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The Extendibility of Diophantine Pairs

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Abstract: Let n be a non-zero integer. A set $\{a_1, a_2, \cdots, a_m\}$ of m distinct positive integers is called a Diophantine $m-tuples$ with the property $D(n)$, if $a_i a_j + n$ is a perfect square for all $1 \leq i < j \leq m$. In this paper,we give some sets of polynomial with integer coefficients, such that the product of any two of them added with a quadratic polynomial in $Z(n)$, is a square of a polynomial with integer coefficients.

MSC: 11D09

Keywords: Diophantine triple, Perfect square, Quadratic Polynomial.

1. Introduction

In this paper, we study Diophantine sets, defined as consisting of linear polynomials with the property that, adding the quadratic polynomial $6\sigma^2 + 11\sigma + 4$ or $15\omega^2 + 31\omega + 15$, (here $\sigma, \omega \in \mathbf{N}$) to the product of any two of them, one gets a perfect square. The idea of extending a Diophantine set by joining an integer that preserves the defining property was employed since Euler. Obtaining Dio-pairs is easy: for any integer $r \geq 2$, find a factor a of $r^2 - 1$ and consider it along with the cofactor $\frac{r^2-1}{a}$. I requires an extensive work to find all Diophantine triples extending a fixed pair (a, b). The problem involves solving of Pellian equation for finding a "c" such that ac and bc also preserves the defined property. The case of extending a triple to quadruple have been studied by many mathematicians. In 1993, Dujella proved that if an integer n does not have the form $n = 4k + 2$ and $n \notin S = \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there exists atleast one Diophantine quadruple with property $D(n)$.

2. Preliminaries

Many parametric families of Diophantine triples and $4 - tuples$ are known. For example.

$$\{(a, b, c) : ab + 1 = q^2, c = a + b + 2q\}. \tag{1}$$

is a family of Diophantine triples for $0 < a < b, q > 0$, and

$$\{(a, b, c, d) : ab + 1 = x^2, bc + 1 = y^2, ca + 1 = z^2, d = a + b + c + 2abc + 2xyz\} \tag{2}$$

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is a family of Diophantine $4-tuples$. Note that (1) shows how to pass from a Diophantine pair (a,b) to a Diophantine triple (a,b,c), and (2) shows how to pass from a Diophantine triple (a,b,c) to a Diophantine 4 - tuple (a,b,c,d).

Let us briefly consider the problem of extending a Diophantine pair (a,b), where $ab+1=q^2$, to a Diophantine triple (a,b,c). We shall assume that a and b are co-prime, and we require an integer c such that $ac+1=X^2$ and $bc+1=Y^2$ for some integer X and Y. If such a c exist, then X and Y must be solutions of the Diophantine equation $bX^2-aY^2=b-a$. Conversely, if X and Y are solutions of this equation then a divides X^2-1 (because a and b are co-prime) and so we can define c by $ac+1=X^2$ and $bc+1=Y^2$ and (a,b,c) is a Diophantine triple. As the Diophantine equation behave erratically with respect to the coefficients, it therefore seems unlikely that we can parameterize all Diophantine triples of the form (a,b,x). In this argument, the Diophantine triples, in (1) correspond to the solutions $X=a+q$ and $Y=b+q$.

3. Main Results

Construction of a Diophantine triple (α,β,γ) of linear polynomials such that the product of any two of them added with quadratic polynomials explained in choices 1 and 2, leaves a perfect square.

3.1. Choice 1

For the different pairs of linear polynomials, we investigate for its extendability as a triple satisfying the property $D(6\sigma^2+11\sigma+4)$.

3.1.1. Case 1

Consider the $\{\alpha,\beta\}$ where $\alpha=2\sigma+3$ and $\beta=5\sigma+7$ satisfying $D(6\sigma^2+11\sigma+4)$. To search for its extendability, let γ be any other polynomial with the same property. This can be written as $\alpha\gamma+6\sigma^2+11\sigma+4=\rho^2$ and $\beta\gamma+6\sigma^2+11\sigma+4=\eta^2$. Applying the linear transformation, $\rho=\mu+(2\sigma+3)\nu$ and $\eta=\mu+(5\sigma+7)\nu$ and eliminating γ, we get the Pellian equation $\mu^2-(10\sigma^2+29\sigma+21)\nu^2=6\sigma^2+11\sigma+4$, with initial solution $\mu_0=4\sigma+5,\nu_0=1$. The values of μ_0 and ν_0 can be employed to find the polynomial γ as $\gamma=15\sigma+20$, satisfying the property $D(6\sigma^2+11\sigma+4)$.

3.1.2. Case 2

Consider the $\{\beta,\gamma\}$ where $\beta=5\sigma+7$ and $\gamma=15\sigma+20$ satisfying $D(6\sigma^2+11\sigma+4)$. To search for its extendability, let δ be any other polynomial with the same property. This can be written as $\beta\delta+6\sigma^2+11\sigma+4=\rho^2$ and $\gamma\delta+6\sigma^2+11\sigma+4=\eta^2$. Applying the linear transformation, $\rho=\mu+(5\sigma+7)\nu$ and $\eta=\mu+(15\sigma+20)\nu$ and eliminating δ, we get the Pellian equation $\mu^2-(75\sigma^2+205\sigma+140)\nu^2=6\sigma^2+11\sigma+4$, with initial solution $\mu_0=9\sigma+12,\nu_0=1$. The values of μ_0 and ν_0 can be employed to find the polynomial δ as $\delta=38\sigma+51$, satisfying the property $D(6\sigma^2+11\sigma+4)$.

3.1.3. Case 3

Consider the $\{\gamma,\delta\}$ where $\gamma=15\sigma+20$ and $\delta=38\sigma+51$ satisfying $D(6\sigma^2+11\sigma+4)$. To search for its extendability, let ξ be any other polynomial with the same property. This can be written as $\gamma\xi+6\sigma^2+11\sigma+4=\rho^2$ and $\delta\xi+6\sigma^2+11\sigma+4=\eta^2$. Applying the linear transformation, $\rho=\mu+(15\sigma+20)\nu$ and $\eta=\mu+(38\sigma+51)\nu$ and eliminating ξ, we get the Pellian equation $\mu^2-(570\sigma^2+1525\sigma+1020)\nu^2=6\sigma^2+11\sigma+4$, with initial solution $\mu_0=24\sigma+32,\nu_0=1$. The values of μ_0 and ν_0 can be employed to find the polynomial ξ as $\xi=101\sigma+135$, satisfying the property $D(6\sigma^2+11\sigma+4)$.

3.1.4. Case 4

Consider the $\{\delta,\xi\}$ where $\delta=38\sigma+51$ and $\xi=101\sigma+135$ satisfying $D(6\sigma^2+11\sigma+4)$. To search for its extendability, let ζ be any other polynomial with the same property. This can be written as $\delta\zeta+6\sigma^2+11\sigma+4=\rho^2$ and $\xi\zeta+6\sigma^2+11\sigma+4=\eta^2$. Applying the linear transformation, $\rho=\mu+(38\sigma+51)\nu$ and $\eta=\mu+(101\sigma+135)\nu$ and eliminating ζ, we get the Pellian equation $\mu^2-(3838\sigma^2+10281\sigma+6885)\nu^2=6\sigma^2+11\sigma+4$, with initial solution $\mu_0=62\sigma+83, \nu_0=1$. The values of μ_0 and ν_0 can be employed to find the polynomial ζ as $\zeta=263\sigma+352$, satisfying the property $D(6\sigma^2+11\sigma+4)$.

3.1.5. Case 5

Consider the $\{\xi,\zeta\}$ where $\xi=101\sigma+135$ and $\xi=263\sigma+352$ satisfying $D(6\sigma^2+11\sigma+4)$. To search for its extendability, let ϕ be any other polynomial with the same property. This can be written as $\xi\phi+6\sigma^2+11\sigma+4=\rho^2$ and $\zeta\phi+6\sigma^2+11\sigma+4=\eta^2$. Applying the linear transformation, $\rho=\mu+(101\sigma+135)\nu$ and $\eta=\mu+(263\sigma+352)\nu$ and eliminating ϕ, we get the Pellian equation $\mu^2-(26563\sigma^2+71057\sigma+47520)\nu^2=6\sigma^2+11\sigma+4$, with initial solution $\mu_0=163\sigma+218, \nu_0=1$. The values of μ_0 and ν_0 can be employed to find the polynomial ϕ as $\phi=690\sigma+923$, satisfying the property $D(6\sigma^2+11\sigma+4)$.

3.2. Choice 2

For the different pairs of linear polynomials, we investigate for its extendability as a triple satisfying the property $D(15\omega^2+31\omega+15)$.

3.2.1. Case 1

Consider the $\{\alpha,\beta\}$ where $\alpha=2\omega+3$ and $\beta=5\omega+7$ satisfying $D(15\omega^2+31\omega+15)$. To search for its extendability, let γ be any other polynomial with the same property. This can be written as $\alpha\gamma+15\omega^2+31\omega+15=\rho^2$ and $\beta\gamma+15\omega^2+31\omega+15=\eta^2$. Applying the linear transformation, $\rho=\mu+(2\omega+3)\nu$ and $\eta=\mu+(5\omega+7)\nu$ and eliminating γ, we get the Pellian equation $\mu^2-(10\omega^2+29\omega+21)\nu^2=15\omega^2+31\omega+15$, with initial solution $\mu_0=5\omega+6, \nu_0=1$. The values of μ_0 and ν_0 can be employed to find the polynomial γ as $\gamma=17\omega+22$, satisfying the property $D(15\omega^2+31\omega+15)$.

3.2.2. Case 2

Consider the $\{\beta,\gamma\}$ where $\beta=5\omega+7$ and $\gamma=17\omega+22$ satisfying $D(15\omega^2+31\omega+15)$. To search for its extendability, let δ be any other polynomial with the same property. This can be written as $\beta\delta+15\omega^2+31\omega+15=\rho^2$ and $\gamma\delta+15\omega^2+31\omega+15=\eta^2$. Applying the linear transformation, $\rho=\mu+(5\omega+7)\nu$ and $\eta=\mu+(17\omega+22)\nu$ and eliminating δ, we get the Pellian equation $\mu^2-(85\omega^2+229\omega+154)\nu^2=15\omega^2+31\omega+15$, with initial solution $\mu_0=10\omega+13, \nu_0=1$. The values of μ_0 and ν_0 can be employed to find the polynomial δ as $\delta=42\omega+55$, satisfying the property $D(15\omega^2+31\omega+15)$.

3.2.3. Case 3

Consider the $\{\gamma,\delta\}$ where $\gamma=17\omega+22$ and $\delta=42\omega+55$ satisfying $D(15\omega^2+31\omega+15)$. To search for its extendability, let ξ be any other polynomial with the same property. This can be written as $\gamma\xi+15\omega^2+31\omega+15=\rho^2$ and $\delta\xi+15\omega^2+31\omega+15=\eta^2$. Applying the linear transformation, $\rho=\mu+(17\omega+22)\nu$ and $\eta=\mu+(42\omega+55)\nu$ and eliminating ξ, we get the Pellian equation $\mu^2-(714\omega^2+1859\omega+1210)\nu^2=15\omega^2+31\omega+15$, with initial solution $\mu_0=27\omega+35, \nu_0=1$. The values of μ_0 and ν_0 can be employed to find the polynomial ξ as $\xi=113\omega+147$, satisfying the property $D(15\omega^2+31\omega+15)$.

3.2.4. Case 4

Consider the $\{\delta,\xi\}$ where $\delta=42\omega+55$ and $\xi=113\omega+147$ satisfying $D(15\omega^2+31\omega+15)$. To search for its extendability, let ζ be any other polynomial with the same property. This can be written as $\delta\zeta+15\omega^2+31\omega+15=\rho^2$ and $\xi\zeta+15\omega^2+31\omega+15=\eta^2$. Applying the linear transformation, $\rho=\mu+(42\omega+55)\nu$ and $\eta=\mu+(113\omega+147)\nu$ and eliminating ζ, we get the Pellian equation $\mu^2-(4746\omega^2+12389\omega+8085)\nu^2=15\omega^2+31\omega+15$, with initial solution $\mu_0=69\omega+90, \nu_0=1$. The values of μ_0 and ν_0 can be employed to find the polynomial ζ as $\zeta=293\omega+382$, satisfying the property $D(15\omega^2+31\omega+15)$.

3.2.5. Case 5

Consider the $\{\xi,\zeta\}$ where $\xi=113\omega+147$ and $\zeta=293\omega+382$ satisfying $D(15\omega^2+31\omega+15)$. To search for its extendability, let ϕ be any other polynomial with the same property. This can be written as $\xi\phi+15\omega^2+31\omega+15=\rho^2$ and $\zeta\phi+15\omega^2+31\omega+15=$

η^2. Applying the linear transformation, $\rho = \mu + (113\omega + 147)\nu$ and $\eta = \mu + (293\omega + 382)\nu$ and eliminating ϕ, we get the Pellian equation $\mu^2 - (33109\omega^2 + 86237\omega + 56154)\nu^2 = 15\omega^2 + 31\omega + 15$, with initial solution $\mu_0 = 182\omega + 237, \nu_0 = 1$. The values of μ_0 and ν_0 can be employed to find the polynomial ϕ as $\phi = 770\sigma + 1003$, satisfying the property $D(15\omega^2 + 31\omega + 15)$.

4. Conclusion

All the triples considered in this paper cannot be extended to a quadruple for the particular choices of quadratic polynomial.

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