Int. J. Math. And Appl., 6(1–C)(2018), 509–513
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of *Mathematics And its Applications*

Inverse Complementary Tree Domination Number of Total Graphs of P_n and C_n

S. Muthammai[1] and P. Vidhya[2,*]

1 Principal, Alagappa Government Arts College, Karaikudi, Tamil Nadu, India.

2 Department of Mathematics, E.M.G. Yadava Women's College, Madurai, Tamil Nadu, India.

Abstract: A non-empty set $D \subseteq V$ of a graph is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality taken over all the minimal dominating sets of G. A dominating set D is called a complementary tree dominating set if the induced subgraph $\langle V - D \rangle$ is a tree. The complementary tree domination number $\gamma_{ctd}(G)$ of G is the minimum cardinality taken over all minimal complementary tree dominating sets of G. Let D be a minimum dominating set of G. If $V - D$ contains a dominating set D', then D' is called the inverse dominating set of G w.r.t to D. The inverse domination number $\gamma'(G)$ is the minimum cardinality taken over all the minimal inverse dominating sets of G. In this paper, inverse complementary tree domination in total graphs of P_n and C_n are obtained.

MSC: 05C69.

Keywords: Dominating set, complementary tree dominating set, inverse complementary dominating set.

1. Introduction

Kulli V.R. et al. [1] introduced the concept of inverse domination in graphs. Let $G(V, E)$ be a simple, finite, undirected, connected graph with p vertices and q edges. A non-empty set $D \subseteq V$ of a graph is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality taken over all the minimal dominating sets of G. A dominating set D is called a complementary tree dominating set if the induced subgraph $\langle V - D \rangle$ is a tree. The complementary tree domination number $\gamma_{ctd}(G)$ of G is the minimum cardinality taken over all minimal complementary tree dominating sets of G. Let D be a minimum dominating set of G. If $V - D$ contains a dominating set D', then D' is called the inverse dominating set of G w.r.t to D. The inverse domination number $\gamma'(G)$ is the minimum cardinality taken over all the minimal inverse dominating sets of G. The total graph of G denoted by $T(G)$ is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent if

(1). they are adjacent edges of G, or

(2). one is a vertex of G and the other is an edge incident with it (or)

(3). they are adjacent vertices of G.

* *E-mail: vidhyaramman@gmail.com*

Let $D \subseteq V$ be a minimum complementary tree dominating (ctd) set of G. If $V - D$ contains a ctd set D' of D, then D' is called an inverse ctd set with respect to D. The inverse complementary tree domination number $\gamma'_{ctd}(G)$ of G is the minimum number of vertices in an inverse ctd set of G. In this paper, inverse ctd number of total graphs of P_n and C_n are obtained.

2. Inverse Complementary Tree Domination Number of Total Graphs P_n and C_n

In the following, complementary tree domination number of total graph of P_n is found.

Theorem 2.1. *Let P_n be a path on n vertices $(n \geq 4)$. Then,*

$$\gamma_{ctd}(T(P_n)) = \begin{cases} 2\left(\frac{n-1}{3}\right) & \text{if } n \equiv 1 \ (mod\ 3) \\ 2\left\lfloor \frac{2n-1}{3} \right\rfloor & \text{if } n \equiv 0, 2 \ (mod\ 3) \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the path P_n and let $x_1 x_2 \ldots x_{n-1}$ be the added vertices corresponding to the edges $e_1 e_2 \ldots e_{n-1}$ of P_n to obtain $T(P_n)$ where $e_i = (v_i, v_{i+1})$ $i = 1, 2, \ldots, n-2$. Thus $V(T(P_n)) = \{v_1, v_2, \ldots, v_n, x_1, x_2, \ldots, x_{n-1}\}$ $T(P_n)$ has $2n-1$ vertices and

$$\begin{aligned} E(T(P_n)) &= n - 1 + n - 2 + 2(n-1) \\ &= 4n - 5 \end{aligned}$$

Let

$$D = \begin{cases} \{v_2, v_5, \ldots, v_{3j-1}\} \cup \{x_3, x_6, \ldots, x_{3k}\},\ 1 \leq j \leq \frac{n-1}{3},\ 1 \leq k \leq \frac{n-1}{3}, & \text{if } n \equiv 1\ (mod\ 3); \\ \{v_2, v_5, \ldots, v_{3j-1}\} \cup \{x_3, x_6, \ldots, x_{3k}\},\ 1 \leq j \leq \frac{n}{3},\ 1 \leq k \leq \frac{n}{3}, & \text{if } n \equiv 0\ (mod\ 3); \\ \{v_2, v_5, \ldots, v_{3j-1}\} \cup \{x_3, x_6, \ldots, x_{3k}\},\ 1 \leq j \leq \frac{n+1}{3},\ 1 \leq k \leq \frac{n-2}{3}, & \text{if } n \equiv 2\ (mod\ 3). \end{cases}$$

If $n \equiv 1$ (mod 3), $|D| = 2\left(\frac{n-1}{3}\right)$ if $n \equiv 0, 2$ (mod 3), $|D| = 2\left\lfloor \frac{n-1}{3} \right\rfloor$.

The above set D is a ctd-set of $T(P_n)$, since each vertex in $V(T(P_n))$ is either in D or is adjacent to a vertex in D or is adjacent to a vertex in D and $\langle V(T(P_n)) - D \rangle$ is a path. For any $v_i \in D$, $D - \{v_i\}$ does not dominate itself if v_i not a support of P_n. If v_i is a support, then $D - \{v_i\}$ does not dominate itself and the pendant vertex and the vertex in $T(P_n)$ correspond to the pendant edge in P_n. Similarly, if $x_j \in D$, then $F - \{x_j\}$ does not dominate itself or the pendant vertex. Therefore, D is a ctd-set of $T(P_n)$.

Claim: D is a minimum ctd-set of $T(P_n)$.

It is to be observed that any two adjacent vertices of $T(P_n)$ (i.e., on a triangle and $\Delta(T(P_n)) = 4$ Since D is a ctd-set of G, $\langle V(T(P_n)) - D \rangle$ is a tree. Let there exist a vertex say v of degree atleast three in $\langle V - D \rangle$. Then $N(v) \supset \{x_i, x_{i+1}\}$ or $N(v) \supset \{v_j, v_{j+1}\}$ for $i = 1, \ldots, n-2$ and $j = 1, 2, \ldots, n-1$ and hence $\langle V(P_n) - D \rangle$ contains a triangle. Therefore, degree of each vertex in $\langle V(T(P_n) - D \rangle$ is either 1 or 2. That is, $\langle V(T(P_n) - D \rangle$ is a path length of length atmost

$$\begin{aligned} &\frac{4n-4}{3} \quad if \quad n \equiv 1 \text{ (mod 3)} \\ &\frac{4n-3}{3} \quad if \quad n \equiv 0 \text{ (mod 3)} \\ &\frac{4n-5}{3} \quad if \quad n \equiv 2 \text{ (mod 3)} \end{aligned}$$

Hence, number of vertices in $\langle V(T(P_n)) - D\rangle$ is atmost

$$\frac{4n-1}{3} \quad if \quad n \equiv 1 \pmod 3$$
$$\frac{4n}{3} \quad if \quad n \equiv 0 \pmod 3$$
$$\frac{4n-2}{3} \quad if \quad n \equiv 2 \pmod 3$$

Therefore, number of vertices in D is atleast

$$\frac{2(n-1)}{3} \quad if \quad n \equiv 1 \pmod 3$$
$$\frac{2n-3)}{3} \quad if \quad n \equiv 0 \pmod 3$$
$$\frac{2n-1}{3} \quad if \quad n \equiv 2 \pmod 3$$

That is

$$|D| \geq 2\left(\frac{n-1}{3}\right), \quad \text{if } n \equiv 1 \pmod 3$$
$$and \quad |D| = \left\lfloor \frac{2n-1}{3} \right\rfloor, \quad \text{if } n \equiv 0, 2 \pmod 3$$

Therefore

$$\gamma_{ctd}(T(P_n)) = 2\left(\frac{n-1}{3}\right), \quad \text{if } n \equiv 1 \pmod 3$$
$$= \left\lfloor \frac{2n-1}{3} \right\rfloor, \quad \text{if } n \equiv 0, 2 \pmod 3$$

□

Theorem 2.2. *For $n \geq 4$*

$$\gamma'(T(P_n)) = \frac{2(n-1)}{3}, \quad if\ n \equiv 1 \ (mod\ 3)$$
$$= \frac{2n}{3}, \quad if\ n \equiv 0 \ (mod\ 3)$$
$$= \frac{2n-1}{3}, \quad if\ n \equiv 2 \ (mod\ 3)$$

Proof. Let $v_1v_2\ldots v_n$ be the vertices of the path P_n, $n \geq 4$ and let $x_1x_2\ldots x_{n-1}$ be the added vertices corresponding to the edges $e_1, e_2, \ldots, e_{n-1}$ of P_n to obtain $T(P_n)$. Thus $V(T(P_n)) = \{v_1, v_2, \ldots, v_n, x_1, x_2, \ldots, x_{n-1}\}$ $T(P_n)$ has $(2n-1)$ vertices and $4n-5$ edges. From previous theorem, the set

$$D = \{v_2, v_5, \ldots, v_{3j-1}\} \cup \{x_3, x_6, \ldots, x_{3k}\}$$
$$1 \leq j \leq \frac{n-1}{3}, 1 \leq k \leq \frac{n-1}{3}, \text{ if } n \equiv 1 \pmod 3$$
$$\{v_2, v_5, \ldots, v_{3j-1}\} \cup \{x_3, x_6, \ldots, x_{3k}\}$$
$$1 \leq j \leq \frac{n}{3}, 1 \leq k \leq \frac{n-1}{3}, \text{ if } n \equiv 0 \pmod 3$$
$$= \{v_2, v_5, \ldots, v_{3j-1}\} \cup \{x_3, x_6, \ldots, x_{3k}\}$$
$$1 \leq j \leq \frac{n+1}{3}, 1 \leq k \leq \frac{n-2}{3}, \text{ if } n \equiv 2 \pmod 3$$

is a minimum ctd-set of $T(P_n)$. Let

$$\begin{aligned} D' &= \{v_3, v_6, \ldots, v_{3j}\} \cup \{x_1, x_4, \ldots, x_{3k-2}\} \\ &\qquad 1 \le j \le \frac{n-1}{3}, 1 \le k \le \frac{n-1}{3}, \text{ if } n \equiv 1 \pmod 3 \\ &\{v_3, v_6, \ldots, v_{3j}\} \cup \{x_1, x_4, \ldots, x_{3k-2}\} \\ &\qquad 1 \le j \le \frac{n}{3}, 1 \le k \le \frac{n}{3}, \text{ if } n \equiv 0 \pmod 3 \\ &= \{v_3, v_6, \ldots, v_{3j}\} \cup \{x_1, x_4, \ldots, x_{3k-2}\} \\ &\qquad 1 \le j \le \frac{n-2}{3}, 1 \le k \le \frac{n+1}{3}, \text{ if } n \equiv 2 \pmod 3 \end{aligned}$$

If

$$\left.\begin{aligned} n \equiv 1 \pmod 3, |D'| = |D| = \tfrac{2(n-1)}{3} \\ n \equiv 0 \pmod 3, |D'| = |D| + 1 = \tfrac{2n}{3} \\ n \equiv 2 \pmod 3, |D'| = |D| = \tfrac{2n-1}{3} \end{aligned}\right\} = \left\lceil \frac{2n-2}{3} \right\rceil$$

The set D' is a inverse ctd-set of $T(P_n)$, since each vertex in $V-D'$ is adjacent to atleast one vertex in D' and $\langle V(T(P_n)-D')\rangle$ is a tree. Also D' is a subset of $V(T(P_n))-D$; $\gamma'_{ctd}(T(P_n)) \le \lceil \frac{2n-2}{3} \rceil$, $n \ge 4$.

To prove $\gamma'_{ctd}(T(P_n)) \ge \lceil \frac{2n-2}{3} \rceil$, $n \ge 4$.

Let D' be a minimum inverse ctd-set of $T(P_n)$. As in theorem, $\langle V(T(P_n)) - D'\rangle$ is a path and is of length atmost

$$\begin{aligned} &\frac{4n-4}{3} \quad if \;\; n \equiv 1 \pmod 3 \\ &\frac{4n-6}{3} \quad if \;\; n \equiv 0 \pmod 3 \\ &\frac{4n-5}{3} \quad if \;\; n \equiv 2 \pmod 3 \end{aligned}$$

Hence, no. of vertices in $\langle V(T(P_n) - D'\rangle$ is atmost

$$\begin{aligned} &\frac{4n-1}{3} \quad if \;\; n \equiv 1 \pmod 3 \\ &\frac{4n-3}{3} \quad if \;\; n \equiv 0 \pmod 3 \\ &\frac{4n-5}{3} \quad if \;\; n \equiv 2 \pmod 3 \end{aligned}$$

No. of vertices is D' is atleast

$$\begin{aligned} \frac{2(n-1)}{3} &\quad if \;\; n \equiv 1 \pmod 3 \\ \frac{2n}{3} &\quad if \;\; n \equiv 0 \pmod 3 \\ \frac{2n-1}{3} &\quad if \;\; n \equiv 2 \pmod 3 \\ |D'| &\ge \left\lceil \frac{2n-2}{3} \right\rceil \\ |D'| &= \left\lceil \frac{2n-2}{3} \right\rceil \end{aligned}$$

if $n \ge 4$, $\gamma_{ctd}(T(P_n)) = \lceil \frac{2n-2}{3} \rceil$. □

In the following, complementary tree domination of total graph of C_n is found.

Theorem 2.3. *Let C_n be a cycle on n vertices $(n \geq 3)$ then $\gamma_{ctd}(T(C_n)) = n$.*

Proof. Let $v_1 v_2 \ldots v_n$ be the vertices of cycle C_n and let $x_1 x_2 \ldots x_n$ be the added vertices corresponding to the edges $e_1 e_2 \ldots e_n$ of C_n to obtain $T(C_n)$ where $e_i = (v_i, v_{i+1})$, $i = 1, 2, \ldots, n-1$. Thus

$$V(T(C_n)) = \{v_1, v_2, \ldots, v_n, x_1, x_2, \ldots, x_n\}$$

$T(C_n)$ has $2n$ vertices and $E(T(C_n)) = n + n + 2n = 4n$. Let $D = \{v_1, v_2, \ldots, v_{n-1}, x_1\}$ then D is a ctd-set of $T(C_n)$, since each vertex in $V(T(C_n))$ is either in D or is adjacent to a vertex in D and $\langle V(T(C_n)) - D\rangle$ is a path. For any $v_i \in D$ then $D - v_i$ forms a cycle which is a contradiction to $\langle V - D\rangle$ is a tree. Therefore, D is a ctd-set of $T(C_n)$.

Claim: D is a minimum ctd-set of $T(C_n)$.

It is observed that $\delta(T(C_n)) = \Delta(T(C_n)) = 4$. Since D is a ctd-set of G. $\langle V(T(C_n) - D\rangle$ is a tree. Let there exist a vertex say v of degree 4 in $< V - D >$. Then $N(v) \supseteq \{x_j, x_{j+1}, v_i, v_{i+1}\}$ for $i = 1, \ldots, n$, $j = 1, 2, \ldots, n$ and hence $\langle V(C_n) - D\rangle$ contains a cycle. Therefore, degree of each vertex in $\langle V(T(C_n) - D\rangle$ is either 1 or 2. That is, $\langle V(T(C_n) - D\rangle$ is a path. i.e., $\langle V(T(C_n) - D\rangle \cong P_n$. Therefore, number of vertices in D is n. $\gamma_{ctd}(T(C_n)) = n$. □

Theorem 2.4. *For $n \geq 3$, $\gamma'_{ctd}(T(C_n)) = n$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the cycle C_n and let $x_1 x_2 \ldots x_n$ be the added vertices corresponding to the edges $e_1, e_2, \ldots, e_n$ of C_n to obtain $T(C_n)$ where $e_i = (v_i, v_{i+1})$; $i = 1, 2, \ldots, n-1$. Thus $V(T(C_n)) = \{v_1, v_2, \ldots, v_n, x_1, x_2, \ldots, x_n\}$. $T(C_n)$ has $2n$ vertices and $|E(T(C_n)| = 4n$ vertices. From the Theorem 2.3 the set $D = \{u_1, u_2, \ldots, u_{n-1}, x_1\}$ is a ctd-set of $T(C_n)$. Let $D' = \{x_2, x_3, \ldots, x_n, v_n\}$ is a inverse ctd-set of $T(C_n)$. Since each vertex in $V(T(C_n))$ is either in D' or is adjacent to a vertex in D' and $\langle V(T(C_n) - D'\rangle$ is a path. Also D' is a subset of $V(T(C_n) - D$. $\gamma'_{ctd}(T(C_n)) = n$, $n \geq 3$. □

References

[1] V.R.Kulli and S.C.Sigarkanti, *Inverse domination in graphs*, Nat. Acad. Sc. Lett., 14(1991), 473-475.

[2] F.Harary, *Graph Theory*, Narosa Publishing House.

[3] S.Muthammai, M.Bhanumathi and P.Vidhya, *Complementary tree domination of a graph*, International Mathematical Forum, 6(2011), 25-28.

[4] O.Ore, *Theory of Graphs*, Amer. Math Soc. Colloq. Publ., 38(1962).

[5] Teresa W.Haynes, Stephen T.Hedetniemi and Peter J.Slater, *Fundamental of Domination in Graphs*, Marcel Dekker, (1998).