal{N}[\phi(X,T;q)]=\phi(X,T;q)\frac{\partial^2\phi(X,T;q)}{\partial X^2}+\left\{\frac{\partial\phi(X,T;q)}{\partial X}\right\}^2-\frac{\partial\phi(X,T;q)}{\partial T} \tag{12}$$

and $\phi(X,T;q)$ be an unknown function which represents height of infiltrated groundwater $\mathcal{H}$ at length X for a given time T for $0\le q\le 1$. Liao [10] constructed the zeroth-order deformation equation

$$(1-q)\mathcal{L}[\phi(X,T;q)-\mathcal{H}_0(X,T)]=c_0 q H(X,T)\mathcal{N}[\phi(X,T;q)] \tag{13}$$

where $q\in[0,1]$ the embedding-parameter, $\mathcal{L}$ the auxiliary linear operator, $\mathcal{H}_0(X,T)$ the initial guess of the solution $\mathcal{H}(X,T)$, c_0 nonzero convergence control parameter, $H(X,T)$ nonzero auxiliary function. We choose the auxiliary linear operator $\mathcal{L}[\phi(X,T;q)] = \frac{\partial^2 \phi(X,T;q)}{\partial X^2}$ and the initial guess of $\mathcal{H}(X,T)$ as $\mathcal{H}_0(X,T) = (h_{max} - TX)(1-X^2)e^{-X}$ which satisfy both boundary conditions. Thus when $q=0$ and $q=1$, (13) reduce to

$$\phi(X,T;0) = \mathcal{H}_0(X,T) \text{ and } \phi(X,T;1) = \mathcal{H}(X,T) \tag{14}$$

respectively. According to (14) as q increases from 0 to 1, $\phi(X,T;q)$ continuously deforms from $\mathcal{H}_0(X,T)$ to $\mathcal{H}(X,T)$. Thus, the solution is considered in the form of series as

$$\phi(X,T;q) = \mathcal{H}_0(X,T) + \sum_{m=1}^{\infty} \mathcal{H}_m(X,T) q^m \tag{15}$$

where

$$\mathcal{H}_m(X,T) = \frac{1}{m!} \frac{\partial^m \phi(X,T;q)}{\partial q^m}\bigg|_{q=0}. \tag{16}$$

Assume the initial guess, the auxiliary linear operator, the auxiliary function and the convergence control parameter are in such a way that the series of $\phi(X,T;q)$ with respect to q converges at $q=1$. i.e.

$$\mathcal{H}(X,T) = \mathcal{H}_0(X,T) + \sum_{m=1}^{\infty} \mathcal{H}_m(X,T). \tag{17}$$

Write $\overrightarrow{\mathcal{H}_n} = \{\mathcal{H}_0(X,T), \mathcal{H}_1(X,T), \ldots, \mathcal{H}_n(X,T)\}$. Differentiating (13) m times with respect to q and setting $q=0$ and then dividing them by $m!$, we have the high order deformation equation of the form

$$\mathcal{L}[\mathcal{H}_m(X,T) - \chi_m \mathcal{H}_{m-1}(X,T)] = c_0 H(X,T) \mathcal{R}_m(\overrightarrow{\mathcal{H}_{m-1}}) \tag{18}$$

subject to the boundary conditions

$$\mathcal{H}_m(0,T) = 0 \text{ and } \mathcal{H}_m(1,T) = 0,\ \ m \geq 1 \tag{19}$$

where

$$\mathcal{R}_m(\overrightarrow{\mathcal{H}_{m-1}}) = \sum_{i=0}^{m-1} \mathcal{H}_i \frac{\partial^2 \mathcal{H}_{m-1-i}}{\partial X^2} + \sum_{i=0}^{m-1} \frac{\partial \mathcal{H}_i}{\partial X} \frac{\partial \mathcal{H}_{m-1-i}}{\partial X} - \frac{\partial \mathcal{H}_{m-1}}{\partial T}, m \geq 1 \tag{20}$$

and

$$\chi_m = \begin{cases} 0 & \text{if } m \leq 1, \\ 1 & \text{if } m > 1. \end{cases} \tag{21}$$

For sake of simplicity, we assume that $H(X,T) = 1$. Thus the solution of (18) is

$$\mathcal{H}_m(X,T) = \chi_m \mathcal{H}_{m-1}(X,T) + c_0 \mathcal{L}^{-1}[\mathcal{R}_m(\overrightarrow{\mathcal{H}_{m-1}})] + C_1 X + C_2 \tag{22}$$

where C_1 and C_2 are constants or functions of T. Hence the homotopy series solution is of the form

$$\mathcal{H}(X,T) = \mathcal{H}_0(X,T) + \mathcal{H}_1(X,T) + \mathcal{H}_2(X,T) + \cdots. \tag{23}$$

i.e.

$$\mathcal{H}(X,T) = e^{-X}\left(h_{max} - h_{max}X^2 - TX + TX^3\right) + c_0 e^{-X}\left\{-22 - 17X - 6X^2 - X^3\right\} + c_0 e^{-2X}\left\{\frac{h_{max}^2}{2}\right.$$

$$
\begin{aligned}
&- h_{max}TX + \frac{T^2X^2}{2} - h_{max}^2X^2 + 2h_{max}TX^3 - T^2X^4 + \frac{h_{max}^2X^4}{2} - h_{max}TX^5 + \frac{T^2X^6}{2}\Bigg\} \\
&+ c_0\left\{22 - \frac{h_{max}^2}{2} + 46Xe^{-1} - 22X + \frac{h_{max}^2X}{2}\right\} + \cdots. \qquad (24)
\end{aligned}
$$

The solution (24) represents the height of the water mound for infiltration phenomenon. The solution is contained convergence control parameter c_0 which help us to control convergence of homotopy series solution. Many researchers have been discussed the convergence of homotopy analysis method using c_0-curve [9, 11–14, 16–20, 25]. With the help of c_0-curve, we choose proper value of $c_0 = -0.014$ from valid region. The c_0-curve of $\mathcal{H}_{XX}(1,1)$ is plotted using Mathematica BVPh package for homotopy analysis method [12] (see figure 2).

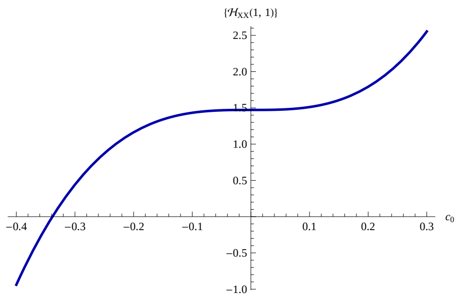

Figure 2. **The c_0-curve of $\mathcal{H}_{XX}(1,1)$ (Thick line) for $h_{max} = 1$.**

The numerical and graphical representations of solution are obtained using Mathematica software [12]. Table 1 indicates the numerical values of the height of infiltrated groundwater $\mathcal{H}$ at different length X and different time T. Figure 3 shows the graphical representation of the height of infiltrated groundwater $\mathcal{H}$ v/s length X for fixed time $T = 0.1, 0.2, ..., 1$.

T	$X = 0.0$	$X = 0.1$	$X = 0.2$	$X = 0.3$	$X = 0.4$	$X = 0.5$	$X = 0.6$	$X = 0.7$	$X = 0.8$	$X = 0.9$	$X = 1.0$
0.1	1.0000000	0.8953185	0.7858909	0.6747501	0.5643396	0.4566071	0.3530792	0.2549220	0.1629947	0.0778948	0.0000000
0.2	1.0000000	0.8874137	0.7717887	0.6563064	0.5434690	0.4352064	0.3329633	0.2377733	0.1503233	0.0710117	0.0000000
0.3	1.0000000	0.8794994	0.7576567	0.6378121	0.5225341	0.4137387	0.3127889	0.2205825	0.1376292	0.0641219	0.0000000
0.4	1.0000000	0.8715756	0.7434946	0.6192669	0.5015348	0.3922039	0.2925558	0.2033493	0.1249121	0.0572254	0.0000000
0.5	1.0000000	0.8636423	0.7293024	0.6006707	0.4804707	0.3706016	0.2722636	0.1860737	0.1121721	0.0503222	0.0000000
0.6	1.0000000	0.8556994	0.7150801	0.5820233	0.4593416	0.3489315	0.2519122	0.1687555	0.0994091	0.0434122	0.0000000
0.7	1.0000000	0.8477470	0.7008276	0.5633245	0.4381471	0.3271934	0.2315013	0.1513945	0.0866231	0.0364956	0.0000000
0.8	1.0000000	0.8397850	0.6865447	0.5445740	0.4168870	0.3053870	0.2110307	0.1339907	0.0738140	0.0295722	0.0000000
0.9	1.0000000	0.8318135	0.6722314	0.5257717	0.3955610	0.2835119	0.1905002	0.1165439	0.0609817	0.0226421	0.0000000
1.0	1.0000000	0.8238324	0.6578875	0.5069174	0.3741689	0.2615679	0.1699094	0.0990539	0.0481262	0.0157053	0.0000000

Table 1. **Numerical values of the height of infiltrated groundwater $\mathcal{H}$ for $h_{max} = 1$.**

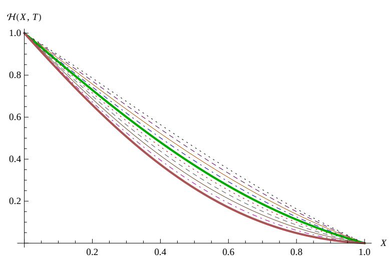

Figure 3. The graph of the height of infiltrated groundwater $\mathcal{H}(X,T)$ v/s length X for fixed time $T = 0.1$(uppermost graph), $0.2, ..., 1$(lowermost graph) ($h_{max} = 1$).

4. Conclusions

We have discussed the infiltration phenomenon for unsaturated porous medium and its mathematical formulation leads Boussinesq equation. The solution of Boussinesq equation is obtained by homotopy analysis method. The convergence of homotopy series solution is discussed by using c_0-curve. The solution $\mathcal{H}$ decreases when length X increases as well as time T increases. We concluded that the height of infiltrated groundwater decreases when length as well as time increases.

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