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# $M_I^*$-closed Sets in Ideal Topological Spaces

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**Abstract:** The concept of generalized closed sets was considered by Levine in 1970 [7]. In this way we introduce a new generalized closed set via $I_\omega$-open set and study its some basic properties. Also, we investigate the relationship with other types of closed sets.

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**Keywords:** $M_I^*$-closed set, $M_I^*$-derived set, $M_I^*$-frontier.


## 1. Introduction

The concept of ideals in topological spaces is treated in the classic text by Kuratowski [6] and Vaidyanathaswamy [10]. Jankovic and Hamlett [5] investigated further properties of ideal spaces. An Ideal $I$ on a topological space $(X,\tau)$ is a nonempty collection of subsets of $X$ which satisfies the following properties: (1). $A \in I$ and $B \subset A$ implies $B \in I$ (2). $A \in I$ and $B \in I$ implies $A \cup B \in I$. An ideal topological space(or an ideal space) is a topological space $(X,\tau)$ with an ideal $I$ on $X$ and is denoted by $(X,\tau,I)$. For a subset $A \subset X$, $A^*(I,\tau) = \{x \in X : A \cap U \notin I \text{ for every } U \in \tau(x)\}$ is called the local function of $A$ with respect to $I$ and $\tau$ [6]. We simply write $A^*$ in case there is no chance for confusion. A Kuratowski closure operator $cl^*(.)$ for a topology $\tau^*(I,\tau)$ called the $*$-topology, finer than $\tau$ is defined by $cl^*(A) = A \cup A^*$ [10].

## 2. Preliminaries

**Definition 2.1** ([4]). *A subset $A$ of a topological space $(X,\tau)$ is called*

*(1). a pre-open set if $A \subseteq int(cl(A))$ and a pre-closed set if $cl(int(A)) \subseteq A$,*

*(2). a semi-open set $A \subseteq cl(int(A))$ and a semi-closed set if $int(cl(A)) \subseteq A$,*

*(3). an $\alpha$-open set if $A \subseteq int(cl(int(A)))$ and an $\alpha$-closed set if $cl(int(cl(A))) \subseteq A$,*

*(4). a semi-preopen set(= $\beta$-open set[]) if $A \subseteq cl(int(cl(A)))$ and a semi-preclosed set (= $\beta$-closed set) if $int(cl(int(A))) \subseteq A$.*

**Definition 2.2.** *A subset $A$ of a topological space $(X,\tau)$ is called*

*(1). an $\omega$-closed [9] ($\hat{g}$-closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $(X,\tau)$,*

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*(2). an $\hat{\eta}^*$-closed [1] if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\omega$-open in $(X, \tau)$.*

*(3). an $\hat{\eta}$-closed [1] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\omega$-open in $(X, \tau)$.*

**Definition 2.3** ([4]). *Let $S$ be a subset of $(X, \tau, I)$. Then $S$ is said to be*

*(1). $\alpha - I-$open if $S \subseteq int(cl^*(int(A)))$,*

*(2). semi$-I-$open if $S \subseteq cl^*(int(S))$,*

*(3). pre$-I-$open if $S \subseteq int(cl^*(S))$,*

*(4). semipre$-I-$open if $S \subseteq cl(int(cl^*(S)))$.*

**Definition 2.4.** *A subset $A$ of a space $(X, \tau)$ is called:*

*(1). a generalized closed set [3] (briefly g-closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open,*

*(2). $\alpha$-generalized closed [3] (briefly $\alpha$g-closed) if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open,*

*(3). a generalized pre-closed set [8] (briefly gp-closed) if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open,*

*(4). a generalized semipre-closed set [3] (briefly gsp-closed) if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open.*

**Definition 2.5** ([2]). *A subset $A$ of an ideal topological space $(X, \tau, I)$ is called an $I_\omega$(or $I_{\hat{g}}$)-closed set if $A^* \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $(X, \tau)$.*

**Proposition 2.6** ([4]). *For a subset of an ideal topological space the following holds:*

*(a). Every $\alpha - I-$open set is $\alpha$-open.*

*(b). Every semi$-I-$open set is semi-open.*

*(c). Every $\beta - I-$open set is $\beta$-open.*

*(d). Every pre$-I-$open set is pre-open.*

**Lemma 2.7** ([3]). *Let $A$ be a subset of a topological space $X$.*

*(a). Then $spcl(A) = spcl(spcl(A))$.*

*(b). Let $F \subset A \subset X$,where $A$ is open in $X$.*

*Then $spcl_A(F) = spcl(F) \cap A$.*

**Theorem 2.8** ([2]). *Let $(X, \tau, I)$ be an ideal space.Then every $\hat{g}$-closed set is an $I_{\hat{g}}$-closed set but not conversely.*

## 3. Comparison of ${M_I}^*$-closed Set with other Closed Sets and its Basic Properties

**Definition 3.1.** *A subset $A$ of an ideal topological space $(X, \tau, I)$ is called an $M_I$-closed if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $I_\omega$-open in $(X, \tau, I)$.*

**Definition 3.2.** *A subset $A$ of an ideal topological space $(X,\tau,I)$ is called an $M_I^*$-closed if $spcl(A)\subseteq U$ whenever $A\subseteq U$ and $U$ is $I_\omega$-open in $(X,\tau,I)$. The class of all $M_I^*$-closed sets in $(X,\tau,I)$ is denoted by $M_I^*cl(\tau,I)$. That is, $M_I^*cl(\tau,I)=\{A\subset X : A \text{ is } M_I^*\text{-closed in } (X,\tau,I)\}$.*

**Proposition 3.3.** *Every $M_I$-closed set is $M_I^*$-closed but not conversely.*

*Proof.* Let $A$ be a $M_I$-closed set and $U$ be an $I_\omega$-open set such that $A\subseteq U$.Since $A$ is $M_I$-closed, then $pcl(A)\subseteq U$. But $spcl(A)\subseteq pcl(A)\subseteq U$. Hence $A$ is $M_I^*$-closed. □

**Example 3.4.** *Let $X=\{a,b,c\},\tau=\{\emptyset,X,\{a\},\{b\},\{a,b\}\}$ and $I=\{\emptyset,\{c\}\}$. Here the set $A=\{a\}$ is $M_I^*$-closed but not $M_I$-closed.*

**Proposition 3.5.** *Every closed (resp. $\alpha$-closed, pre-closed, semi-closed, semi-preclosed) set is $M_I^*$-closed but not conversely.*

*Proof.* Let $A$ be a closed set and $U$ be an $I_\omega$-open set such that $A\subseteq U$. Then $cl(A)\subseteq U$. But $spcl(A)\subseteq cl(A)\subseteq U$. Thus $A$ is $M_I^*$-closed. The proof follows from the facts that $spcl(A)\subseteq scl(A)\subseteq U$ and $spcl(A)\subseteq pcl(A)\subseteq \alpha cl(A)\subseteq cl(A)$. □

**Example 3.6.** *Let $X=\{a,b,c\},\tau=\{\emptyset,X,\{a\}\}$ and $I=\{\emptyset,\{a\}\}$. Here the set $A=\{b\}$ is $M_I^*$-closed but not closed.*

**Example 3.7.** *Let $X=\{a,b,c\},\tau=\{\emptyset,X,\{a\},\{a,c\}\}$ and $I=\{\emptyset\}$.Here the set $A=\{a,b\}$ is $M_I^*$-closed but not semi-closed (resp. $\alpha$-closed, pre-closed, semipre-closed).*

**Proposition 3.8.** *Every $\alpha-I-$closed (resp. semi$-I-$closed, pre$-I-$closed, semipre$-I-$closed) set is $M_I^*$-closed but not conversely.*

*Proof.* The proof is follows from Proposition 3.5 and Proposition 2.6. □

**Example 3.9.** *Let $X=\{a,b,c\},\tau=\{\emptyset,X,\{a\}\}$ and $I=\{\emptyset,\{a\}\}$. Here the set $A=\{a,b\}$ is $M_I^*$-closed but not $\alpha-I-$closed (resp. semi$-I-$closed, pre$-I-$closed, semipre$-I-$closed).*

**Proposition 3.10.** *Every $M_I^*$-closed set is generalized semi-preclosed (briefly gsp-closed) but not conversely.*

*Proof.* Let $A$ be a $M_I^*$-closed set and $U$ be an open set such that $A\subseteq U$. By Remark 2.21 [3], every open set is $I_\omega$-open and since $A$ is $M_I^*$-closed,we have $spcl(A)\subseteq U$ and hence $A$ is generalized semi-preclosed. □

**Example 3.11.** *Let $X=\{a,b,c,d\},\tau=\{\emptyset,X,\{c\},\{d\},\{a,c\},\{c,d\},\{a,c,d\}\}$ and $I=\{\emptyset,\{a\},\{b\},\{a,b\}\}$. Here the set $A=\{b,c\}$ is gsp-closed but not $M_I^*$-closed.*

**Proposition 3.12.** *Every $M_I^*$-closed set is $\hat{\eta}^*$-closed but not conversely.*

*Proof.* Let $A$ be a $M_I^*$-closed set and $U$ be an $\omega$-open set such that $A\subseteq U$. By Theorem 2.8, every $\omega$-open set is $I_\omega$-open and since $A$ is $M_I^*$-closed, we have $spcl(A)\subseteq U$ and hence $A$ is $\hat{\eta}^*$-closed. □

**Example 3.13.** *Let $X=\{a,b,c,d\},\tau=\{\emptyset,X,\{c\},\{d\},\{a,c\},\{c,d\},\{a,c,d\}\}$ and $I=\{\emptyset,\{a\},\{b\},\{a,b\}\}$. Here the set $A=\{b,c,d\}$ is $\hat{\eta}^*$-closed but not $M_I^*$-closed.*

**Remark 3.14.** *The concept of g-closedness (resp. gp-closedness) and $M_I^*$-closedness are independent concepts as we illustrate by means of the following example.*

**Example 3.15.** *Let $X=\{a,b,c,d\},\tau=\{\emptyset,X,\{c\},\{d\},\{a,c\},\{c,d\},\{a,c,d\}\}$ and $I=\{\emptyset,\{a\},\{b\},\{a,b\}\}$. Here the set $A=\{b,c\}$ is g-closed (resp. gp-closed) but not $M_I^*$-closed and the set $B=\{d\}$ is $M_I^*$-closed but not g-closed (resp. gp-closed).*

**Remark 3.16.** *The concept of $\hat{\eta}$-closedness and $M_I^*$-closedness are independent concepts as we illustrate by means of the following example.*

**Example 3.17.**

*(a). Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}\}$ and $I = \{\emptyset, \{a\}\}$. Here the set $\{a\}$ is $\hat{\eta}$-closed but not $M_I^*$-closed.*

*(b). Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $I = \{\emptyset\}$. Here the set $\{a\}$ is $M_I^*$-closed but not $\hat{\eta}$-closed.*

**Remark 3.18.** *The union (intersection) of any two $M_I^*$-closed sets is not $M_I^*$-closed.*

**Example 3.19.** *Let $X, \tau$ and $I$ be defined as in Example 3.15. Here the set $\{a, c\}$ and $\{a, d\}$ are $M_I^*$-closed but the union $\{a, c\} \cup \{a, d\} = \{a, c, d\}$ is not $M_I^*$-closed.*

**Example 3.20.** *Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}\}$ and $I = \{\emptyset, \{a\}\}$. Here the set $\{a, b\}$ and $\{a, c\}$ are $M_I^*$-closed but the intersection $\{a, b\} \cap \{a, c\} = \{a\}$ is not $M_I^*$-closed.*

**Proposition 3.21.** *Let $A$ be an $M_I^*$-closed set in $(X, \tau, I)$. Then $spcl(A) - A$ does not contain any non-empty $I_\omega$-closed set but not conversely.*

*Proof.* Suppose that $A$ is $M_I^*$-closed and let $F$ be an $I_\omega$-closed set with $F \subset spcl(A) - A$. Then $A \subset F^c$ and so $spcl(A) \subset F^c$. Hence $F \subset (spcl(A))^c$.Thus $F \subset spcl(A) \cap (spcl(A))^c = \emptyset$. □

**Example 3.22.** *Let $X = \{a, b, c, d\}, \tau = \{\emptyset, X, \{c\}, \{d\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}$ and $I = \{\emptyset\}$. For the set $A = \{c\}$, $spcl(A) - A = \{a, c\} - \{c\} = \{a\}$ does not contain any non-empty $I_\omega$-closed set but $A = \{c\}$ is not $M_I^*$-closed.*

**Proposition 3.23.** *Let $A$ and $B$ be any two subsets of an ideal topological space $(X, \tau, I)$. If $A$ is $M_I^*$-closed such that $A \subset B \subset spcl(A)$, then $B$ is $M_I^*$-closed.*

*Proof.* Let $U$ be an $I_\omega$-open set of $(X, \tau, I)$ such that $B \subset U$. Then $A \subset U$, $A$ is $M_I^*$-closed, we get $spcl(A) \subset U$. Now $spcl(B) \subset spcl(spcl(A)) = spcl(A) \subset U$. Thus $B$ is $M_I^*$-closed. □

**Remark 3.24.** *The converse of the above Proposition 3.23 is not true in general.*

**Example 3.25.** *Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}\}$ and $I = \{\emptyset, \{a\}\}$. Here the set $A = \{b\}$ and $B = \{b, c\}$ are $M_I^*$-closed and $A \subset B$ but $B$ is not a subset of $spcl(A)$.*

**Proposition 3.26.** *If $A$ is $I_\omega$-open and $M_I^*$-closed, then $A$ is semi-preclosed.*

*Proof.* Since $A \subset A$ and $A$ is both $I_\omega$-open and $M_I^*$-closed, we get $spcl(A) \subset A$. Since always $A \subset spcl(A)$. Thus $A$ is semi-preclosed. □

**Proposition 3.27.** *For each $x \in X$,either $\{x\}$ is $I_\omega$-closed or $\{x\}^c$ is $M_I^*$-closed in $(X, \tau, I)$.*

*Proof.* Suppose that $\{x\}$ is not $I_\omega$-closed in $(X, \tau, I)$. Then $\{x\}^c$ is not $I_\omega$-open and the only $I_\omega$-open set containing $\{x\}^c$ is the space $X$ itself. Therefore $spcl(\{x\}^c) \subset X$ and so $\{x\}^c$ is $M_I^*$-closed. □

**Proposition 3.28.** *If a subset $A$ of $(X, \tau, I)$ is $M_I^*$-closed, then $I_\omega cl(\{x\}) \cap A \neq \emptyset$ for each $x \in spcl(A)$.*

*Proof.* Suppose that $x \in spcl(A)$ and $I_\omega cl(\{x\}) \cap A = \emptyset$. Then $A \subset (I_\omega cl(\{x\}))^c$ and $(I_\omega cl(\{x\}))^c$ is $I_\omega$-open. By assumption, $spcl(A) \subset (I_\omega cl(\{x\}))^c$ which is a contradiction to $x \in spcl(A)$. □

# 4. $M_I^*$-closure

In this section, we define $M_I^*$-closure of a set and we prove that $M_I^*$-closure is a Kuratowski closure operator on $X$ under the certain condition.

**Definition 4.1.** *For every set $E \subset X$, we define the $M_I^*$-closure of $E$ to be the intersection of all $M_I^*$-closed sets containing $E$. In symbols, $M_I^*\text{-}cl(E) = \bigcap\{A : E \subset A, A \in M_I^*c(\tau, I)\}$.*

**Lemma 4.2.** *For any $E \subset X, E \subset M_I^*cl(E) \subset cl(E)$.*

*Proof.* Follows from Proposition 3.5. □

**Lemma 4.3.** *If $A \subset B$ then $M_I^*cl(A) \subset M_I^*cl(B)$.*

*Proof.* Clearly follows from Definition 4.1. □

**Remark 4.4.** *$M_I^*$-closure of a set need not be $M_I^*$-closed.*

**Example 4.5.** *Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}\}$ and $I = \{\emptyset, \{a\}\}$. Consider the set $A = \{a\}, M_I^*cl(A) = \{a\}$ but $A$ is not $M_I^*$-closed.*

**Lemma 4.6.** *If $E$ is $M_I^*$-closed, then $M_I^*cl(E) = E$ but not conversely.*

*Proof.* From Definition 4.1,the proof follows. □

**Example 4.7.** *In Example 4.5, $M_I^*cl(\{a\}) = \{a\}$ but $\{a\}$ is not $M_I^*$-closed.*

**Theorem 4.8.** *If $M_I^*c(\tau, I)$ is closed under finite union, then $M_I^*$-closure is a Kuratowski closure operator on $X$.*

*Proof.* Since $\emptyset$ and $X$ are $M_I^*$-closed, by Lemma 4.6 we get

(1). $M_I^*cl(\emptyset) = \emptyset, M_I^*cl(X) = X$.

(2). $E \subset M_I^*cl(E)$, by Lemma 4.2.

(3). Suppose $E$ and $F$ are two subsets of $X$, then by Lemma 4.3, we get $M_I^*cl(E) \subset M_I^*cl(E \cup F)$ and $M_I^*cl(F) \subset M_I^*cl(E \cup F)$. Hence $M_I^*cl(E) \cup M_I^*cl(F) \subset M_I^*cl(E \cup F)$. If $x \notin M_I^*cl(E) \cup M_I^*cl(F)$, then there exist $A, B \in M_I^*cl(\tau, I)$ such that $E \subset A, x \notin A, F \subset B$ and $x \notin B$. Hence $E \cup F \subset A \cup B$ and $x \notin A \cup B$. By hypothesis $A \cup B$ is $M_I^*$- -closed. Thus $x \notin M_I^*cl(E \cup F)$. Hence $M_I^*cl(E) \cup M_I^*cl(F) \supset M_I^*cl(E \cup F)$. From the above discussions we have $M_I^*cl(E \cup F) = M_I^*cl(E) \cup M_I^*cl(F)$.

(4). Let $E$ be a subset of $X$ and $A$ be an $M_I^*$-closed set containing $E$. Then by Definition 4.1, $M_I^*cl(E) \subset A$ and $M_I^*cl(M_I^*cl(E)) \subset A$. Since $M_I^*cl(M_I^*cl(E)) \subset A$, we have $M_I^*cl(M_I^*cl(E)) \subset \bigcap\{A : E \subset A, A \in M_I^*c(\tau, I)\} = M_I^*cl(E)$. By Lemma 4.2, $M_I^*cl(E) \subset M_I^*cl(M_I^*cl(E)$ and therefore $M_I^*cl(E) = M_I^*cl(M_I^*cl(E)$. Hence, $M_I^*$-closure is a Kuratowski closure operator on $X$. □

**Definition 4.9.** *Let $\tau_{M_I^*}$ be the topology on $X$ generated by $M_I^*$-closure in the usual manner. That is, $\tau_{M_I^*} = \{U : M_I^*cl(U^c) = U^c\}$.*

**Proposition 4.10.** *If $M_I^*cl(\tau, I)$ is closed under finite union,then $\tau_{M_I^*}$ is a topology for $X$.*

*Proof.* By Theorem 4.8,$M_I^*$-closure satisfies the Kuratowski closure axioms, $\tau_{M_I^*}$ is a topology on $X$. □

# 5. $M_I^*$-open Set

**Definition 5.1.** *A subset A in $(X, \tau, I)$ is called a $M_I^*$-open if $A^c$ is $M_I^*$-closed set in $(X, \tau, I)$. We denote the family of all $M_I^*$-open sets in $(X, \tau, I)$ by $M_I^*o(\tau, I)$.*

The following five Propositions are analogue of Propositions 3.3, 3.5, 3.8, 3.10, 3.12.

**Proposition 5.2.** *Every $M_I$-open set is $M_I^*$-open.*

**Proposition 5.3.** *Every open (resp. $\alpha$-open, pre-open, semi-open, semi-preopen) set is $M_I^*$-open.*

**Proposition 5.4.** *Every $\alpha - I-$open (resp. semi$-I-$open, semipre$-I-$open) set is $M_I^*$-open.*

**Proposition 5.5.** *Every $M_I^*$-open set is generalized semipre-open(briefly gsp-open).*

**Proposition 5.6.** *Every $M_I^*$-open set is $\hat{\eta}^*$-open.*

**Remark 5.7.** *The union (intersection) of any two $M_I^*$-open sets is not $M_I^*$-open.*

**Proposition 5.8.** *A subset A of an ideal topological space $(X, \tau, I)$ is $M_I^*$-open if and only if $F \subset spint(A)$ whenever $A \supset F$ and F is $I_\omega$-closed in $(X, \tau, I)$.*

*Proof.* Suppose that $A$ is $M_I^*$-open in $(X, \tau, I)$ and $A \supset F$,where $F$ is $I_\omega$-closed in $(X, \tau, I)$. Then $A^c \subset F^c$, where $F^c$ is $I_\omega$-open in $(X, \tau, I)$. Hence, we get $spcl(A^c) \subset F^c$ implies $(spint(A))^c \subset F^c$. Thus we have $spint(A) \supset F$.
Conversely, suppose that $A^c \subset U$ and $U$ is $I_\omega$-open in $(X, \tau, I)$. Then $A \supset U^c$ and $U^c$ is $I_\omega$-closed and by hypothesis $sp(int(A) \supset U^c$ implies that $(sp(int(A))^c \subset U$. Hence $spcl(A^c) \subset U$ implies that $A^c$ is $M_I^*$-closed. □

**Proposition 5.9.** *If $spint(A) \subset B \subset A$ and if A is $M_I^*$-open, then B is $M_I^*$-open.*

*Proof.* Suppose that $spint(A) \subset B \subset A$ and $A$ is $M_I^*$-open. Then $A^c \subset B^c \subset spcl(A^c)$ and since $A^c$ is $M_I^*$-closed. By Proposition 3.23, $B^c$ is $M_I^*$-closed. Hence, $B$ is $M_I^*$-open. □

**Proposition 5.10.** *If a set A is $M_I^*$-open, then $spcl(A) - A$ is $M_I^*$-closed but not conversely.*

*Proof.* Suppose that $A$ is $M_I^*$-open. Let $U$ be an $I_\omega$-open set such that $spcl(A) - A \subset U$. Now $spcl(spcl(A) - A) = spcl(A) - spcl(A) = \emptyset \subset U$. Hence $spcl(A) - A$ is $M_I^*$-closed. □

**Example 5.11.** *Let $X = \{a, b, c, d\}, \tau = \{\emptyset, X, \{c\}, \{d\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}$ and $I = \{\emptyset\}$. Consider the set $A = \{c\}, spcl(A) - A = \{a, c\} - \{c\} = \{a\}$ is $M_I^*$-closed but not $M_I^*$-open.*

**Proposition 5.12.** *Let A be a subset of an ideal topological space $(X, \tau, I)$. For any $x \in X, x \in M_I^*cl(A)$ if and only if $U \cap A \neq \emptyset$ for every $M_I^*$-open set U containing x.*

*Proof.* **Necessity:** Suppose that $x \in M_I^*cl(A)$. Let U be an $M_I^*$-open set containing $x$ such that $U \cap A = \emptyset$ and so $A \subset U^c$. But $U^c$ is $M_I^*$-closed and hence $M_I^*cl(A) \subset U^c$. Since $x \notin U^c$ we obtain $x \notin M_I^*cl(A)$, which is contrary to the hypothesis.
**Sufficiency:** Suppose that every $M_I^*$-open set of $(X, \tau, I)$ containing $x$ meets $A$. If $x \notin M_I^*cl(A)$, then there exist an $M_I^*$-closed $F$ of $(X, \tau, I)$ such that $A \subset F$ and $x \notin F$. Therefore, $x \in F^c$ and $F^c$ is an $M_I^*$-open set containing $x$. But $F^c \cap A = \emptyset$. This is contrary to the hypothesis. □

**Definition 5.13.** *For any $A \subset X$, $M_I^*int(A)$ is defined as the union of all $M_I^*$-open sets contained in A. That is, $M_I^*int(A) = \bigcup\{U : U \subset A$ and $U \in M_I^*o(\tau, I)\}$.*

**Proposition 5.14.** *For any set $A \subset X$,$int(A) \subset M_I^*int(A)$.*

*Proof.* The proof follows from Proposition 5.3. ☐

**Proposition 5.15.** *For any two subsets $A_1$ and $A_2$ of $X$,*

*(1). If $A_1 \subset A_2$, then $M_I^*int(A_1) \subset M_I^*int(A_2)$.*

*(2). $M_I^*int(A_1 \cup A_2) \supset M_I^*int(A_1) \cup M_I^*int(A_2)$.*

**Proposition 5.16.** *If $A$ is $M_I^*$-open, then $A = M_I^*int(A)$.*

*Proof.* Clearly follows from Definition 5.13. ☐

**Remark 5.17.** *The converse of the Proposition 5.16 is not true as seen from the following example.*

**Example 5.18.** *Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}\}$ and $I = \{\emptyset, \{a\}\}$. Here $M_I^*o(\tau, I) = P(X) - \{b, c\}$. Now consider the set $A = \{b, c\}$. Then $M_I^*int(A) = \{b\} \cup \{c\} = \{b, c\} = A$ but $A$ is not $M_I^*$-open.*

**Proposition 5.19.** *Let $A$ be a subset of an ideal space $(X, \tau, I)$,then the followings are true.*

*(1). $(M_I^*int(A))^c = M_I^*cl(A^c)$*

*(2). $M_I^*int(A) = (M_I^*(cl(A^c))^c$*

*(3). $M_I^*clA) = (M_I^*int(A))^c$.*

*Proof.*

(1). Let $x \in (M_I^*int(A))^c$. Then $x \notin M_I^*int(A)$. That is, every $M_I^*$-open set $U$ containing $x$ is such that $U \nsubseteq A$. Thus every $M_I^*$-open set $U$ containing $x$ is such that $U \cap A \neq \emptyset$. By Proposition 5.12, $x \in M_I^*cl(A^c)$ and therefore, $(M_I^*int(A))^c \subset M_I^*cl(A^c)$.
Conversely, let $x \in M_I^*cl(A^c)$. Then by Proposition 5.12, every $M_I^*$-open set $U$ containing $x$ is such that $U \cap A^c \neq \emptyset$. By Definition 5.13, $x \notin M_I^*int(A)$, hence $x \in (M_I^*int(A))^c$ and so $M_I^*cl(A^c) \subset (M_I^*int(A))^c$. Thus $(M_I^*int(A))^c = M_I^*cl(A^c)$.

(2). Follows by taking complements in (1).

(3). Follows by replacing $A$ by $A^c$ in (1). ☐

**Proposition 5.20.** *For a subset $A$ of an ideal topological space $(X, \tau, I)$, the following conditions are equivalent:*

*(1). $M_I^*o(\tau, I)$ is closed under any union,*

*(2). $A$ is $M_I^*$-closed if and only if $M_I^*cl(A) = A$,*

*(3). $A$ is $M_I^*$-open if and only if $M_I^*int(A) = A$.*

*Proof.* (1)⇒(2): Let $A$ be a $M_I^*$-closed set. Then by definition of $M_I^*$-closure, $M_I^*cl(A) = A$. Conversely, assume that $M_I^*cl(A) = A$. For each $x \in A^c, x \notin M_I^*cl(A)$. By Proposition 5.12, there exist a $M_I^*$-open set $G_x$ such that $G_x \cap A = \emptyset$ and hence $x \in G_x \subset A^c$. Therefore, we obtain $A^c = \bigcup x \in A^c$. By (1) $A^c$ is $M_I^*$-open and hence $A$ is $M_I^*$-closed.
(2)⇒(3): Follows by (2) and Proposition 5.19.
(3)⇒(1): Let $\{U_\alpha/\alpha \in \wedge\}$ be a family of $M_I^*$-open sets of $X$. Put for each $x \in U$, there exist $\alpha(x) \in \wedge$ such that $x \in U_\alpha(x) \subset U$. Since $U_\alpha(x)$ is $M_I^*$-open, $x \in M_I^*int(U)$ and so $U = M_I^*int(U)$. By (3), $U$ is $M_I^*$-open. Thus $M_I^*o(\tau, I)$ is closed under any union. ☐

**Proposition 5.21.** *In an ideal topological space $(X,\tau,I)$, assume that $M_I^*o(\tau,I)$ is closed under any union. Then $M_I^*cl(A)$ is an $M_I^*$-closed set for every subset $A$ of $X$.*

*Proof.* Since $M_I^*cl(A) = M_I^*cl(M_I^*cl(A))$ and by Proposition 5.20, we get $M_I^*cl(A)$ is a $M_I^*$-closed set. □

## 6. $M_I^*$-derived Set

**Definition 6.1.** *Let $A$ be a subset of a space $X$. A point $x \in X$ is said to be an $M_I^*$ limit point of $A$ if for each $M_I^*$-open set $U$ containing $x, U \cap (A - \{x\}) \neq \emptyset$. The set of all $M_I^*$ limit points of $A$ is called an $M_I^*$-derived set of $A$ and is denoted by $D_{M_I^*}(A)$.*

**Theorem 6.2.** *For subsets $A, B$ of a space $X$, the following statements hold:*

*(1). $D_{M_I^*}(A) \subset D(A)$, where $D(A)$ is the derived set of $A$.*

*(2). If $A \subset B$, then $D_{M_I^*}(A) \subset D_{M_I^*}(B)$.*

*(3). $D_{M_I^*}(A) \cup D_{M_I^*}(B) \subset D_{M_I^*}(A \cup B)$ and $D_{M_I^*}(A \cap B) \subset D_{M_I^*}(A) \cap D_{M_I^*}(B)$.*

*(4). $D_{M_I^*}(D_{M_I^*}(A)) - A \subset D_{M_I^*}(A)$.*

*(5). $D_{M_I^*}(A \cup D_{M_I^*}(A)) \subset A \cup D_{M_I^*}(A)$.*

*Proof.*

(1). Since every open set is $M_I^*$-open,the proof follows.

(2). Follows by Definition 6.1.

(3). Follows by (2).

(4). If $x \in D_{M_I^*}(D_{M_I^*}(A)) - A$ and $U$ is a $M_I^*$-open set containing $x$, then $U \cap (D_{M_I^*}(A) - \{x\}) \neq \emptyset$. Let $y \in U \cap (D_{M_I^*}(A) - \{x\})$. Then since $y \in D_{M_I^*(A)}$ and $y \in U, U \cap (D_{M_I^*}(A) - \{y\}) \neq \emptyset$. Let $z \in U \cap (A - \{y\})$. Then $z \neq x$ for $z \in A$ and $x \notin A$. Hence, $U \cap (A - \{x\}) \neq \emptyset$. Therefore $x \in D_{M_I^*}(A)$.

(5). Let $x \in D_{M_I^*}(A \cup D_{M_I^*}(A))$. If $x \in A$, the result is obvious. So let $x \in D_{M_I^*}(A \cup D_{M_I^*}(A)) - A$, then for an $M_I^*$-open set $U$ containing $x$ such that $U \cap ((A \cup D_{M_I^*}(A)) - \{x\}) \neq \emptyset$. Thus $U \cap (A - \{x\}) \neq \emptyset$ or $U \cap (D_{M_I^*}(A) - \{x\}) \neq \emptyset$. Now, it follows similarly from (4) that $U \cap (A - \{x\}) \neq \emptyset$. Hence, $x \in D_{M_I^*}(A)$. Therefore, in any case $D_{M_I^*}(A \cup D_{M_I^*}(A)) \subset A \cup D_{M_I^*}(A)$. □

**Example 6.3.** *Let $X = \{a,b,c\}, \tau = \{\emptyset, X, \{a,b\}\}$ and $I = \{\emptyset, \{c\}\}$. Then $M_I^*o(\tau,I) = P(X) - \{c\}$. Consider the set $A = \{a,b\}$, we get $D_{M_I^*}(A) = \{c\}$ and $D(A) = X$. Hence $D(A) \nsubseteq D_{M_I^*}(A)$. Also consider the set $A = \{a\}$ and $B = \{b\}$. Then $D_{M_I^*}(A) = \{\emptyset\}, D_{M_I^*}(B) = \{\emptyset\}$ and $D_{M_I^*}(A \cup B) = \{c\}$. Hence $D_{M_I^*}(A \cup B) \nsubseteq D_{M_I^*}(A) \cup D_{M_I^*}(B)$. The converse of the Proposition 6.2 is not true in general.*

**Theorem 6.4.** *For any subset $A$ of a space $X, M_I^*cl(A) = A \cup D_{M_I^*}(A)$.*

*Proof.* Since $D_{M_I^*}(A) \subset M_I^*cl(A), A \cup D_{M_I^*}(A) \subset M_I^*cl(A)$. On the other hand, let $x \in M_I^*cl(A)$.If $x \in A$, then the proof is complete. If $x \notin A$, each $M_I^*$-open set $U$ containing $x$ intersects $A$ at a point distinct from $x$, so $x \in D_{M_I^*}(A)$. Thus $M_I^*cl(A) \subset A \cup D_{M_I^*}(A)$. □

**Definition 6.5.** *$b_{M_I^*}(A) = A - M_I^*int(A)$ is said to be the $M_I^*$-border of $A$.*

**Theorem 6.6.** *For a subset $A$ of a space $X$, the following statements hold:*

*(1). $b_{M_I^*}(A) \subset b(A)$, where $b(A)$ denotes the border of $A$.*

*(2). $A = M_I^*int(A) \cup b_{M_I^*}(A)$.*

*(3). $M_I^*int(A) \cap b_{M_I^*}(A) = \emptyset$.*

*(4). If $A$ is $M_I^*$-open, then $b_{M_I^*}(A) = \emptyset$.*

*(5). $M_I^*int(b_{M_I^*}(A)) = \emptyset$.*

*(6). $b_{M_I^*}(b_{M_I^*}(A)) = b_{M_I^*}(A)$.*

*(7). $b_{M_I^*}(A) = A \cap M_I^*cl(A^c)$.*

*Proof.* (1), (2) and (3) clearly follows.
(4) If $A$ is $M_I^*$-open, then $A = M_I^*int(A)$.
(5) Suppose $x \in M_I^*int(b_{M_I^*}(A))$, then $x \in b_{M_I^*}(A)$. On the other hand, since $b_{M_I^*}(A) \subset A, x \in M_I^*int(b_{M_I^*}(A)) \subset M_I^*int(A)$. Hence, $x \in M_I^*int(A) \cap b_{M_I^*}(A)$, which contradicts (3). Thus $M_I^*int(b_{M_I^*}(A)) = \emptyset$.
(6) Follows by (5).
(7) $b_{M_I^*}(A) = A - M_I^*int(A) = A - (M_I^*cl(A^c))^c = A \cap M_I^*cl(A^c)$. □

**Example 6.7.** *Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}\}$ and $I = \{\emptyset, \{a\}\}$. Here $M_I^*o(\tau, I) = \{\emptyset, X\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$. If $A = \{b\}$, then $b_{M_I^*}(A) = \{b\} - \{b\} = \emptyset, b(A) = \{b\} - \emptyset = \{b\}$. Hence, $b(A) \nsubseteq b_{M_I^*}(A)$. Consider the set $A = \{b, c\}, b_{M_I^*}(A) = \{b, c\} - \{b, c\} = \emptyset$, but $A$ is not $M_I^*$-open, thus in general the converse of the Theorem 6.6 may not be true.*

**Definition 6.8.** *$Fr_{M_I^*}(A) = M_I^*cl(A) - M_I^*int(A)$ is said to be the $M_I^*$-frontier of $A$.*

**Theorem 6.9.** *For a subset $A$ of a space $X$, the following statements are hold:*

*(1). $Fr_{M_I^*}(A) \subset Fr(A)$, where $Fr(A)$ denotes the frontier of $A$.*

*(2). $M_I^*cl(A) = M_I^*int(A) \cup Fr_{M_I^*}(A)$.*

*(3). $M_I^*int(A) \cap Fr_{M_I^*}(A) = \emptyset$.*

*(4). $b_{M_I^*}(A) \subset Fr_{M_I^*}(A)$.*

*(5). $Fr_{M_I^*}(A) = b_{M_I^*}(A) \cup D_{M_I^*}(A)$.*

*(6). If $A$ is $M_I^*$-open, then $Fr_{M_I^*}(A) = D_{M_I^*}(A)$.*

*(7). $Fr_{M_I^*}(A) = M_I^*cl(A) \cap M_I^*cl(A^c)$.*

*(8). $Fr_{M_I^*}(A) = Fr_{M_I^*}(A^c)$.*

*(9). $Fr_{M_I^*}(M_I^*int(A)) \subset Fr_{M_I^*}(A)$.*

*(10). $Fr_{M_I^*}(M_I^*cl(A)) \subset Fr_{M_I^*}(A)$.*

*Proof.*

(1). Since every open set is $M_I^*$-open,we get the proof.

(2). $M_I^*int(A) \cup Fr_{M_I^*}(A) = M_I^*int(A) \cup (M_I^*cl(A) - M_I^*int(A)) = M_I^*cl(A)$.

(3). $M_I^*int(A) \cap Fr_{M_I^*}(A) = M_I^*int(A) \cap (M_I^*cl(A) - M_I^*int(A)) = \emptyset$.

(4). Clearly follows from Definitions.

(5). Since $M_I^*int(A) \cup Fr_{M_I^*}(A) = M_I^*int(A) \cup b_{M_I^*}(A) \cup D_{M_I^*}(A)$, we get $Fr_{M_I^*}(A) = b_{M_I^*}(A) \cup D_{M_I^*}(A)$.

(6). If $A$ is $M_I^*$-open, then $b_{M_I^*}(A) = \emptyset$, then by (5), $Fr_{M_I^*}(A) = D_{M_I^*}(A)$.

(7). $Fr_{M_I^*}(A) = M_I^*cl(A) - M_I^*int(A) = M_I^*cl(A) - (M_I^*cl(A^c))^c = M_I^*cl(A) \cap M_I^*cl(A^c)$.

(8). Follows by (7).

(9). Clearly follows.

(10). $Fr_{M_I^*}(M_I^*cl(A)) = M_I^*cl(M_I^*cl(A)) - M_I^*int(M_I^*cl(A)) \subset M_I^*cl(A) - M_I^*int(A) = Fr_{M_I^*}(A)$. □

**Example 6.10.** *Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}\}$ and $I = \{\emptyset, \{a\}\}$. Thus $M_I^*o(\tau, I) = P(X) - \{b, c\}$. Consider the set $A = \{b\}$, then $Fr_{M_I^*}(A) = \emptyset, Fr(A) = \{b, c\}$. Hence, $Fr(A) \nsubseteq Fr_{M_I^*}(A)$. In general, the converse of Theorem 6.9 need not be true.*

**Definition 6.11.** *$M_I^*Ext(A) = M_I^*int(A^c)$ is said to be the $M_I^*$-exterior of $A$.*

**Theorem 6.12.** *For a subset $A$ of a space $X$, the following statements are hold:*

*(1). $Ext(A) \subset M_I^*Ext(A)$, where $Ext(A)$ denotes the exterior of $A$.*

*(2). $M_I^*Ext(A) = M_I^*int(A^c) = (M_I^*cl(A))^c$.*

*(3). $M_I^*Ext(M_I^*Ext(A)) = M_I^*int(M_I^*cl(A))$.*

*(4). If $A \subset B$, then $M_I^*Ext(A) \supset M_I^*Ext(B)$.*

*(5). $M_I^*Ext(A \cup B) \subset M_I^*Ext(A) \cup M_I^*Ext(B)$.*

*(6). $M_I^*Ext(A \cap B) \supset M_I^*Ext(A) \cap M_I^*Ext(B)$.*

*(7). $M_I^*Ext(A) = \emptyset$.*

*(8). $M_I^*Ext(\emptyset) = X$.*

*(9). $M_I^*int(A) \subset M_I^*Ext(M_I^*Ext(A))$.*

*Proof.* (1) and (2) clearly follows from Definition 6.11.

(3) $M_I^*Ext(M_I^*Ext(A)) = M_I^*Ext(M_I^*int(A^c)) = M_I^*Ext(M_I^*cl(A))^c = M_I^*int(M_I^*cl(A))$.

(4) If $A \subset B$, then $A^c \supset B^c$. Hence $M_I^*int(A^c) \supset M_I^*int(B^c)$ and so $M_I^*Ext(A) \supset M_I^*Ext(B)$.

(5) and (6) follows from (4).

(7) and (8) follows from Definition 6.11.

(9) $M_I^*int(A) \subset M_I^*int(M_I^*cl(A)) = M_I^*int(M_I^*int(A^c))^c = M_I^*int(M_I^*Ext(A))^c = M_I^*Ext(M_I^*Ext(A))$. □

**Example 6.13.** *Let* $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}\}$ *and* $I = \{\emptyset, \{a\}\}$*. Then* $M_I^* o(\tau, I) = P(X) - \{b, c\}$*. Consider the set,* $A = \{a\}$*,* $B = \{c\}$*,* $M_I^* Ext(A) = X, M_I^* Ext(B) = \{a, b\}$ *and* $M_I^* Ext(A \cup B) = \{b\}$*. Hence,* $M_I^* Ext(A \cup B) \neq M_I^* Ext(A) \cup M_I^* Ext(B)$*. Also consider the set* $A = \{a\}$*,* $B = \{a, b\}$*,* $M_I^* Ext(A) = X$*,* $M_I^* Ext(B) = \{c\}$ *and* $M_I^* Ext(A \cap B) = X$*. Hence,* $M_I^* Ext(A \cap B) \neq M_I^* Ext(A) \cap M_I^* Ext(B)$*.*

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