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Stability of Functional Equation in Banach Space: Using Two Different Methods

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Abstract: Using Direct and fixed point method, we prove the Ulam-Hyers stability of a generalized n-dimensional additive functional equation of the form

$$f\left(\sum_{i=1}^{n} kx_i\right) + \sum_{j=1}^{n} f\left(-kx_j + \sum_{\substack{i=1 \\ i\neq j}}^{n} kx_i\right) = (n-1)\left[\sum_{i=1}^{n}(2i-1)\, f(x_i)\right]$$

Where n is the positive integer with $\mathbb{N}-\{0,1,2\}$ and k is the only odd positive integers in Banach Space is discussed.

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1. Introduction

One of the most famous functional equation is the additive functional equation

$$f(x+y) = f(x) + f(y) \tag{1}$$

In 1821, it was first solved by A.L. Cauchy in the class of continuous real-valued functions. It is often called Cauchy additive functional equation in honor of A.L. Cauchy. The theory of additive functional equations is frequently applied to the development of theories of other functional equations. Moreover, the properties of additive functional equations are powerful tools in almost every field of natural and social sciences. Every solution of the additive functional equation (1) is called an additive function. The quadratic function $f(x) = cx^2$ satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{2}$$

And therefore, the equation (2) is called quadratic functional equation. The Hyers-Ulam stability theorem for the quadratic functional equation (2) was proved by F. Skof for the functions $f : E_1 \to E_2$ where E_1 a normed space and E_2 be a Banach space.

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 2f(2x) - 4f(x) \tag{3}$$

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$$f\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=1}^{n} f\left(-x_j + \sum_{\substack{i=1,\\ i\neq j}} x_i\right) = (n-5) \sum_{1\le i<j\le k\le n} f\left(x_i + x_j + x_k\right) + \left(-n^2 + 8n - 11\right) \sum_{\substack{i=1,\\ i\neq j}} f\left(x_i x_j\right) - \sum_{j=1}^{n} f\left(2x_j\right) + \frac{1}{2}\left(n^3 - 10n^2 + 23n + 2\right) \sum_{i=1}^{n} f\left(x_i\right) \tag{4}$$

$$\begin{aligned} &g\left(ax_1 + bx_2 + cx_3 + dx_4\right) + g\left(-ax_1 + bx_2 + cx_3 + dx_4\right) + g\left(ax_1 + bx_2 - cx_3 + dx_4\right) + g\left(ax_1 + bx_2 + cx_3 - dx_4\right) \\ &= g\left(ax_1 + bx_2\right) + g\left(ax_1 + cx_3\right) + g\left(ax_1 + dx_4\right) g\left(bx_2 + cx_3\right) + g\left(bx_2 + dx_4\right) + g\left(cx_3 + dx_4\right) \\ &+ g\left(ax_1\right) - g\left(-ax_1\right) + g\left(bx_2\right) - g\left(-bx_2\right) + g\left(cx_3\right) - g\left(-cx_3\right) + g\left(dx_4\right) - g\left(-dx_4\right) \\ &- \left(ag\left(x_1\right) - ag\left(-x_1\right) + bg\left(x_2\right) - bg\left(-x_2\right) + cg\left(x_3\right) - cg\left(-x_3\right) + dg\left(x_4\right) - dg\left(-x_4\right)\right) \\ &+ a^2\left(g\left(x_1\right) + g\left(-x_1\right)\right) + b^2\left(g\left(x_2\right) + g\left(-x_2\right)\right) + c^2\left(g\left(x_3\right) + g\left(-x_3\right)\right) + d^2\left(g\left(x_4\right) + g\left(-x_4\right)\right) \end{aligned} \tag{5}$$

$$f\left(\sum_{i=1}^{n} x_i\right) \sum_{i=1}^{n} f\left(-x_j + \sum_{i=1, i\neq j}^{n} x_i\right) = \sum_{i=1}^{n} f\left(x_i + x_j\right) + 12 \sum_{1\le i<j\le k\le n} f\left(\sqrt[4]{x_i^2 x_j x_k}\right) + 24 \sum_{1\le i<j\le k\le l\le n} f\left(\sqrt[4]{x_i x_j x_k x_l}\right) - (n+14) \sum_{i=1}^{n} f\left(x_i\right) + \sum_{j=1}^{n} f\left(2x_j\right) \tag{6}$$

$$f\left(\sum_{i=1}^{n} x_i\right) = \sum_{i,j\neq 1} f\left(x_i + x_j\right) + 6 \sum_{1\le i<j<k\le n} f\left(\sqrt[3]{x_i x_j x_k}\right) - (n+6) \sum_{i=1}^{n} f\left(x_i\right) + \sum_{j=1}^{n} f\left(2x_j\right) \tag{7}$$

$$\begin{aligned} &f\left(nx + n^2y + n^3z\right) + f\left(nx - n^2y + n^3z\right) + f\left(nx + n^2y - n^3z\right) + f\left(-nx + n^2y + n^3z\right) \\ &= n\left[f(x) - f(-x)\right] + n^2\left[f(y) - f(-y)\right] + n^3\left[f(z) - f(-z)\right] \\ &+ 2n^2\left[f(x) + f(-x)\right] + 2n^4\left[f(y) + f(-y)\right] + 2n^6\left[f(z) + f(-z)\right] \end{aligned} \tag{8}$$

Were discussed by M. Arunkumar and S. Karthikeyan [3], R. Badora [5], S. Czerwi [7, 8], V. Govindan et.al [14–17], A. Najati and M. B. Moghimi [20]. Motivated by the above findings in this paper, we introduce and investigate that the general solution and generalized Ulam-Hyers stability of a generalized n-dimensional Additive functional equation of the form

$$f\left(\sum_{i=1}^{n} kx_i\right) + \sum_{j=1}^{n} f\left(-kx_j + \sum_{\substack{i=1\\ i\neq j}}^{n} kx_i\right) = (n-1)\left[\sum_{i=1}^{n} (2i-1) f\left(x_i\right)\right] \tag{9}$$

Where k is the only odd positive integers, in Banach Space using Direct and Fixed point methods.

2. General Solution of the Functional Equation (9): When f is Odd

Theorem 2.1. *If an odd mapping $f : X \to Y$ satisfies the functional equation*

$$f(x+y) = f(x) + f(y) \tag{10}$$

For all $x, y \in X$ iff $f : X \to Y$ satisfies the functional equation

$$f\left(\sum_{i=1}^{n} kx_i\right) + \sum_{j=1}^{n} f\left(-kx_j + \sum_{\substack{i=1\\ i\neq j}}^{n} kx_i\right) = (n-1)\left[\sum_{i=1}^{n} (2i-1) f\left(x_i\right)\right] \tag{11}$$

For all $x_1, x_2, \ldots x_n \in X$. Where k is the only odd positive integers.

Proof. Let $f : X \to Y$ satisfies the functional equation (10). Letting $x = y = 0$ in (10), we get $f(0) = 0$. Substituting y by $-x$ in (10), we obtain $f(-x) = -f(x)$ for all $x \in X$. Hence f is an odd function. Replacing (x, y) by (x, x) and $(2x, x)$ in (10), we get

$$f(2x) = 2f(x) \quad and \quad f(3x) = 3f(x) \tag{12}$$

for all $x \in X$. In general, for any positive integer a, we get

$$f(ax) = af(x) \tag{13}$$

One can easy to verify from (10) that

$$f(x_1 + 3x_2 + 5x_3 + \cdots + kx_n) = f(x_1) + 3f(x_2) + 5f(x_3) + \cdots + kf(x_n) \tag{14}$$

For all $x_1, x_2, \ldots x_n \in X$, where k is the greatest finite odd positive integer. Replacing x_1 by $-x_1$, x_2 by $-x_2$, x_3 by $-x_3$,. . . ,x_n by $-x_n$ respectively in (14), we have the following equations

$$\begin{aligned}
f(-x_1 + 3x_2 + 5x_3 + \cdots + kx_n) &= -f(x_1) + 3f(x_2) + 5f(x_3) + \cdots + kf(x_n) \\
f(x_1 - 3x_2 + 5x_3 + \cdots + kx_n) &= f(x_1) - 3f(x_2) + 5f(x_3) + \cdots + kf(x_n) \\
f(x_1 + 3x_2 - 5x_3 + \cdots + kx_n) &= f(x_1) + 3f(x_2) - 5f(x_3) + \cdots + kf(x_n) \\
f(x_1 + 3x_2 + 5x_3 + \cdots - kx_n) &= f(x_1) + 3f(x_2) + 5f(x_3) + \cdots - kf(x_n)
\end{aligned}$$

for all $x_1, x_2, \ldots x_n \in X$. Adding all the above n-equations, we arrive (11) as desired.
Conversely, $f : X \to Y$ satisfies the functional equation (11). Replacing $(x_1, x_2, \ldots, x_n)$ by $(0, 3x, 0, \ldots, 0)$ in (11), we get

$$f(3x) = 3f(x) \tag{15}$$

for all $x \in X$. It is easy to verify from (11) that

$$f\left(\frac{x}{3^i}\right) = \frac{1}{3^i} f(x), \quad i = 1, 2, \ldots, n \tag{16}$$

for all $x \in X$. Replacing $(x_1, x_2, \ldots, x_n)$ by $\left(x, \frac{y}{3}, 0, \ldots, 0\right)$ in (11) and using (**??**), we get the result of (10). □

3. Stability Results for (9): Direct Method

Theorem 3.1. *Let $j \in \{-1, 1\}$ and $\propto$: $X^n \to [0, \infty)$ be a function such that $\sum_{m=0}^{\infty} \frac{\propto(3^{mj}x_1, 3^{mj}x_2, \ldots, 3^{mj}x_n)}{3^{mj}}$ converges in R and*

$$\lim_{k\to\infty} \frac{\propto(3^{mj}x_1, 3^{mj}x_2, \ldots, 3^{mj}x_n)}{3^{mj}} = 0 \tag{17}$$

for all $x_1, x_2, \ldots x_n \in X$. Let $f : X \to Y$ be an odd function satisfying the inequality

$$\|Df(x_1, x_2, \ldots, x_n)\| \leq \propto (x_1, x_2, \ldots, x_n) \tag{18}$$

for all $x_1, x_2, \ldots x_n \in X$. Then there exists a unique additive mapping $A : X \to Y$ which satisfies the functional equation (9) and

$$\|f(x) - A(x)\| \leq \frac{1}{3(n-1)} \sum_{m=\frac{1-j}{2}}^{\infty} \frac{\propto(0, 3^{mj}x, 0, \ldots, 0)}{3^{mj}} \tag{19}$$

for all $x \in X$. The mapping $A(x)$ is defined by

$$A(x) = \lim_{m\to\infty} \frac{f(3^{mj}x)}{3^{mj}} \tag{20}$$

for all $x \in X$.

Proof. Assume that $j = 1$. Replacing $(x_1, x_2, \dots, x_n)$ by $(0, x, 0, \dots, 0)$ in (18) and using oddness of f, we get

$$\|(n-1) f(3x) - 3(n-1) f(x)\| \leq \propto (0, x, 0, \dots, 0) \tag{21}$$

for all $x \in X$. It follows from (21) that

$$\left\| \frac{f(3x)}{3} - f(x) \right\| \leq \frac{\propto (0, x, 0, \dots, 0)}{3(n-1)} \tag{22}$$

for all $x \in X$. Replacing x by $3x$ in (22), we get

$$\left\| \frac{f(3^2x)}{3} - f(3x) \right\| \leq \frac{\propto (0, 3x, 0, \dots, 0)}{3(n-1)} \tag{23}$$

for all $x \in X$. It follows from (23) that

$$\left\| \frac{f(3^2x)}{3^2} - \frac{f(3x)}{3} \right\| \leq \frac{\propto (0, 3x, 0, \dots, 0)}{3^2(n-1)} \tag{24}$$

for all $x \in X$. It follows from (22) and (24) that

$$\left\| \frac{f(3^2x)}{3^2} - f(x) \right\| \leq \frac{1}{3(n-1)} \left[\propto (0, x, 0, \dots, 0) + \frac{1}{3} \propto (0, 3x, 0, \dots, 0) \right]$$

for all $x \in X$. Generalizing we have,

$$\left\| \frac{f(3^m x)}{3^m} - f(x) \right\| \leq \frac{1}{3(n-1)} \sum_{m=0}^{n-1} \frac{\propto (0, 3^m x, 0, \dots, 0)}{3^m}$$

$$\left\| \frac{f(3^m x)}{3^m} - f(x) \right\| \leq \frac{1}{3(n-1)} \sum_{m=0}^{\infty} \frac{\propto (0, 3^m x, 0, \dots, 0)}{3^m} \tag{25}$$

for all $x \in X$. In order to prove convergence of the sequence $\left\{ \frac{f(3^m x)}{3^m} \right\}$, replace x by $3^l x$ and dividing 3^l in (25) for any $m, l > 0$ to deduce

$$\begin{aligned}
\left\| \frac{f(3^{m+l}x)}{3^{m+l}} - \frac{f(3^l x)}{3^l} \right\| &= \frac{1}{3^l} \left\| \frac{f(3^{m+l}x)}{3^m} - f\left(3^l x\right) \right\| \\
&\leq \frac{1}{3(n-1)} \sum_{m=0}^{n-1} \frac{\propto \left(0, 3^{m+l}x, 0, \dots, 0\right)}{3^{m+l}} \\
&\leq \frac{1}{3(n-1)} \sum_{m=0}^{\infty} \frac{\propto \left(0, 3^{m+l}x, 0, \dots, 0\right)}{3^{m+l}} \to 0 \quad as \quad l \to \infty
\end{aligned} \tag{26}$$

for all $x \in X$. Hence the sequence $\left\{ \frac{f(3^m x)}{3^m} \right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $A : X \to Y$ such that

$$A(x) = \lim_{m\to\infty} \frac{f(3^m x)}{3^m}$$

For all $x \in X$. Letting $m \to \infty$ in (25), we get the result is (19) holds for all $x \in X$. To prove that A satisfies (9), replacing $(x_1, x_2, \ldots, x_n)$ by $(3^m x_1, 3^m x_2, \ldots, 3^m x_n)$ and dividing 3^m in (18), we get

$$\frac{1}{3^m} \|Df\left((3^m x_1, 3^m x_2, \ldots, 3^m x_n)\right)\| \leq \frac{1}{3^m} \propto (3^m x_1, 3^m x_2, \ldots, 3^m x_n)$$

for all $x_1, x_2, \ldots x_n \in X$. Letting $m \to \infty$ in the above inequality and using the definition of A(x), we see that $DA\left(x_1, x_2, \ldots, x_n\right) = 0$ for all $x_1, x_2, \ldots x_n \in X$. Hence A satisfies (8). To show that A is unique. Let $B(x)$ be an another additive mapping satisfying (9) and (19), then

$$\begin{aligned} \|A\left(x\right) - B\left(x\right)\| &\leq \frac{1}{3^l} \left\|A\left(3^l x\right) - f\left(3^l x\right)\right\| + \left\|f\left(3^l x\right) - B\left(3^l x\right)\right\| \\ &\leq \frac{1}{3\left(n-1\right)} \sum_{m=0}^{\infty} \frac{\propto \left(0, 3^{m+l} x, 0, \ldots, 0\right)}{3^{m+l}} \to 0 \quad as \qquad l \to \infty \end{aligned}$$

for all $x \in X$. Hence A is unique. Now, replacing x by $\frac{x}{3}$ in (21), we get

$$\left\|(n-1) f\left(x\right) - 3\left(n-1\right) f\left(\frac{x}{3}\right)\right\| \leq \propto \left(0, \frac{x}{3}, 0, \ldots, 0\right) \tag{27}$$

for all $x \in X$. It follows from (27) that

$$\left\|f\left(x\right) - 3f\left(\frac{x}{3}\right)\right\| \leq \frac{\propto \left(0, \frac{x}{3}, 0, \ldots, 0\right)}{(n-1)} \tag{28}$$

for all $x \in X$. The rest of the proof is similar to that j=1.Hence for $j = -1$ also the theorem is true.This completes the proof of the theorem. $\square$

Corollary 3.2. *Let λ and s be a non negative real numbers. Let an odd function $f : X \to Y$ satisfying the inequality*

$$\|Df\left(x_1, x_2, \ldots, x_n\right)\| \leq \begin{cases} \lambda, & \\ \lambda\left(\sum\limits_{i=1}^{n} \|x_i\|^s\right), & s \neq 1; \\ \lambda\left\{\left(\prod\limits_{i=1}^{n} \|x_i\|^s\right) + \left(\sum\limits_{i=1}^{n} \|x_i\|^{ns}\right)\right\}, & s \neq \frac{1}{n} \end{cases} \tag{29}$$

for all $x_1, x_2, \ldots x_n \in X$. Then there exist a unique additive function $A : X \to Y$ such that

$$\|f\left(x\right) - A\left(x\right)\| \leq \begin{cases} \frac{\lambda}{2(n-1)}, & \\ \frac{\lambda\|x\|^s}{(n-1)(3-3^s)}, & s \neq 1; \\ \frac{\lambda\|x\|^{ns}}{(n-1)(3-3^{ns})}, & s \neq \frac{1}{n} \end{cases}$$

for all $x \in X$.

4. Fixed Point Stability of (9): Odd Case-Fixed Point Method

The following theorems are useful to prove our fixed point stability results.

Theorem A (Banach contraction principle). *Let (X, d) be a complete metric spaces and consider a mapping $T : X \to X$ which is strictly contractive mapping, that is,*

(A_1) $d(Tx, Ty) \leq d(x, y)$ for some (Lipschtiz constant) $L < 1$, then,

(i). The mapping T has one and only fixed point $x^* = T(x^*)$.

(ii). The fixed point for each given element x^* is globally contractive that is

(A_2) $\lim_{n\to\infty} T^n x = x^*$ for any starting point $x \in X$.

(iii). One has the following estimation inequalities,

(A_3) $d(T^n x, x^*) \leq \frac{1}{1-L} d\left(T^n x, T^{n+1} x\right), \quad \forall n \geq 0, \quad \forall x \in X$.

(A_4) $d(x, x^*) = \frac{1}{1-L} d(x, x^*), \quad \forall x \in X$.

Theorem B (The Alternative Fixed Point)**.** *Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $T : X \to X$ with Lipschtiz constant L, then for each given element $x \in X$ either,*

(B_1) $d(T^n x, T^{n+1} x) = \infty, \quad \forall n \geq 0$.

(B_2) There exists a natural number n_0 such that,

(i). $d(T^n x, T^{n+1} x) < \infty$ for all $\forall n \geq 0$.

(ii). The sequence $\{T^n x\}$ is convergent to a fixed point y^* of T,

(iii). y^* is the unique fixed point of Tin the set $Y = \{y \in Y; d(T^{n_o} x, y) < \infty\}$.

(iv). $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in Y$.

Theorem 4.1. *Let $f : W \to B$ be an odd mapping for which there exists a function $\propto : W^n \to [0, \infty)$ with the condition*

$$\lim_{m\to\infty} \frac{\propto (\psi_i^m x_1, \psi_i^m x_2, \ldots, \psi_i^m x_n)}{\psi_i^m} = 0 \tag{30}$$

where

$$\psi_i = \begin{cases} 3, & i = 0 \\ \frac{1}{3}, & i = 1 \end{cases}$$

such that the functional inequality

$$\|Df(x_1, x_2, \ldots, x_n)\| \leq \propto (x_1, x_2, \ldots, x_n) \tag{31}$$

For all $x_1, x_2, \ldots x_n \in X$. If there exist $L = L(i)$ such that the function

$$x \to \beta(x) = \frac{\propto \left(0, \frac{x}{3}, 0, \ldots, 0\right)}{(n-1)}$$

has the property,

$$\frac{1}{\psi_i} \beta(\psi_i x) = L\beta(x) \tag{32}$$

for all $x \in W$. Then there exists a unique additive function $A : W \to B$ satisfying the functional equation (9) and

$$\|f(x) - A(x)\| \leq \frac{L^{1-i}}{1-L} \beta(x) \tag{33}$$

holds for all $x \in W$.

Proof. Consider the set $X = \{P/P : W \to B,\ P(0) = 0\}$ and introduce the generalized metric on X. $d(p,q) = \inf\{K \in (0,\infty) : \|\, p(x) - q(x)\,\| \le K\beta(x), x \in W\}$. It is easy to see that (X,d) is complete. Define $T : X \to X$ by

$$T_p\,(x) = \frac{1}{\psi_i} p\,(\psi_i x)$$

for all $x \in W$. Now $p,q \in X$,

$$\begin{aligned} d(p,q) \le K &\Rightarrow \|p(x) - q(x)\,\| \le K\beta(x), \quad x \in W; \\ &\Rightarrow \left\| \frac{1}{\psi_i} p(\psi_i x) - \frac{1}{\psi_i} q(\psi_i x) \right\| \le \frac{1}{\psi_i} K\beta(\psi_i x), \quad x \in W \\ &\Rightarrow \|Tp(x) - Tq(x)\| \le LK\beta(x), \quad x \in W \\ &\Rightarrow d(Tp, Tq) \le LK. \end{aligned}$$

This implies $d(Tp,Tq) \le Ld(p,q)$ for all $p,q \in X$. (i,e.,) T is strictly contractive mapping on X with Lipschtiz constant L. It is follows from (21) that

$$\|(n-1)\, f\,(3x) - 3\,(n-1)\, f\,(x)\| \le \propto\ (0, x, 0, \ldots, 0) \tag{34}$$

for all $x \in W$. It is follows from (34) that

$$\left\| \frac{f\,(3x)}{3} - f\,(x) \right\| \le \frac{\propto(0, x, 0, \ldots, 0)}{3\,(n-1)} \tag{35}$$

for all $x \in W$. Using (32), for the case $i = 0$, it reduces to

$$\left\| \frac{f\,(3x)}{3} - f\,(x) \right\| \le \frac{1}{3} \beta\,(x)$$

for all $x \in W$. (i.e.,) $d(f, Tf) \le \frac{1}{3} \Rightarrow d(f, Tf) \le \frac{1}{3} = L = L^1 < \infty$. Again replacing $x = \frac{x}{n}$ in (34), we get

$$\begin{aligned} \left\|(n-1)\, f\,(x) - 3\,(n-1)\, f\left(\frac{x}{3}\right)\right\| &\le \propto \left(0, \frac{x}{3}, 0, \ldots, 0\right) \\ \left\| f\,(x) - 3f\left(\frac{x}{3}\right)\right\| &\le \frac{\propto\left(0, \frac{x}{3}, 0, \ldots, 0\right)}{(n-1)} \end{aligned} \tag{36}$$

for all $x \in W$. Using (32) for the case $i = 1$, it reduces to,

$$\left\| 3f\left(\frac{x}{3}\right) - f\,(x)\right\| \le\ (x) \tag{37}$$

for all $x \in W$. (i.e.,) $d(f, Tf) \le 1 \Rightarrow d(f, Tf) \le 1 = L^0 < \infty$. In above case, we arrive

$$d(f, Tf) \le L^{1-i}$$

Therefore $(B_2(i))$ holds. By $(B_2(ii))$, it follows that there exists a fixed point A of T in X, such that

$$A(x) = \lim_{m\to\infty} \frac{f_a\,(\psi_i^m x)}{\psi_i^m}, \qquad \forall\, x \in W. \tag{38}$$

In order to prove $A : W \to B$ is additive. Replacing $(x_1, x_2, \ldots, x_n)$ by $(\psi_i^m x_1, \psi_i^m, \ldots, \psi_i^m x_n)$ in (22) and dividing by ψ_i^m, it follows from (30) and (38), we see that A satisfies (9) for all $x_1, x_2, \ldots x_n \in X$. Hence A satisfies the functional equation (9).

By $(B_2(iii))$, A is the unique fixed point of T in the set, $Y = \{f \in X\ ; d\,(Tf,\ A) < \infty\}$. Using the fixed point alternative result, A is the unique function such that,

$$\|f(x) - A(x)\| \leq K\beta(x)$$

for all $x \in W$, and $k > 0$.Finally by $(B_2(iv))$, we obtain

$$d\,(f, A) \leq \frac{1}{1-L} d\,(f, Tf)$$

(i.e.,) $d\,(f, A) \leq \frac{L^{1-i}}{1-L}$. Hence, we conclude that

$$\|f(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\beta(x)$$

for all $x \in W$. This completes the proof of the theorem. □

Corollary 4.2. *Let $f : W \to B$ be an odd mapping and there exists a real numbers λ and s such that*

$$\|Df\,(x_1, x_2, \ldots, x_n)\| \leq \begin{cases} \lambda & \\ \lambda\left(\sum\limits_{i=1}^{n} \|x_i\|^s\right), & s \neq 1 \\ \lambda\left\{\left(\prod\limits_{i=1}^{n} \|x_i\|^s\right) + \left(\sum\limits_{i=1}^{n} \|x_i\|^{ns}\right)\right\}, & s \neq \frac{1}{n} \end{cases} \tag{39}$$

for all $x_1, x_2, \ldots x_n \in W$. Then there exist a unique additive function $A : W \to B$ such that

$$\|f\,(x) - A\,(x)\| \leq \begin{cases} \frac{\lambda}{2(n-1)} & \\ \frac{\lambda\|x\|^s}{(n-1)(3-3^s)}, & s \neq 1 \\ \frac{\lambda\|x\|^{ns}}{(n-1)(3-3^{ns})}, & s \neq \frac{1}{n} \end{cases} \tag{40}$$

for all $x \in W$.

Proof. Setting

$$\propto (x_1, x_2, \ldots, x_n) \leq \begin{cases} \lambda & \\ \lambda\left(\sum\limits_{i=1}^{n} \|x_i\|^s\right) & s \neq 1 \\ \lambda\left\{\left(\prod\limits_{i=1}^{n} \|x_i\|^s\right) + \left(\sum_{i=1}^{n} \|x_i\|^{ns}\right)\right\}, & s \neq \frac{1}{n} \end{cases}$$

for all $x_1, x_2, \ldots x_n \in W$. Now

$$\begin{aligned} \frac{\propto (\psi_i^m x_1, \psi_i^m x_2, \ldots, \psi_i^m x_n)}{\psi_i^m} &= \begin{cases} \frac{\lambda}{\psi_i^m} \\ \frac{\lambda}{\psi_i^m}\left\{\|\psi_i^m x_1\|^s + \|\psi_i^m x_2\|^s + \cdots + \|\psi_i^m x_n\|^s\right\} \\ \frac{\lambda}{\psi_i^m}\left\{\|\psi_i^m x_1\|^s \|\psi_i^m x_2\|^s \ldots \|\psi_i^m x_n\|^s + \{\|\psi_i^m x_1\|^{ns} + \|\psi_i^m x_2\|^{ns} + \ldots + \|\psi_i^m x_n\|^{ns}\}\right\} \end{cases} \\ &= \begin{cases} \to 0 \ \text{ as } m \to \infty \\ \to 0 \ \text{ as } m \to \infty \\ \to 0 \ \text{ as } m \to \infty \end{cases} \end{aligned}$$

i.e., (34) is holds. But we have $\beta\,(x) = \frac{1}{(n-1)} \propto \left(0, \frac{x}{3}, \ldots, 0\right)$. Hence

$$\beta\,(x) = \frac{1}{(n-1)} \propto \left(0, \frac{x}{3}, \ldots, 0\right) = \begin{cases} \frac{\lambda}{(n-1)} \\ \frac{\lambda}{(n-1)3^s}\|x\|^s \\ \frac{\lambda}{(n-1)3^{ns}}\|x\|^{ns} \end{cases}$$

$$\frac{1}{\psi_i}\beta\left(\psi_i x\right)=\begin{cases}\frac{1}{\psi_i}\frac{\lambda}{(n-1)}\\ \frac{1}{\psi_i}\frac{\lambda\|x\|^s\psi_i^s}{(n-1)3^s}\\ \frac{1}{\psi_i}\frac{\lambda\|x\|^{ns}\psi_i^{ns}}{(n-1)3^{ns}}\end{cases}$$

for all $x \in W$. Hence the inequality (9) holds for $L = 3^{-1}$ if $i = 0$ and $L = \frac{1}{3^{-1}}$ if $i = 1$. $L = 3^{s-1}$ for $s < 1$ if $i = 0$ and $L = \frac{1}{3^{s-1}}$ for $s > 1$ if $i = 1$. $L = 3^{ns-1}$ for $s < \frac{1}{n}$ if $i = 0$ and $L = \frac{1}{3^{ns-1}}$ for $s > \frac{1}{n}$ if $i = 1$. Now, from (34) we prove the following cases:

Case 1: $L = 3^{-1}$ if $i = 0$

$$\|f(x)-A(x)\| \le \frac{L^{1-i}}{1-L}\beta(x)=\frac{\left(3^{-1}\right)}{1-3^{-1}}\frac{\lambda}{n-1}=\frac{\lambda}{2(n-1)}$$

Case 2: $L = \frac{1}{3^{-1}}$ if $i = 1$

$$\|f(x)-A(x)\| \le \frac{L^{1-i}}{1-L}\beta(x)=\frac{1}{1-3}\frac{\lambda}{n-1}=\frac{\lambda}{2(1-n)}$$

Case 3: $L = 3^{s-1}$ for $s < 1$ if $i = 0$

$$\|f(x)-A(x)\| \le \frac{L^{1-i}}{1-L}\beta(x)=\frac{3^{s-1}}{1-3^{s-1}}\frac{\lambda\|x\|^s}{(n-1)3^s}=\frac{\lambda\|x\|^s}{(n-1)(3-3^s)}$$

Case 4: $L = \frac{1}{3^{s-1}}$ for $s > 1$ if $i = 1$

$$\|f(x)-A(x)\| \le \frac{L^{1-i}}{1-L}\beta(x)=\frac{1}{1-\frac{1}{3^{s-1}}}\frac{\lambda\|x\|^s}{(n-1)3^s}=\frac{\lambda\|x\|^s}{(n-1)(3^s-3)}$$

Case 5: $L = 3^{ns-1}$ for $s < \frac{1}{n}$ if $i = 0$

$$\|f(x)-A(x)\| \le \frac{L^{1-i}}{1-L}\beta(x)=\frac{3^{ns-1}}{1-3^{ns-1}}\frac{\lambda\|x\|^{ns}}{(n-1)3^{ns}}=\frac{\lambda\|x\|^{ns}}{(n-1)(3-3^{ns})}$$

Case 6: $L = \frac{1}{3^{ns-1}}$ for $s > \frac{1}{n}$ if $i = 1$

$$\|f(x)-A(x)\| \le \frac{L^{1-i}}{1-L}\beta(x)=\frac{1}{1-\frac{1}{3^{ns-1}}}\frac{\lambda\|x\|^{ns}}{(n-1)3^{ns}}=\frac{\lambda\|x\|^{ns}}{(n-1)(3^{ns}-3)}$$

□

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