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Fixed Point Theorems for Weakly Generalized w-Contraction Mappings and Applications

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Abstract: In this paper, we prove fixed point theorems for weakly generalized w-contraction mappings in the setting of w-distances which generalize results of Wongyat and Sintunavarat [16]. As an application, we obtain the existence and uniqueness solutions for nonlinear Fredholm and Volterra integral equations.

Keywords: Fixed point, w-distances, Contraction mappings.

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1. Introduction

In 1922, Polish mathematician Banach [1] proved a very important result regarding a contraction mapping, known as the Banach contraction principle. It is one of the fundamental results in fixed point theory. Due to its importance and simplicity, several authors have obtained many interesting extensions and generalizations of the Banach contraction principle (see [2–6] and references therein).

In 1984, the notion of an altering distance function was introduced and studied by Khan et al. [7], applying it to define weak contractions. They also proved the existence and uniqueness of a fixed point for mappings satisfying such a contraction condition. Afterward, some fixed point results for generalized weak contraction mappings were proved by Choudhury et al. [8] by using some control function along with the notion of an altering distance function. Moreover, the notion of weak contraction mappings was extended in many different directions (see [9, 10] and references therein). On the other hand, the notion of a w-distance on a metric space was introduced and investigated by Kada et al. [11]. Using this concept, they also improved many famous theorems. Afterward, Du [12] proved the existence of a fixed point for some nonlinear mappings by using a specific w-distance, called a w^0-distance. From this trend, several mathematicians extended fixed point results for weak contraction mappings and generalized weak contraction mappings with respect to w-distances on metric spaces (see [13, 14] and references therein).

Wongyat and Sintunavarat[16], by introducing the concept called w-generalized weak contraction mappings and proved new fixed point theorems for w-generalized weak contraction mappings with respect to w-distances in complete metric spaces by using the concept of an altering distance function.

In this paper, we introduce the concept called weakly generalized w-contraction mappings which generalize the definition of w-generalized weak contraction mappings and obtain new fixed point theorems for weakly generalized w-contraction

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mappings in the setting of w-distance in complete metric spaces which generalize results of Wongyat and Sintunavarat [16]. Also we obtain existence and uniqueness of the solution for nonlinear Fredholm integral equations and Volterra integral equations.

1.1. Preliminaries

Definition 1.1 ([11]). *Let (X, d) be a metric space. A function $q : X \times X \to [0, \infty)$ is called a w-distance on X if it satisfies the following three conditions for all $x, y, z \in X$:*

(W1) $q(x, y) \leq q(x, z) + q(z, y)$;

(W2) $q(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous;

(W3) for each $\epsilon > 0$, there exists $\delta > 0$ such that $q(x, y) \leq \delta$ and $q(x, z) \leq \delta$ imply $d(y, z) \leq \epsilon$.

Definition 1.2 ([12]). *Let (X, d) be a metric space. A function $q : X \times X \to [0, \infty)$ is called a w^0-distance on X if it is a w-distance on X with $q(x, x) = 0$ for all $x \in X$.*

Next, we give the definition of an altering distance function.

Definition 1.3 ([7]). *A function $\phi : [0, \infty) \to [0, \infty)$ is said to be an altering distance function if it satisfies the following conditions:*

(a). ψ is continuous and nondecreasing;

(b). $\psi(t) = 0$ if and only if $t = 0$.

Definition 1.4 ([16]). *A w-distance q on a metric space (X, d) is said to be a ceiling distance of d if and only if*

$$q(x, y) \geq d(x, y) \tag{1}$$

for all $x, y \in X$.

Now we give some examples of a ceiling distance of itself.

Example 1.5. *Each metric on a nonempty set X is a ceiling distance of itself.*

Example 1.6. *Let $X = \mathbb{R}$ with the metric $d : X \times X \to \mathbb{R}$ defined by $d(x, y) = |x - y|$ for all $x, y \in X$, and let $a, b \geq 1$. Define the w-distance $q : X \times X \to [0, \infty)$ by*

$$q(x, y) = \max\{a(y - x), b(x - y)\}$$

for all $x, y \in X$. For all $x, y \in X$, we get

$$\begin{aligned} d(x, y) &= |x - y| \\ &= \begin{cases} x - y, & x \geq y, \\ y - x, & x \leq y, \end{cases} \\ &\leq \max\{a(y - x), b(x - y)\} \\ &= q(x, y). \end{aligned}$$

Thus q is a ceiling distance of d.

2. Main Section

In this section, we introduce the new concept called weakly generalized w-contraction mapping along with w-distance in metric spaces. Furthermore, we investigate the sufficient condition for the existence and uniqueness of a fixed point for a self-mapping on a metric space satisfying the weakly generalized contractive condition.

Definition 2.1. *Let q be a w-distance on a metric space (X,d). A mapping $T : X \to X$ is said to be a weakly generalized w-contraction mappings if*

$$\psi(q(Tx,Ty)) \leq \psi(M_T(x,y)) - \phi(q(x,y)) \tag{2}$$

for all $x, y \in X$, where

$$M_T(x,y) : = \max\left\{q(x,y), \frac{q(x,Tx)+q(y,Ty)}{2}, \frac{q(x,Ty)+q(Tx,y)}{2}\right\}$$

$\psi : [0,\infty) \to [0,\infty)$ is an altering distance function, and $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$.

Note that if T is w-generalized weak contraction, then

$$\begin{aligned}\psi(q(Tx,Ty)) &\leq \psi\left(\max\left\{q(x,y), \frac{q(x,Ty)+q(y,Tx)}{2}\right\}\right) - \phi(q(x,y))\\ &\leq \psi\left(\max\left\{q(x,y), \frac{q(x,Tx)+q(y,Ty)}{2}, \frac{q(x,Ty)+q(Tx,y)}{2}\right\}\right) - \phi(q(x,y)).\end{aligned}$$

Hence T is weakly generalized w-contraction mapping. First, we prove the following lemma.

Lemma 2.2. *Let (X,d) be a metric space and, $q : X \times X \to [0,\infty)$ be a w^0-distance on X and suppose that $T : X \to X$ is a continuous weakly generalized w-contraction mappings. If $\{x_n\}$ is a sequence in X defined by $x_{n+1} = T(x_n)$ and if $d(x_n, x_{n+1})$ is decreasing and*

$$\lim_{n\to\infty} q(x_n, x_{n+1}) = 0 \text{ and } \lim_{n\to\infty} q(x_{n+1}, x_n) = 0. \tag{3}$$

Then $\{x_n\}$ is a Cauchy sequence.

Proof. Assume that $\{x_n\}$ is not a Cauchy sequence. So there exist $\epsilon > 0$, and subsequence $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with $n_k > m_k > k$ such that

$$q(x_{m_k}, x_{n_k}) \geq \epsilon \qquad for\ all\ k \in \mathbb{N}. \tag{4}$$

Let n_k be the smallest integer such that $n_k > m_k$ and $q(x_{m_k}, x_{n_k}) \geq \epsilon$. Then

$$q(x_{m_k}, x_{n_k-1}) < \epsilon. \tag{5}$$

From (4), (5), and (W1) we obtain

$$\epsilon \leq q(x_{m_k}, x_{n_k}) \leq q(x_{m_k}, x_{n_k-1}) + q(x_{n_k-1}, x_{n_k}) < \epsilon + q(x_{n_k-1}, x_{n_k}).$$

Taking the limit as $k \to \infty$ in this inequality and using (3), we have

$$\lim_{n\to\infty} q(x_{m_k}, x_{n_k}) = \epsilon. \tag{6}$$

By using (W1), we have

$$q(x_{m_k}, x_{n_k}) \leq q(x_{m_k}, x_{m_k+1}) + q(x_{m_k+1}, x_{n_k+1}) + q(x_{n_k+1}, x_{n_k})$$

and

$$q(x_{m_k+1}, x_{n_k+1}) \leq q(x_{m_k+1}, x_{m_k}) + q(x_{m_k}, x_{n_k}) + q(x_{n_k}, x_{n_k+1}).$$

Taking the limit as $k \to \infty$ in the last two inequalities and using (3) and (6) we have

$$\lim_{n\to\infty} q(x_{m_k+1}, x_{n_k+1}) = \epsilon. \tag{7}$$

Again, by using (W1) we obtain

$$\begin{aligned} q(x_{m_k}, x_{n_k}) &\leq q(x_{m_k}, x_{n_k+1}) + q(x_{n_k+1}, x_{n_k}) \\ &\leq q(x_{m_k}, x_{n_k}) + q(x_{n_k}, x_{n_k+1}) + q(x_{n_k+1}, x_{n_k}) \end{aligned}$$

and

$$\begin{aligned} q(x_{m_k}, x_{n_k}) &\leq q(x_{m_k}, x_{m_k+1}) + q(x_{m_k+1}, x_{n_k}) \\ &\leq q(x_{m_k}, x_{m_k+1}) + q(x_{m_k+1}, x_{m_k}) + q(x_{m_k}, x_{n_k}) \end{aligned}$$

Taking the limit as $k \to \infty$ in the last two inequalities and using (3) and (6) we have

$$\lim_{n\to\infty} q(x_{m_k}, x_{n_k+1}) = \epsilon, \quad \lim_{n\to\infty} q(x_{m_k+1}, x_{n_k}) = \epsilon. \tag{8}$$

Now

$$\begin{aligned} \psi(q(x_{m_k+1}, x_{n_k+1})) &= \psi(q(Tx_{m_k}, Tx_{n_k})) \\ &\leq \psi(\max\{q(x_{m_k}, x_{n_k}), \frac{q(x_{m_k}, x_{m_k+1}) + q(x_{n_k}, x_{n_k+1})}{2} + \\ &\quad \frac{q(x_{m_k}, x_{n_k+1}) + q(x_{m_k+1}, x_{n_k})}{2}) - \phi(q(x_{m_k}, x_{n_k}) \end{aligned} \tag{9}$$

Taking the limit as $k \to \infty$ in (9) and using (8), we get

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon).$$

Hence $\phi(\epsilon) = 0$ which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. □

Theorem 2.3. *Let (X, d) be a complete metric space, and $q : X \times X \to [0, \infty)$ be a w^0-distance on X and a ceiling distance of d. Suppose that $T : X \to X$ is a continuous weakly generalized w-contraction mappings. Then T has a unique fixed point in X. Moreover, for each $x \in X$, the Picard iteration $\{x_n\}$ defined by $x_n = T^n x$ for all $n \in \mathbb{N}$ converges to a unique fixed point of T.*

Proof. Suppose that $\psi, \phi : [0,\infty) \to [0,\infty)$ are two functions contractive condition (2). Starting from a fixed arbitrary point $x \in X$, we put $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If $x_{n^*} = x_{n^*+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then x_n is a fixed point of T. Thus we will assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, that is, $d(x_n, x_{n+1}) > 0$, for all $n \in \mathbb{N} \cup \{0\}$. Since q is a ceiling distance of d, we obtain $q(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. From the contractive condition (2), for all $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
\psi(q(x_{n+1}, x_{n+2})) &= \psi(q(Tx_n, Tx_{n+1})) \\
&\leq \psi(M_T(x_n, x_{n+1})) - \phi(q(x_n, x_{n+1})) \\
&= \psi\Big(\max\Big\{ q(x_n, x_{n+1}), \frac{q(x_n, Tx_n) + q(x_{n+1}, Tx_{n+1})}{2}, \frac{q(x_n, Tx_{n+1}) + q(Tx_n, x_{n+1})}{2} \Big\} - \\
&\qquad \phi(q(x_n, x_{n+1})) \\
&= \psi\Big(\max\Big\{ q(x_n, x_{n+1}), \frac{q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2})}{2}, \frac{q(x_n, x_{n+2}) + q(x_{n+1}, x_{n+1})}{2} \Big\} - \\
&\qquad \phi(q(x_n, x_{n+1})) \\
&= \psi\big(\max\Big\{ q(x_n, x_{n+1}), \frac{q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2})}{2} \Big\} - \phi(q(x_n, x_{n+1})). \qquad (10)
\end{aligned}$$

Suppose that $q(x_n, x_{n+1}) \leq q(x_{n+1}, x_{n+2})$ for some $n \in \mathbb{N} \cup \{0\}$. From (10) we have

$$\psi(q(x_{n+1}, x_{n+2})) \leq \psi(q(x_{n+1}, x_{n+2}) - \phi(q(x_n, x_{n+1})),$$

which yields that $\phi(q(x_n, x_{n+1})) = 0$, and so, $q(x_n, x_{n+1}) = 0$, which is a contradiction. Therefore $q(x_{n+1}, x_{n+2}) < q(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$, and hence $q(x_n, x_{n+1})$ is decreasing and bounded below. Therefore, there exists $L \geq 0$ such that

$$\lim_{n\to\infty} q(x_n, x_{n+1}) = L. \qquad (11)$$

We claim that $L = 0$. We have

$$\lim_{n\to\infty} M_T(x_n, x_{n+1}) = L. \qquad (12)$$

Recall that, for all $n \in \mathbb{N} \cup \{0\}$,

$$\psi(q(x_{n+1}, x_{n+2})) \leq \psi(M_T(x_n, x_{n+1})) - \phi(q(x_n, x_{n+1})).$$

Taking the limit as $n \to \infty$ in this inequality and using the continuity of ϕ and ψ, we have

$$\psi(L) \leq \psi(L) - \phi(L),$$

which is a contradiction unless $L = 0$. Hence

$$\lim_{n\to\infty} q(x_n, x_{n+1}) = 0.$$

Similarly, we can prove that

$$\lim_{n\to\infty} q(x_{n+1}, x_n) = 0.$$

Then by Lemma 2.2, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, there exists $p \in X$ such that $x_n \to p$ as $n \to \infty$. Since T is continuous and since $x_{n+1} = Tx_n$, we get $p = Tp$. Thus, T has fixed point. Finally, we will show that the fixed point is unique. Suppose that p and p^* are two distinct points fixed points of T. Then

$$\psi(q(p, p^*)) = \psi(q(Tp, Tp^*)) \leq \psi\Big(\max\Big\{ q(p, p^*), \frac{q(p, Tp) + q(p^*, Tp^*)}{2}, \frac{q(p, Tp^*) + q(Tp, p^*)}{2} \Big\}\Big) - \phi(q(p, p^*))$$

$$\leq \psi(q(p,p^*)) - \phi(q(p,p^*)),$$

which is a contradiction by the property ϕ. Therefore, $p = p^*$, and hence the fixed point is unique. This complete the proof. □

Corollary 2.4. *Let (X,d) be a complete metric space, and $q : X \times X \to [0,\infty)$ be a w-distance on X and a ceiling distance of d. Suppose that $T : X \to X$ is a continuous w-generalized weak contraction mapping. Then T has a unique fixed point in X. Moreover, for each $x_0 \in X$, the Picard iteration x_n defined by $x_n = T^n x_0$ for all $n \in \mathbb{N}$ converges to a unique fixed point of T.*

Proof. The proof of the corollary follows by the fact that every w-generalized weak contraction mapping is weakly generalized w-contraction mappings. □

We can extend the condition of w^0-distance in Theorem 2.3 to w-distances if we replace the contractive condition (2) by some stronger condition. Here we give the results.

Corollary 2.5. *Let (X,d) be a complete metric space, and $q : X \times X \to [0,\infty)$ be a w-distance on X and ceiling distance of d. Suppose that $T : X \to X$ is a continuous mapping such that, for all $x,y \in X$,*

$$\psi(q(Tx,Ty)) \leq \psi(q(x,y)) - \phi(q(x,y)), \tag{13}$$

where $\psi : [0,\infty) \to [0,\infty)$ is an altering distance function, and $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point in X. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$ defined by $x_n = T^n x_0$ for all $n \in \mathbb{N}$ converges to a unique fixed point of T.

Proof. The proof of the corollary follows by the fact that

$$\psi(q(x,y)) \leq \psi\left(\max\left\{q(x,y), \frac{q(x,Tx)+q(y,Ty)}{2}, \frac{q(x,Ty)+q(Tx,y)}{2}\right\}\right).$$

□

3. Existence of a Solution for Nonlinear Integral Equations

The aim of this section is to present as application of our theoretical results in the previous section for guaranteeing the existence and uniqueness of a solution for various problems regarded by the following equations.

- nonlinear Fredholm integral equations;
- nonlinear Volterra integral equations.

3.1. Nonlinear Fredholm Integral Equations

In this subsection, we prove the existence and uniqueness of a solution for nonlinear Fredholm integral and nonlinear Volterra integral equations by using Theorem

Theorem 3.1. *Consider the nonlinear Fredholm integral equation*

$$x(t) = \phi(t) + \int_a^b K(t,s,x(s))ds, \tag{14}$$

where $a,b \in \mathbb{R}$ with $a < b$, and $\phi : [a,b] \to \mathbb{R}$ and $K : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$ are given continuous mappings. Suppose that the following conditions hold:

(i). Let $(C[a,b], \|\cdot\|_\infty)$ be a complete metric space and the mapping $T : C[a,b] \to C[a,b]$ defined by

$$(Tx)(t) = \phi(t) + \int_a^b K(t,s,x(s))ds \quad \textit{for all } x \in C[a,b] \textit{ and } t \in [a,b]$$

is a continuous mapping;

(ii). there are two functions $\psi, \phi : [0,\infty) \to [0,\infty)$ with ψ is an altering distance function and ϕ is continuous function such that $\psi(t) < t$, for all $t > 0$, $\phi(t) = 0$ if and only if $t = 0$, and for all $x, y \in C[a,b]$, we have

$$\sup_{t\in[a,b]} |K(t,s,y(t)| \leq \frac{\psi(\sup_{t\in[a,b]}|y(t)|) - \phi(\sup_{t\in[a,b]}|y(t)|) - \|\phi\|}{b-a}$$

for all $t, s \in [a,b]$.

Then the nonlinear integral equation (14) has a unique solution. Moreover, for each $x_0 \in C[a,b]$, the Picard iteration $\{x_n\}$ defined by

$$(x_n)(t) = \phi(t) + \int_a^b K(t,s,x_{n-1}(s))ds$$

for all $n \in \mathbb{N}$ converges to a unique solution of the nonlinear integral equation (14).

Proof. Let $X = C[a,b]$. Clearly, X with the metric $d : X \times X \to [0,\infty)$ given by

$$d(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|$$

for all $x, y \in X$ is a complete metric space. Next, we define the function $q : X \times X \to [0,\infty)$ by

$$q(x,y) = \|y\|_\infty = \sup_{t\in[a,b]} |y(t)|$$

for all $x, y \in X$. Clearly, q is a w-distance on X and a ceiling distance of d. Here, we will show that T satisfies the contractive condition (2). Assume that $x, y \in X$ and $t \in [a,b]$. Then we get

$$\begin{aligned}
|(Ty)(t)| &= \sup_{t\in[a,b]} |\phi(t) + \int_a^b K(t,s,y(t))ds| \\
&\leq \|\phi\| + \sup_{t\in[a,b]} \int_a^b |K(t,s,y(t))|ds \\
&\leq \|\phi\| + \int_a^b \left(\frac{\psi(\sup_{t\in[a,b]}|y(t)|) - \phi(\sup_{t\in[a,b]}|y(t)|) - \|\phi\|}{b-a} \right) ds \\
&= \|\phi\| + \frac{1}{b-a} \int_a^b (\psi(q(x,y)) - \phi(q(x,y)) - \|\phi\|)ds \\
&= \psi(q(x,y)) - \phi(q(x,y).
\end{aligned}$$

This implies that

$$\sup_{t\in[a,b]} |(Ty)(t)| \leq \psi(q(x,y)) - \phi(q(x,y)$$

and so

$$q(Tx,Ty) \leq \psi(q(x,y)) - \phi(q(x,y)$$

for all $x, y \in X$. Hence we have

$$\psi(q(Tx,Ty)) \leq q(Tx,Ty) \leq \psi(q(x,y)) - \phi(q(x,y)$$

for all $x, y \in X$. It follows that T satisfies condition (13). Therefore, all conditions of Corollary 2.5 are satisfied, and thus T had a unique fixed point. This implies that there exists a unique solution of the nonlinear Fredholm integral equation (14). This completes the proof. □

Theorem 3.2. *Consider the nonlinear Fredholm integral equation*

$$x(t) = \phi(t) + \int_a^b K(t,s,x(s))ds, \tag{15}$$

where $a, b \in \mathbb{R}$ with $a < b$, and $\phi : [a,b] \to \mathbb{R}$ and $K : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$ are given continuous mappings. Suppose that the following conditions hold:

(i). the mapping $T : C[a,b] \to C[a,b]$ defined by

$$(Tx)(t) = \phi(t) + \int_a^b K(t,s,x(s))ds \quad \textit{for all } x \in C[a,b] \textit{ and } t \in [a,b]$$

is a continuous mapping;

(ii). there are two functions $\psi, \phi : [0,\infty) \to [0,\infty)$ with ψ is an altering distance function and ϕ is continuous function such that $\psi(t) < t$, for all $t > 0$, $\phi(t) = 0$ if and only if $t = 0$, and for all $x, y \in C[a,b]$, if and only if $t = 0$, and for all $x, y \in C[a,b]$, where $a_1, b_1 > 1$ we have

$$\begin{aligned}&\sup\nolimits_{t\in[a,b]} \max\{a_1|K(t,s,x(s) - K(t,s,y(s))|,\\ &\qquad b_1|K(t,s,y(s) - K(t,s,x(s))|\} \le \frac{1}{b-a}\Big(\psi(\sup\nolimits_{t\in[a,b]}\{\max\{a_1(x(t)-y(t)),\\ &\qquad b_1(y(t)-x(t))\}\}) - \phi(\sup\nolimits_{t\in[a,b]}\{\max\{a_1(x(t)-y(t)), b_1(y(t)-x(t))\}\})\Big)\end{aligned}$$

for all $t, s \in [a,b]$.

Then the nonlinear integral equation (15) has a unique solution. Moreover, for each $x_0 \in C[a,b]$, the Picard iteration $\{x_n\}$ defined by

$$(x_n)(t) = \phi(t) + \int_a^b K(t,s,x_{n-1}(s))ds$$

for all $n \in \mathbb{N}$ converges to a unique solution of the nonlinear integral equation (15).

Proof. Let $X = C[a,b]$. Clearly, X with the metric $d : X \times X \to [0,\infty)$ given by

$$d(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|$$

for all $x, y \in X$ is a complete metric space. Next, we define the function $q : X \times X \to [0,\infty)$ by

$$q(x,y) = \sup_{t\in[a,b]} \{\max\{a_1(x(t)-y(t)), b_1(y(t)-x(t))\}\}$$

for all $x, y \in X$. Clearly, q is a w-distance on X and a ceiling distance of d. Here, we will show that T satisfies the contractive condition (2). Assume that $x, y \in X$ and $t \in [a,b]$. Then we get

$$\begin{aligned} q(Tx,Ty) &= \sup_{t\in[a,b]}\{\max\{\int_a^b a_1[K(t,s,x(s) - K(t,s,y(s)]ds, \int_a^b b_1[K(t,s,y(s) - K(t,s,x(s)]ds\}\} \\ &\le \int_a^b \sup_{t\in[a,b]}\{\max\{a_1[K(t,s,x(s) - K(t,s,y(s)]ds, \int_a^b b_1[K(t,s,y(s) - K(t,s,x(s)]ds\}\} \\ &= \int_a^b \big(\sup_{t\in[a,b]}\{\max a_1[K(t,s,x(s) - K(t,s,y(s)], b_1[K(t,s,y(s) - K(t,s,x(s)]\}\}\big)ds \\ &\le \frac{1}{b-a}\int_a^b \Big(\psi(\sup_{t\in[a,b]}\{\{\max\{a_1(x(t)-y(t)), b_1(y(t)-x(t))\}\}) - \end{aligned}$$

$$\phi(\sup_{t\in[a,b]}\{\{\max\{a_1(x(t)-y(t)),b_1(y(t)-x(t))\}\}\}\Big)ds$$

$$\begin{aligned}&= \frac{1}{b-a}\int_a^b \big(\psi(q(x,y))-\phi(q(x,y))\big)ds\\ &= \psi(q(x,y))-\phi(q(x,y),\end{aligned}$$

and so

$$q(Tx,Ty)\le\psi(q(x,y))-\phi(q(x,y)$$

for all $x,y\in X$. Hence we have

$$\psi(q(Tx,Ty))\le q(Tx,Ty)\le\psi(q(x,y))-\phi(q(x,y),$$

for all $x,y\in X$. It follows that T satisfies condition (13). Therefore, all conditions of Corollary 2.5 are satisfied, and thus T had a unique fixed point. This implies that there exists a unique solution of the nonlinear Fredholm integral equation (15). This completes the proof. □

3.2. Nonlinear Volterra Integral Equations

By using the identical method in the proof of Theorem 3.1 and Theorem 3.2, we get the following results.

Theorem 3.3. *Consider the nonlinear Volterra integral equation*

$$x(t)=\phi(t)+\int_a^t K(t,s,x(s))ds, \tag{16}$$

where $a,b\in\mathbb{R}$ with $a<b$, and $\phi:[a,b]\to\mathbb{R}$ and $K:[a,b]^2\times\mathbb{R}\to\mathbb{R}$ are given continuous mappings. Suppose that the following conditions hold:

(i). Let $(C[a,b],\|\cdot\|_\infty)$ be a complete metric space and the mapping $T:C[a,b]\to C[a,b]$ defined by

$$(Tx)(t)=\phi(t)+\int_a^t K(t,s,x(s))ds \quad \text{for all } x\in C[a,b] \text{ and } t\in[a,b]$$

is a continuous mapping;

(ii). there are two functions $\psi,\phi:[0,\infty)\to[0,\infty)$ with ψ is an altering distance function and ϕ is continuous function such that $\psi(t)<t$, for all $t>0$, $\phi(t)=0$ if and only if $t=0$, and for all $x,y\in C[a,b]$, if and only if $t=0$, and for all $x,y\in C[a,b]$, we have

$$\sup_{t\in[a,b]}|K(t,s,y(t)|\le\frac{\psi(\sup_{t\in[a,b]}|y(t)|)-\phi(\sup_{t\in[a,b]}|y(t)|)-\|\phi\|}{b-a}$$

for all $t,s\in[a,b]$.

Then the nonlinear integral equation (16) has a unique solution. Moreover, for each $x_0\in C[a,b]$, the Picard iteration $\{x_n\}$ defined by

$$(x_n)(t)=\phi(t)+\int_a^t K(t,s,x_{n-1}(s))ds$$

for all $n\in\mathbb{N}$ converges to a unique solution of the nonlinear integral equation (16).

Theorem 3.4. *Consider the nonlinear Volterra integral equation*

$$x(t) = \phi(t) + \int_a^t K(t, s, x(s))ds, \tag{17}$$

where $a, b \in \mathbb{R}$ with $a < b$, and $\phi : [a, b] \to \mathbb{R}$ and $K : [a, b]^2 \times \mathbb{R} \to \mathbb{R}$ are given continuous mappings. Suppose that the following conditions hold:

(i). the mapping $T : C[a, b] \to C[a, b]$ defined by

$$(Tx)(t) = \phi(t) + \int_a^t K(t, s, x(s))ds \quad \text{for all } x \in C[a, b] \text{ and } t \in [a, b]$$

is a continuous mapping;

(ii). there are two functions $\psi, \phi : [0, \infty) \to [0, \infty)$ with ψ is an altering distance function and ϕ is continuous function such that $\psi(t) < t$, for all $t > 0$, $\phi(t) = 0$ if and only if $t = 0$, and for all $x, y \in C[a, b]$, if and only if $t = 0$, and for all $x, y \in C[a, b]$, where $a_1, b_1 > 1$ we have

$$\begin{aligned} &\sup_{t\in[a,b]} \max\{a_1|K(t, s, x(s) - K(t, s, y(s))|,\\ &\qquad b_1|K(t, s, y(s) - K(t, s, x(s))|\} \leq \frac{1}{b-a}\Big(\psi(\sup_{t\in[a,b]}\{\max\{a_1(x(t) - y(t)),\\ &\qquad b_1(y(t) - x(t))\}\}) - \phi(\sup_{t\in[a,b]}\{\max\{a_1(x(t) - y(t)), b_1(y(t) - x(t)))\}\}\Big) \end{aligned}$$

for all $t, s \in [a, b]$.

Then the nonlinear integral equation (17) has a unique solution. Moreover, for each $x_0 \in C[a, b]$, the Picard iteration $\{x_n\}$ defined by

$$(x_n)(t) = \phi(t) + \int_a^t K(t, s, x_{n-1}(s))ds$$

for all $n \in \mathbb{N}$ converges to a unique solution of the nonlinear integral equation (17).

References

[1] S.Banach, *Surles opérations das les ensembles abstraits et leur applications aux équations intégrales*, Fundam. Math., 3(1922), 133-181.

[2] T.Suzuki, *A new type of fixed point theorem in metric spaces*, Nonlinear Anal., 71(2009), 5313-5317.

[3] T.Suzuki, *Generalized distance and existence theorems in complete metric spaces*, J. Math. Anal. Appl., 253(2001), 440-458.

[4] T.Suzuki, *Several fixed point theorems concerning τ-distance*, Fixed Point Theory Appl., 2004(2004), 195-209.

[5] D.Tataru, *Viscosity solutions of Hamilton-Jacobi equations with unbounded nonlinear terms*, J. Math. Anal. Appl., 163(1992), 345-392.

[6] I.Vályi, *A general maximality principle and a fixed point theorem in uniform space*, Period. Math. Hung., 16(1985), 127-134.

[7] M.S.Khan, M.Swaleh and S.Sessa, F*ixed point theorems by altering distances between the points*, Bull. Aust. Math. Soc., 30(1)(1984), 1-9.

[8] B.S.Choudhury, P.Konar, B.E.Rhoades and N.Metiya, *Fixed point theorems for generalized weakly contractive mappings*, Nonlinear Anal., 74(6)(2011), 2116-2126.

[9] H.Aydi, H.K.Nashine, B.Samet and H.Yazidi, *Coincidence and common fixed point results in partially ordered cone metric spaces and applications to integral equations*, Nonlinear Anal., 74(17)(2011), 6814-6825.

[10] H.Aydi, *On common fixed point theorems for* (ψ, ϕ)*-generalized* f *-weakly contractive mappings*, Miskolc Math. Notes, 14(1)(2013), 19-30.

[11] O.Kada, T.Suzuki and W.Takahashi, *Nonconvex minimization theorems and fixed point theorems in complete metric spaces*, Math. Jpn., 44(1996), 381-391.

[12] W.S.Du, *Fixed point theorems for generalized Hausdorff metrics*, Int. Math. Forum, 3(2008), 1011-1022.

[13] C.Alegre, J.Marn and S.Romaguera, *A fixed point theorem for generalized contractions involving w-distances on complete quasi-metric spaces*, Fixed Point Theory Appl., 2014(2014), 40.

[14] H.Lakzian, H.Aydi and B.E.Rhoades, *Fixed points for* (ϕ, ψ, p)*-weakly contractive mappings in metric spaces with w-distance*, Appl. Math. Comput., 219(12)(2013), 6777-6782.

[15] H.Piri and P.Kumam, *Wardowski type fixed point theorems in complete metric spaces*, Fixed Point Theory Appl., 2016(2016), 45.

[16] Teerawat Wongyat and Wutiphol Sintunavarat, *The existence and uniqueness of the solution for nonlinear Fredholm and Volterra integral equations together with nonlinear fractional differential equations via w-distances*, Advances in Differential Equations, 2017(2017), 211.