Int. J. Math. And Appl., 6(1–C)(2018), 549–555
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Edge-Vertex Domination in Graphs

D. K. Thakkar[1] and Neha P. Jamvecha[1,*]

1 Department of Mathematics, Saurashtra University, Rajkot, Gujarat, India.

Abstract: In this paper we continue the study of ev-domination (edge-vertex domination) in graphs. We give a characterization of minimal ev-dominating sets in graphs. In particular we prove that in a graph with minimum vertex degree greater than or equal to 2, the complement of a minimal ev-dominating set is an edge dominating set. We also state and prove necessary and sufficient condition under which the ev-domination number increases or decreases when a vertex is removed from the graph. We also consider the operation of removing an edge from the graph and prove that the ev-domination number does not decrease when an edge is removed from the graph.

MSC: 05C69.

Keywords: ev-dominating set, minimal ev-dominating set, minimum ev-dominating set, ev-domination number, edge dominating set, private vertex neighbourhood of edge.

1. Introduction

The concept of edges dominate vertices was introduced in 1985 by Laskar, R. and Peters, K. [3, 5] and then in 1992 by Sampathkumar, E. and Kamath, S. S. [4]. An edge $e = uv$ m-dominates a vertex x if $x \in N(u) \cup N(v)$. A set F of edges of a graph G is said to be an ev-dominating set if every vertex of the graph is m-dominated by some edge in F. The minimum cardinality of an ev-dominating set is called the ev-domination number of a graph. It is obvious that every edge cover of a graph is an ev-dominating set of a graph. Thus ev-domination is a generalization of the concept of edge cover for graphs. For any undefined terminologies we refer [1]. We will consider the operations of vertex removal, edge removal and will prove some theorems regarding the change in the ev-domination number when these operations are performed. In particular, we will prove that the ev-domination number of a graph may increase, decrease or remain unchanged when a vertex is removed from the graph. Also we prove that the ev-domination number does not decrease when an edge is removed from the graph.

2. Preliminaries and Notations

If G is a graph then $E(G)$ denotes the edge set and $V(G)$ denotes the vertex set of the graph. S is any set then $|S|$ denotes the cardinality of S and $E(G)\backslash S$ is a subgraph of G obtained by removing the edges of S. If v is a vertex of G then $G\backslash v$ denotes the subgraph of G obtained by removing the vertex v and the edges incident to v. If f is an edge of G then $G\backslash f$ denotes the subgraph of G obtained by removing the edge f. If G is a graph then $\delta(G)$ denotes the minimum degree of graph G.

* *E-mail: jamvechaneha30@gmail.com*

Definition 2.1 (ev-dominating set)**.** *A set $T \subset E(G)$ is an ev-dominating set if every vertex in G is m-dominated by an edge in T.*

Definition 2.2 (minimal ev-dominating set)**.** *An ev-dominating set T for a graph G is said to be a minimal ev-dominating set for G if no proper subset T' of T is an ev-dominating set for the graph G.*

Definition 2.3 (minimum ev-dominating set)**.** *An ev-dominating set of minimum cardinality is called a minimum ev-dominating set.*

Definition 2.4 (ev-domination number)**.** *The ev-domination number for the graph G is denoted by $\gamma_{ev}(G)$ and is the cardinality of a minimum ev-dominating set.*

Definition 2.5 (edge dominating set)**.** *Let G be a graph and T be a set of edges of G then T is said to be an edge dominating set of G if for every e in $E(G)\backslash T$ there is some f in T such that e is adjacent to f.*

Definition 2.6 (ev-neighbourhood of a vertex)**.** *Let G be a graph and $v \in V(G)$. Then ev-neighbourhood of a vertex v is* $N_m(v) = \{e \in E(G) \text{ such that } e \text{ m} - \text{dominates } v\}$.

First we prove a characterization of a minimal ev-dominating set.

Theorem 2.7. *Let G be a graph and F be an ev-dominating set of G. Then F is a minimal ev-dominating set of G if and only if for every edge e in F there is a vertex which is m-dominated by e but it is not m-dominated by any other edge of F.*

Proof. Suppose F is a minimal ev-dominating set of G and let $e \in F$. Now, $F\backslash\{e\}$ is not an ev-dominating set of G. Therefore, there is a vertex x in G which is not m-dominated by any edge of $F\backslash\{e\}$. But x is m-dominated by some edge of F. Therefore, e is the only edge of F which m-dominates x.

Conversely, suppose the condition holds. Let $e \in F$. There is a vertex x which is m-dominated by e but is not m-dominated by any other edge of F. Therefore, x is not m-dominated by any edge of $F\backslash\{e\}$. Thus F is a minimal ev-dominating set of G. □

Corollary 2.8. *Let G be a graph with $\delta(G) \geq 2$. If F be a minimal ev-dominating set of G then $E(G)\backslash F$ is an edge dominating set of G.*

Proof. Suppose F is a minimal ev-dominating set of G. Let $h = uv$ be any edge of F. Since F is a minimal ev-dominating set, there is a vertex x which is m-dominated by h but it is not m-dominated by any other edge of F.

Case (1): $x = u$ or $x = v$.

Since $\delta(G) \geq 2$, there is some other edge xz incident at x. Since x is not m-dominated by any other edge except h, $xz \notin F$ because xz m-dominates x. Thus, xz is an edge which is adjacent to $h$and$xz \in E(G)\backslash F$.

Case (2): $x \neq u$ and $x \neq v$.

Now, x is m-dominated by h. Therefore, xu is an edge or xv is an edge. Suppose xu is an edge. Since x is m-dominated by $x$uand $xu \neq h$, $xu \in E(G)\backslash F$. Similarly, if xv is an edge then $xv \in E(G)\backslash F$. Note that xu or xv are adjacent to h. From both the cases it follows that $E(G)\backslash F$ is an edge dominating set. □

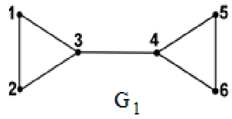
Note that every edge dominating set of G is an ev-dominating set of G. Thus we have the following corollary.

Corollary 2.9. *Let G be a graph with $\delta(G) \geq 2$. If F be a minimal ev-dominating set of G then $E(G)\backslash F$ is an ev-dominating set of G.*

Proof. $E(G)\backslash F$ is an edge dominating set. Therefore, it is an ev-dominating set. □

Now, we consider the operation of removing a vertex from a graph on ev-domination number.

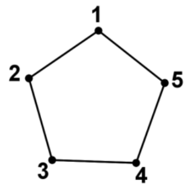
Example 2.10. *Consider the graph whose vertices are* $\{1, 2, 3, 4, 5, 6\}$.



G_1

Now, consider the subgraph $G_1\backslash\{3\}$. *Then,* $\gamma_{ev}(G_1) = 1$ *and* $\gamma_{ev}(G_1\backslash\{3\}) = 2$. *Thus,* $\gamma_{ev}(G_1\backslash\{3\}) > \gamma_{ev}(G)$.

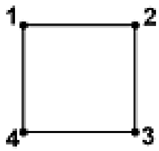
Example 2.11. *Consider the cycle graph with vertices* $\{1, 2, 3, 4, 5\}$.



C_5

If we remove vertex 5 from the graph then $\gamma_{ev}(C_5\backslash\{5\}) = 1$ *while* $\gamma_{ev}(C_5) = 2$. *Thus,* $\gamma_{ev}(C_5\backslash\{5\}) < \gamma_{ev}(C_5)$.

Example 2.12. *Consider the cycle graph with 4 vertices* $\{1, 2, 3, 4\}$



Then, $\gamma_{ev}(C_4) = 1$ *and* $\gamma_{ev}(C_4\backslash\{1\}) = 1$. *Thus,* $\gamma_{ev}(C_4\backslash\{1\}) = \gamma_{ev}(C_4)$.

3. Main Results

Now we prove necessary and sufficient condition under which the removal of a vertex increases the ev-domination number of the graph.

Theorem 3.1. *Let* G *be a graph and* $v \in V(G)$. *Then* $\gamma_{ev}(G\backslash v) > \gamma_{ev}(G)$ *if and only if following two conditions are satisfied.*

(1). For every minimum ev-dominating set F *of* G*, there is an edge* e *containing the vertex* v *such that* $e \in F$.

(2). There is no subset S *of* $G\backslash v$ *such that* $S \cap N_m(v) = \phi$, $|S| \leq \gamma_{ev}(G)$ *and* S *is an ev-dominating set of* $G\backslash v$.

Proof. Suppose $\gamma_{ev}(G\backslash v) > \gamma_{ev}(G)$.

(1). Suppose there is a minimum ev-dominating set F of G such that no edge containing v is a member of F. Then F is a set of edges of $G\backslash v$. Let x be any vertex of $G\backslash v$. Then x is m-dominated by some edge f of F in G. Then x is also m-dominated by f in $G\backslash v$. Therefore, F is an ev-dominating set of $G\backslash v$. Thus, $\gamma_{ev}(G\backslash v) \leq |F| = \gamma_{ev}(G)$ which is a contradiction. Therefore condition (1) is proved.

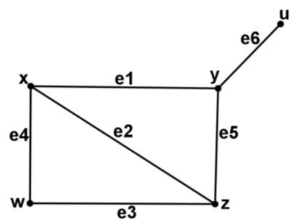
(2). Suppose there is a set F of edges of $G\backslash v$ such that $|F| \leq \gamma_{ev}(G)$, $F \cap N_m(v) = \phi$ and F is an ev-dominating set of $G\backslash v$. Then $\gamma_{ev}(G\backslash v) \leq \gamma_{ev}(G)$ which is again a contradiction. Thus condition (2) is established.

Conversely, suppose condition (1) and (2) are satisfied. Suppose that $\gamma_{ev}(G\backslash v) = \gamma_{ev}(G)$. Let F be a minimum ev-dominating set of $G\backslash v$. Suppose F is also an ev-dominating set of G. Then F is a minimum ev-dominating set of G not containing any edge containing v. This contradicts condition (1). Suppose F is not an ev-dominating set of G. Then no edge of F can m-dominate v. Therefore, $F \cap N_m(v) = \phi$. Also $|F| \leq \gamma_{ev}(G)$ and F is an ev-dominating set of $G\backslash v$. This contradicts condition (2). Thus it follows that $\gamma_{ev}(G\backslash v) = \gamma_{ev}(G)$ is not possible.

Suppose$\gamma_{ev}(G\backslash v) < \gamma_{ev}(G)$. Let F be a minimum ev-dominating set of $G\backslash v$. Then $|F| < \gamma_{ev}(G)$ implies that F cannot be an ev-dominating set of G. Therefore, no edge of F can m-dominate v. Thus $F \cap N_m(v) = \phi$, $|F| \leq \gamma_{ev}(G)$ and F is an ev-dominating set of $G\backslash v$ which again contradicts condition (2). Thus $\gamma_{ev}(G\backslash v) < \gamma_{ev}(G)$ is also not possible. Therefore, $\gamma_{ev}(G\backslash v) > \gamma_{ev}(G)$. □

Definition 3.2 (private vertex neighbourhood of edge)**.** *Let G be a graph, F be a set of edges and $e \in F$. Then private vertex neighbourhood of e with respect to F is the set $prnv[e, F] = \{u \in V(G)$ such that u is m-dominated by v and u is not m-dominated by any other member of $F\}$.*

Example 3.3. *Consider the following graph G*



G

For this graph G, let $F = \{e_1,\ e_2\}$, where $e_1 = \{\text{xy}\}$, $e_2 = \{\text{xz}\}$ then $prnv[e_1, F] = \{\text{u}\}$.

Now, we state and prove a necessary and sufficient condition under which the ev-domination number of a graph decreases when a vertex is removed from a graph.

Theorem 3.4. *Let G be a graph and $v \in V(G)$ then $\gamma_{ev}(G\backslash v) < \gamma_{ev}(G)$ if and only if there is a minimum ev-dominating set of F and edge e in F such that $prnv[e, F] = \{v\}$.*

Proof. Suppose that $\gamma_{ev}(G\backslash v) < \gamma_{ev}(G)$. Let F_1 be a minimum ev-dominating set of $G\backslash v$. Then F_1 cannot be an ev-dominating set of G. Therefore, v is the only vertex which is not m-dominated by any edge of F_1. Let $e = vw$ be any edge and $F = F_1 \cup \{e\}$. Then obviously F is an ev-dominating set of G. Since $|F| = |F_1| + 1$, F is a minimum ev-dominating set of G.

Obviously, v is m-dominated by the edge e which is in F. Since v is not m-dominated by any edge of F_1, v is not m-dominated by any other edge of F. Therefore, $v \in prnv[e, F]$.

(1). Consider the vertex w. Obviously, w is m-dominated by e and $e \in F$. Since w is a vertex of $G\backslash v$, w is also m-dominated by some edge h in F_1. Thus, w is m-dominated by two distinct edges of F. Therefore, $w \notin prnv[e, F]$.

(2). Let z be any vertex of G such that $z \neq v$ and $z \neq w$. If z is m-dominated by e then again by similar argument z is m-dominated by some other edge of F_1 and therefore $z \notin prnv[e, F]$.

Thus, $prnv[e, F] = \{v\}$.

Conversely, suppose there is a minimum ev-dominating set F of G and $e \in F$ such that $prnv[e, F] = \{v\}$. Now consider the graph $G\backslash v$ and the set $F_1 = F\backslash\{e\}$. Then, $|F_1| < |F|$. Let z be any vertex of $G\backslash v$. Then z is also a vertex of G. If z

is m-dominated by e in G then z is also m-dominated by some other edge h in F because $z \notin prnv[e, F]$. Therefore, z is m-dominated by h which is in F_1. If z is not m-dominated by e in G then z must be m-dominated by some other edge h' in F. Then $h' \in F$. Thus, z is m-dominated by some member of F_1. Therefore, F_1 is an ev-dominating set of $G \backslash v$. Therefore, $\gamma_{ev}(G \backslash v) \leq |F_1| < |F| = \gamma_{ev}(G)$. Thus the theorem is proved. □

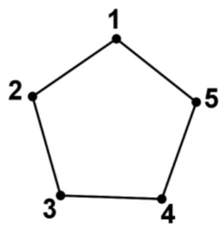
Corollary 3.5. *Let G be a graph and $v \in V(G)$. If $\gamma_{ev}(G \backslash v) < \gamma_{ev}(G)$ then $\gamma_{ev}(G \backslash v) = \gamma_{ev}(G) - 1$.*

Corollary 3.6. *Let G be a graph. $v \in V(G)$ and suppose $\gamma_{ev}(G \backslash v) < \gamma_{ev}(G)$. Then there is a minimum ev-dominating set F of G such that F contains an edge e whose one end vertex is v.*

Proof. Let F_1 be a minimum ev-dominating set of $G \backslash v$. Let $e = vw$ be any edge then from the proof of the above theorem $F = F_1 \cup \{e\}$ is a minimum ev-dominating set of G, $e \in F$ and v is an end vertex of e. □

Remark 3.7. *Suppose, $\gamma_{ev}(G \backslash v) < \gamma_{ev}(G)$. Then it is not necessary that every minimum set should contain an edge whose one end vertex is v.*

For example, consider the cycle graph C_5 with vertices $\{1, 2, 3, 4, 5\}$.



C_5

The set $F = \{12, 34\}$ is a minimum ev-dominating set but it does not contain either $\{45\}$ or $\{15\}$. According to the above theorem there is a minimum set which contains vertex 5 as an end vertex.
For example $\{23, 45\}$ is a minimum ev-dominating set of G. Note that $\gamma_{ev}(G \backslash 5) = 1$ and $\gamma_{ev}(G) = 2$. Now we consider the operation of removing an edge from a graph on the ev-domination number.

Proposition 3.8. *Let G be a graph and e be an edge of G. Then, $\gamma_{ev}(G \backslash e) \geq \gamma_{ev}(G)$.*

Proof. Let F be a minimum ev-dominating set of $G \backslash e$. Then every vertex of $G \backslash e$ is m-dominated by some edge of F. Since $V(G) \backslash \{e\} = V(G)$, every vertex of G is also m-dominated by some member of F. Therefore, $\gamma_{ev}(G) \leq |F| = \gamma_{ev}(G \backslash e)$. □

Now, we state and prove a necessary and sufficient condition under which the ev-domination number of a graph increases when an edge is removed from the graph.

Theorem 3.9. *Let G be a graph and $e = uv$ be an edge of G. Then $\gamma_{ev}(G \backslash e) > \gamma_{ev}(G)$ if and only if following conditions are satisfied by any minimum ev-dominating set T of G.*

(1). if $e \in T$ then there is a vertex z such that $z \in prne[e, T]$.

(2). if $e \notin T$ then every edge in T which m-dominates u is adjacent to v or every edge in T which m-dominates v is adjacent to u.

Proof. Suppose, $\gamma_{ev}(G \backslash e) > \gamma_{ev}(G)$.

(1). Suppose, $e \in T$. Now consider $T_1 = T\backslash\{e\}$ then $|T_1| < \gamma_{ev}(G) < \gamma_{ev}(G\backslash e)$. Therefore T_1 cannot be an ev-dominating set of $G\backslash e$. Therefore, there is a vertex z of $G\backslash e$ such that z is not m-dominated by any member of T_1. But T is an m-dominating set of G. Therefore, z is m-dominated by some member of T. Thus, z is m-dominated by e but not m-dominated by any other member of T. Therefore, $z \in prne[e, T]$.

(2). Suppose, $e \notin T$. Since $\gamma_{ev}(G\backslash e) > \gamma_{ev}(G)$, T cannot be an ev-dominating set of $G\backslash e$. Therefore, there is a vertex w of $G\backslash e$ which is not m-dominated in $G\backslash e$ by any member of T. But w is m-dominated in G by some member of T. Therefore, $w = u$or $w = v$.

Suppose, $w = u$. Any edge of T which m-dominates u cannot have u as an end vertex because this edge will m-dominate u in $G\backslash e$ also which is not true. Suppose f is an edge of T which m-dominates u in G. Suppose, v is not an end vertex of f. Then, $f = xy$ where $\{x, y\} \cap \{u, v\} = \phi$. Then u is m-dominated by f in $G\backslash e$ also which is not true. Therefore, one end vertex of f must be v. Thus every edge in T which m-dominates u does not have u as an end vertex. Similarly, if $w = v$ then any edge of T which m-dominates v does not have v as an end vertex but it does have u as an end vertex. Therefore, condition (2) is also satisfied.
Conversely, suppose condition (1) and (2) are satisfied.

(a). Let T be a set of vertices of $G\backslash e$ such that $|T| < \gamma_{ev}(G)$. If T is an ev-dominating set of $G\backslash e$ then it is also an ev-dominating set of G with $|T| < \gamma_{ev}(G)$, which is a contradiction. Therefore, T cannot be an ev-dominating set of $G\backslash e$.

(b). Let T be a set of vertices of $G\backslash e$ such that $|T| = \gamma_{ev}(G)$. If T is an ev-dominating set of $G\backslash e$. Then T is also an ev-dominating set of G.

Now, $e \notin T$. Therefore, condition (2) is satisfied. Suppose, every edge in T which m-dominates u has an end vertex v then it is obvious that u is not m-dominated by any edge of T in $G\backslash e$. This contradicts our assumption that T is an ev-dominating set of $G\backslash e$. Similarly, if every edge in T which m-dominates v has an end vertex u then v is not m-dominated by any edge of T which is a contradiction. Thus, T cannot be an ev-dominating set of $G\backslash e$. Thus we have proved that if T is a set of vertices of $G\backslash e$ with $|T| \leq \gamma_{ev}(G)$ then T cannot be an ev-dominating set of $G\backslash e$. Thus, any ev-dominating set of $G\backslash e$ has cardinality greater than $\gamma_{ev}(G)$. □

4. Concluding Remarks

The complement of an edge cover of a graph is also an interesting object of study in graph theory which is called an edge stable set. A set F of edges is said to be an edge stable set if for every vertex x there is an edge e containing x such that $e \notin F$ [2]. We can introduce a new concept called m-edge stable set which is the complement of an ev-dominating set. There is a possibility of interesting theorems for this new concept.

References

[1] T.W.Haynes, S.T.Hedetniemi and P.J.Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker Inc., New-York, (1998).

[2] B.M.Kakrecha, *A Study of Some Graph Properties*, Ph.D. thesis submitted to Saurashtra University, Rajkot.

[3] R.Laskar and K.Peters, *Vertex and edge domination parameters in graphs*, Congressus Numerantium, 48(1985), 291-261.

[4] E.Sampathkumar and S.S.Kamath, *Mixed Domination in Graphs*, The Indian Journal of Statistics, 54(1992), 399-402.

[5] E.Sampathkumar and P.S.Neeralagi, *The neighbourhood number of a graph*, Journal of Pure and Applied Mathematics, 16(2)(1985), 126-132.