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Commutativity of Periodic Rings with Some Identities in the Center

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Abstract: Let R be an periodic ring. In this paper, we prove that an $(n+1)n$-tortion free periodic ring satisfying the properties $(ab)^n - ba \in Z(R)$, $(ab)^{n+1} - ba \in Z(R)$, $a^n(ab) - (ba)a^n \in Z(R)$ for all $a, b \in R$ is commutative, then R is commutative.

Keywords: Commutativity of Periodic rings, Periodic rings, Center.

1. Introduction

Herstein [6] shown that if R is a ring satisfying $(ab)^n = a^n b^n$, where $n > 1$ then R has a nil commentator ideal. Also Abu-Khuzam [1] proved that if R is an identity element 1 satisfying $(ab)^n = a^n b^n$ $\forall$ $a, b \in R$ and $n > 1$, then R is commutative . and then Gupta [5] proved that if R is a semi prime ring with centre $Z(R)$ satisfying $(ab)^2 - a^2b^2 \in Z(R)$ $\forall$ $a, b \in R$, then R is commutative. In [2] it is proved that R is commutative if R is a semi prime ring satisfying $(ab)^n - a^n b^n \in Z(R)$ and $(a^2 b^n) - a^{2n} b^n \in Z(R)$ $\forall$ $a, b \in R$. In this direction we prove that, if R is an $(n+1)n$-torsion free periodic ring such that $(ab)^n - ba \in Z(R)$ and $(ab)^{n+1} - ba \in Z(R)$ or $(ab)^{n+1} - ab \in Z(R)$ and if the set of nilpotent elements of R is commutative Then R is commutative. We prove that an n-torsion free periodic ring (not necessarily with unity) for which $(ab)^n - (ba)^n$ is in the center is commutative which provided that the set A of nilpotent's of R form a commutative set. If $a \in aR \cap Ra$, $\forall$ $a \in R$, then the ring R is called s-unital. In [8] it is proved that if R is an S-unital ring satisfying the following identities. (1) $[a^n, b^n] = 0$, (2) $n[a.b] = 0 \Rightarrow [a, b] = 0$ (3) $[a, a^n(ab)^n - (ba)^n a^n] = 0$ $\forall$ $a, b \in R$, then R is commutative. Using this we prove that if R is an $n(n+1)$-torsion free periodic ring (not necessarily with unity) satisfying the identity $a^n(ab) - (ba)a^n \in Z(R)$ $\forall$ $a, b \in R$ and if A is commutative, then R is commutative .We also prove that $(ab)^n - b^n a^n \in Z(R)$ $\forall$ $a \in R/A$, $b \in R/A$ and A is commutative then R is commutative.

2. Preliminaries

Throughout in this paper R represents a periodic ring not necessarily with unity and A denotes the set of nilpotent elements of R. $Z(R)$ is the center of R. We start the results with following lemmas :

Lemma 2.1. *If $[a, [a, b]] = 0$, then $[a^t, b] = ta^{t-1}[a, b]$ $\forall$ integers $t \geq 1$.*

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Proof. For $t = 1$, identity $[a^t, b] = t.a^{t-1}[a, b]$ is true. If we assume that

$$[a^t, b] = ta^{t-1}[a, b] \tag{1}$$

then

$$\begin{aligned} [a^{t+1}, b] &= [a^t a, b] \\ &= a^t[a, b] + [a^t, b]a \\ &= a^t[a, b] + ta^{t-1}[a, b]a, \quad \text{by } (1) \\ &= a^t[a, b] + ta^t[a, b], \quad \text{since } [a, [a, b]] = 0 \\ &= (t + 1)a^t[a, b] \ \forall\ t > 1 \\ [a^{t+1}, b] &= (t + 1)a^t[a, b] \end{aligned}$$

Hence, by induction $[a^t, b] = ta^{t-1}[a, b]$ for all integers $t \geq 1$. □

Lemma 2.2. *A periodic ring R has following axioms.*

(1). For every a in R, some powers of a is idempotent.

(2). For every a in R, ∃ an integer $n > 1$ such that $a - a^{n(x)}$ is nilpotent.

(3). If $p : R \to R'$ is an epimorphism, then $p(A)$ is equals to the set of nilpotent elements of R'.

(4). If A is central, then R is commutative.

Proof.

(1). If $a^n = a^m$ with $n > m$, then $a^{j+t(n-m)} = a^j$ for all positive integer t and every $j \geq m$. So we assume that $n-m+1 \geq m$. Then it follows that $a^{n-m+1} = (a^{n-m+1})^{n-m+1}$. Thus $(a^{n-m+1})^{n-m}$ is idempotent.

(2). Consider $a^n = a^m$, $n > m > 1$. Then $a^{m-1}(a - a^{n-m+1}) = 0 = a^{m-2}a(a - a^{n-m+1}) = a^{m-2}a^{n-m+1}(a - a^{n-m+1})$. So $a^{m-2}(a - a^{n-m+1})^2 = 0$. By induction the result follows :

(3). By Lemma 1 [3], we know that $(a + Z)$ is a non zero nilpotent element if R', where Z is an ideal of R, then R has a nilpotent element V such that $a = V(mod\ Z)$. Hence if $p : R \to R'$ is an epimorphison, then $P(N)$ is equals to the set of nilpotent elements of R^1.

(4). If A is the set of nilpotent elements of R, then it is easily seen that A is an ideal. Also if $a \in R$ and e is an idempotent R, then both $ea = eae$ and $ae - eae$ exist in R. So commute with e. Therefore idempotent in R are central.

We know that homomorphic images inherit the hypothesis on R. Thus we consider only the case of sub directly irreducible R. As per this assumption, part (1) of the Lemma 2.2 prove that R is either nil and hence commutative or R has a multiplicative identity element 1 which is unique non-zero central idempotent. From (2) of Lemma 2.2 that every element of R is either nilpotent or invertible. Therefore the set D of zero divisors is equal to A is a central ideal. Moreover by (2) of Lemma 2.2 $\overline{R} = R/D$ has a property of Jacobson is $x^n = x$, so that R is commutative and R has torsion group which is additive group. Thus if $x, y \in R/D$ is a finite field which is generated by $\overline{x} = x + D$, and $\overline{y} = y + D$. R/D has cyclic multiplicative group. Hence there exist $g \in R$ and $d_1, d_2 \in D$ such that

$$x = g^i + d_1$$

$$y = g^j + d_2,$$

i, j are positive integers. It follows that x, y must commute. □

Lemma 2.3. *If commutator ideal $C(R) \in R$ is nil, then the nilpotent $A \in R$ form an ideal.*

Proof. Let $x, y \in A$ and $\overline{R} = R/C(R)$. Then $\overline{x} = x + C(R)$ and $\overline{y} = y + C(R)$ are also nilpotent. Therefore $\overline{x} - \overline{y}$ is nilpotent. [$\because$ R is commutative]. Assuming $(\overline{x} - \overline{y})^t = 0$. $(x - y)^t$ of $C(R) \subseteq A$. By hypothesis, $(x - y) \in A$. Let $a \in R$ be an arbitrary element $\Rightarrow xa$ is also nilpotent. Let $(\overline{xa})^m = \overline{0} \Rightarrow (xa)^m \in C(R) \subseteq A$. Thus $xa \in A$. Similarly $ax \in A \; \forall x \in A$, $a \in A$. Hence A is an ideal of R. □

Lemma 2.4. *Let R be a periodic ring and A is commutative, thus the commutator ideal of R is nil, and A forms an ideal of R.*

Proof. Let A be the set of nilpotent elements. We have to prove that (i) $V_1 - V_2 \in A$, $V_1, V_2 \in A$ by using the standard argument for the commutative and also we prove by induction on t that if $v_t = 0$ and $r \in R$, it is true that $(vr)^t = (rv)^t = 0$. First consider $t = 2$. Let $V \in R$ such that $V^2 = 0$ and k be a positive integer for which $(vr)^k = e$ is idempotent. Then $(re - ere)$ is nilpotent and commutative with V. i.e.

$$(vr)(vr)^k - v(vr)^k r(vr)^k = r(vr)^k v - (vr)^k r(vr)^k v \tag{2}$$

multiplying above by V on right, we get, $vr(vr)^k v = 0$. Hence $(vr)^{k+2} = (rv)^{k+2} = 0$, which implies that v commutes with both rv and vr. Hence $(rv)^2 = (vr)^2 = 0$. Suppose the above result is true for all b with $b^m = 0$, $m < t$, and $v^t = 0$, $t \geq 3$ multiplying (2) by r on left and v on right side by defining k as above, we obtain

$$(rv)^{k+2} = rv^2\alpha + \beta v^2 \tag{3}$$

α, β are elements of subring generated by r and v. Since $(v^2)^{t+1} = 0 \Rightarrow rv^2\alpha$ and βv^2 are nilpotent. Hence (3) represents rv and vr are nilpotent. Again the element v must commute with vr and rv. Therefore $(rv)^t = (vr)^t = 0$ and A is an ideal. □

Lemma 2.5. *If R be a periodic ring such that A is commutative and for every $a \in R$ and $x \in A$, $\exists$ an integer $n = n(a, x) \geq 1$ such that $[a^n, [a^n, x]] = 0$ and $[a^{n+1}, [a^{n+1}, x]] = 0$. Then R is commutative.*

Proof. By Lemma 2.2 (3), it is enough to prove that if R is sub directly irreducible, then R equals to A or R is a commutative local ring with radical A, so that we can consider that R is subdirectly irreducible and $R \neq A$. Let $a \in R/A$ be an arbitrary element. Then $\exists$ a positive integer t such that $e = a^t (\neq 0)$ is an idempotent. By hypothesis, there exists a positive integer $n = n(a)$ such that

$$(a^n, [a^n, x]] = 0 \;\; and \;\; [a^{n+1}, [a^{n+1}, x]] = 0$$

According to $[e, [e, x]] = 0 \; \forall \, x \in A$, e is central and equal to 1, which proves that a is invertible and R is a local ring with radical A. We can easily seen, R is not p^j-torsion free, where p is a prime and $\overline{a} = \overline{a} + A$ generates a sub field of $\overline{R} = R/A$. Let $x \in A$, since $((\overline{a})^n)^{\lceil p^{it} \rceil} = ((\overline{a})^n)$ and A is commutative. By (1), we can prove that $[a^n, x] = [(a^n)^{p^{it}}, x] = p^{it}(a^n)^{p^{it}}[a^n, x] \neq 0$. Also we can also prove that $a^n[a, x] = a^n[a, x] + [a^n, x]a = (a^{n+1}, x) = 0$. Since 'a' is invertible, $[a, x] = 0$, which proves that A is central. Hence by Lemma 2.2 (4) R is commutative. □

Lemma 2.6. *Let $p : R \to R'$ be an epiomolphism, R be a m-torsion free ring and n be a positive integer. If R is n-torsion free, then R' is n-torsion free.*

Proof. Let $\mathcal{D}$ be the greatest common divisor of m and $n \Rightarrow m = t_1\mathcal{D}$ and $n = t_2\mathcal{D}$ where t_1, t_2 are positive integers. If $\mathcal{D} \neq 1$, then R is not $m(\neq t_1)$-torsion free and also exist element $b \in R$ such that $t_1 b \neq 0$, now $n(t_1 b) = (t_2\mathcal{D})t_1 b = t_2 m b = 0$ it contradicts our assumption that R is m-torsion free. So $\mathcal{D} = 1$ and $(m, n) = 1$. Since $P : R \to R'$ is an epimorphism then for all $a' \in R'$, there exists an element $a \in R$ such that $a' = p(x)$. Now,

$$
\begin{aligned}
mx' &= mp(x) \\
&= p(mx) \\
&= p(0) \\
&= 0 \ \ \forall \ \ a' \in R'
\end{aligned}
$$

Hence char $R' = m'$, where m' divides m. So, $(m', n) = 1$, since $(m, n) = 1 \Rightarrow rm' + sn = 1$, where r, s are integers. If $nb' = 0$ for some $b' \in R'$. Then $b' = (rm' + sn)b' = r(m'b') + s(nb') = 0$. Hence R' is n-torsion free. □

3. Main Results

Theorem 3.1. *Let n be a positive integer and R be an $(n+1)n$-tortion free periodic such that $(ab)^n - ba \in Z(R)$ and $(ab)^{n+1} - ba \in Z(R)$. If A is commutative, then R is commutative.*

Proof. By Lemma 2.4, the set A of nilpotent elements of R is an ideal of R and since A is commutative. We know that

$$A^2 \subseteq Z(R). \tag{4}$$

Let e be an idempotent element of R and ‘a’ be any element of R. From the hypothesis,

$$
\begin{aligned}
&[e(e + ea - eae)]^n - (e + ea - eae)e \in Z(R). \\
&[e(e + ea - exe)] - (e + ex - exe)e \in Z(R).
\end{aligned}
$$

Hence $(ea - eae)e \in Z(R) \Rightarrow e(ea - eae) = (ea - eae)e \Rightarrow ea = eae$. Similarly

$$ae = eae \tag{5}$$

Thus $ea = ae$ and the idempotent of R are central. Let $a, b \in R$ be the two elements. Then by the hypothesis,

$$(ab)^n - ba = W_1 \in Z(R) \ \ \textit{and} \ \ (ba)^n - ab = W_2 \in Z(R) \tag{6}$$

Now, $(ab)^n a = a(ba)^n$ and using (4)we get,

$$
\begin{aligned}
(W_1 + ba)a &= a(ab + W_2) \\
W_1 a + ba^2 &= a^2 b + aW_2 \\
a^2 b - ba^2 &= (W_1 - W_2)a \\
\Rightarrow [a^2, [a^2, b]] &= 0 \ \ \forall \ \ a, b \in R
\end{aligned} \tag{7}
$$

Let $x \in A$, $b = x + 1$ in (7) and use the fact that to $A^2 \subseteq Z(R)$ in (4), to get that

$$[x^2, [x^2, a]] = 0 \;\; \forall \;\; a \in R, \;\; x \in A \tag{8}$$

Repeating the above process from (6) using the hypothesis, $(ab)^{n+1} - ba \in Z(R)$, we get

$$[a^3, [a^3, x]] = 0 \;\; \forall \;\; a \in R, x \in A. \tag{9}$$

Using (8), (9) and Lemma (5) we see that R must be commutative. Hence the theorem proved. ☐

Theorem 3.2. *If R is an n-torsion free periodic ring satisfying $(ab)^n - (ba)^n \in Z(R)$ and A is commutative, then R is commutative.*

Proof. First we consider that R has unity by Lemma 2.4, since A is an ideal of R and commutative, $A^2 \subseteq Z(R)$. Consider $x \in A$, $y \in R$, put $a = (1+x)y$, $b = (1+x)^{-1}$, then hypothesis $(ab)^n - (ba)^n \in Z(R)$ becomes

$$(1+x)y^n(1+x)^{-1} - y^n \in Z(R) \tag{10}$$

Hence

$$\begin{aligned} [(1+x)y^n(1+x)^{-1} - y^n](1+x) &= (1+x)[(1+x)y^n(1+x)^{-1} - b^n] \\ (1+x)y^n - y^n(1+x) &= (1+x)[(1+x)y^n(1+x)^{-1} - y^n] \\ xy^n - yx^n &= (1+x)[(1+x)y^n(1+x)^{-1} - y^n] \end{aligned} \tag{11}$$

Since A is a commutative ideal, $(1+x)(xy^n - y^n x) = (1+x)[(1+x)y^n(1+x)^{-1} - y^n]$, since $x \in A$, $(1+x)$ is a unit in R. Thus $xy^n - y^n x = (1+x)y^n(1+x)^{-1} \in Z(R)$ by (10),

$$[x, y^n] \in Z(R), \;\; (x \in N, y \in R) \tag{12}$$

Now, if $a_1, a_2, \ldots, a_k \in R$, since $R/Z(R)$ is commutative. $(a_1, a_2, \ldots, a_k)^n = (a_1^n a_2^n \ldots a_k^n) \in Z(R) \subseteq A$ by Lemma 2.3. But A is commutative, and therefore

$$[x, (a_1, a_2, \ldots, a_k)^n] = [x, (a_1^n, a_2^n, \ldots, a_k^n)] \;\; (x \in A) \tag{13}$$

Adding (12) and (13), we get

$$[x, (a_1^n, a_2^n, \ldots, a_k^n)] \in Z(R), \;\; (x \in A, a_1, a_2, \ldots, a_k \in R \;\; and \;\; K \geq 1) \tag{14}$$

Let s' be the subring of R generated by the n^{th} powers of elements of R. Then by (14),

$$[x, a] \in Z(s') \;\; \forall \;\; x \in A(s), a \in s'. \tag{15}$$

(Hence $Z(s')$ and $A(s')$ denote the centre of s' and the set of nilpotent's of s', respectively). Therefore $A(s')$ is commutative and (15), a Theorem of 2.2 shows that s' is commutative and hence

$$[a^n, b^n] = 0 \;\; \forall \;\; a, b \in R \tag{16}$$

Observe that R is an n-torsion free ring with unity satisfying (16) and $(ab)^n - (ba)^n$ is always central. Therefore R is commutative. ☐

Lemma 3.3. *Let R is s-unital ring satisfying the identities.*

(1). $[a^n, b^n] = 0 \ \forall \ a, b \in R$

(2). $n[a, b] = 0$ *implies* $[a, b] = 0$, *for all* $a, b \in R$

(3). $[a, a^n(ab) - (ba)a^n] = 0 \ \forall \ a, b \in R$ *then R is commutative.*

Theorem 3.4. *If R is an* $(n+1)n$*-torsion free periodic ring satisfying identity.* $a^n(ab) - (ba)a^n \in Z(R) \ \forall \ a, b \in R$ *and if A is commutative, then R is commutative.*

Proof. we prove the theorem, when R has unity 1. By Lemma 2.3, A is an ideal of R and commutative $A^2 \subseteq Z(R)$. Let $x \in A$ gives that x is quasi regularly and it has quasi inverse, hence $(1+x)$ has inverse in R. Now for $x, y \in A$, we consider $a = 1+x$, $b = y(1+x)^{-1}$ in the above identity, we get,

$$(1+x)^n[(1+x)y(1+x)^{-1}] - [y(1+x)^{-1}(1+x)](1+x)^n \in Z(R)$$
$$(1+x)^{n+1}y(1+x)^{-1} - y(1+x)^n \in Z(R) \tag{17}$$

In particular

$$\{(1+x)^{n+1}y(1+x)^{-1} - y(1+x)^n\{(1+x) = (1+x)\{(1+x)^{n+1}y(1+x)^{-1} - y(1+x^n)\}$$
$$(1+x)^{n+1}y - y(1+x)^{n+1} = (1+x)\{(1+x)^{n+1}y(1+x)^{-1} - y(1+x)^n\}$$

Using binomial expansion and the $N(R)^2 \subseteq Z(R)$, we get,

$$\{1+(n+1)x+\cdots+(n+1)x^n+x^{n+1}\}y-y\{1+(n+1)x+\cdots+(n+1)x^n+x^{n+1}\} = (1+x)\{(1+x)^{n+1}y(1+x)^{-1}-y(1+x)^n\}$$

Hence

$$(n+1)(xy - yx) = (1+x)\{(1+x)^{n+1}y(1+x)^{-1} - y(1+x)^n\} \tag{18}$$

since A is commutative ideal, $(x+1)(xy-yx) = xy-yx$ and hence (18) gives, $(n+1)(x+1)(xy-yx) = (1+x)\{(1+x)^{n+1}y(1+x)^{-1}-y(1+x)^n\}$, since $x \in A$, $(1+x)$ is unit in R and by (9), we get, $(n+1)(xy-yx) = \{(1+x)^{n+1}y(1+x)^{-1}y(1+x)^n\} \in Z(R)$. $(n+1)(xy-yx) \in Z(R)$, since R is $(n+1)n$-torsion free, we get,

$$[x, y] \in Z(R) \ \forall \ x \in A, y \in R \tag{19}$$

Now consider $a_1, a_2, \ldots, a_k \in R$. Since $R/Z(R)$ is commutative, $(a_1, a_2, \ldots, a_k)^n - (a_1^n, a_2^n, \ldots, a_k^n) \in Z(R) \subseteq A(R)$ by Lemma 2.3, therefore $N(R)$ is commutative yields that

$$[x, (a_1, a_2, \ldots, a_n)^n] = [x, (a_1^n, a_2^n, \ldots, a_k^n)] \ \forall \ x \in A(R) \tag{20}$$

By combining (19) and (20), we get

$$[x, (a_1^n, a_2^n, \ldots, a_k^n)] \in \ Z(R) \ \forall \ x \in A, (a_1, a_2, \ldots, a_k) \in \ R, K = 1 \tag{21}$$

let S^* be the subring generated by the n^{th} power of the elements of R then by (21), we have,

$$[x, a] \in Z(S*) \ \forall \ x \in A(S^*), a \in S^* \tag{22}$$

Here $z(S^*)$ and $A(S^*)$ denote the center of S* and set of nilpotent element of S^*. Combining the fact that $A(S^*)$ is commutative, S^* is periodic and (22), Lemma 2.4 shows that S^* is commutative. Hence $[a^n, b^n] = 0 \ \forall \ a, b \in R$, since every commutator in R is $(n+1)n$-torsion free and R satisfies the axiom

$$a^n(ab) - (ba)a^n \in Z(R) \ \forall \ a, b \in R \tag{23}$$

Equation (23) and Lemma 3.3 give that R is commutative by Lemma 3.3, we get required result. It shows that for each non zero idempotent e, eR is commutative. Hence $e[a, b] = 0 \ \forall \ a, b \in R$. Thus if $x \in R$ is potent with $x^n = x$, $n > 1$, then $x^{n-1}[x, y] = 0 = [x, y] \ \forall \ y \in R$ since every element of periodic ring equals to the sum of a potent element and nilpotent element, this implies $A \in Z(R)$ and R is commutative. □

Theorem 3.5. *If $n \geq 1$ is a fixed integer, R is $(n+1)n$-torsion free periodic ring A is commutative and $(ab)^n - b^n a^n \in Z(R) \ \forall \ a \in R/A, b \in R/A$. Then R is commutative.*

Proof. Since A is commutative and R is periodic, by Lemma 2.4, the commutator ideal of R is nil and A forms an ideal of R. Also, since $A^2 \subseteq Z(R)$ and the ideal commutative. *Let*$1 \in R$, $a \in A$, $b \in R/A$, put $a = x + 1$ in identity, then $[(x+1)y]^n - y^n(x+1)^n \in Z(R)$ and $[y(x+1)]^n - (x+1)^n y^n \in Z(R)$. By subtractions above results and using $A^2 \subseteq Z(R)$, we get, $(n+1)(x, y^n) \in Z(R)$ and since R is $(n+1)n$-torsion free

$$[x, y^n] \in Z(R) \ \forall \ x \in A, y \in R/A \tag{24}$$

since a is commutative, (24) implies

$$[x, y^n] \in Z(R) \ \forall \ x \in A, y \in R \tag{25}$$

Now consider $a_1, a_2, \ldots, a_k \in R$, since $R/Z(R)$ is commutative, $(a_1, a_2, \ldots, a_k)^n - (a_1^n{}^, a_2^n, \ldots, a_k^n) \in Z(R) \subseteq A$. But A is commutative and therefore

$$[x, (a_1, a_2, \ldots, a_k)^n] = [x, (a_1^n a_2^n \ldots a_k^n)] \ \ (x \in A) \tag{26}$$

By combining (25) and (26), we get,

$$[x, (a_1^n{}^, a_2^n, \ldots, a_k^n)] \in Z(R) \ \ (x \in A, a_1, a_2, \ldots, a_k \in R \ \forall \ K \geq 1) \tag{27}$$

Let S' be the subring of R generated by n^{th} power of elements of R, using (27),

$$[x, a] \in Z*(R) \ \forall \ x \in A^*, a \in S' \tag{28}$$

where $Z^*(R)$ and A^* denote the center of S' and the set of nilpotent of S' respectively. Combining the facts that S' is periodic, A* is commutative and (28), Lemma 2.5 gives that S' is commutative. Hence

$$[a^n, b^n] = 0 \ \forall \ a, b \in R \tag{29}$$

Further we assume that $x \in A$, $y \in R/A$. Then, by hypothesis and facts that $R/Z(R)$ is commutative and $Z(R) \in A$. We have

$$[(1+x)y]^n - y^n(1+z)^n = w \in Z(R) \cap A \tag{30}$$

$$[y(1+x)]^n - [(1+x)^n y^n] = w' \in Z(R) \cap A \tag{31}$$

Equation (30) and (31) gives

$$y[(1+x)y]^n(1+x) - y^{n+1}(1+x)^{n+1} = yw(1+x) \tag{32}$$

$$(1+x)[y(1+x)]^n y - (1+x)^{n+1}y^{n+1} = (1+x)w'y \tag{33}$$

Observe that $w \in A$ by (30) and $A^2 \in Z(R)$. Hence $ywx \in Z(R)$. Similarly, $xw'y \in Z(R) \Rightarrow w \in Z(R)$ and $w' \in Z(R)$ we see that $[yw(1+x), y] = 0$ and $[(1+x)w^1 y, y] = 0$. Hence by (32) and (33), we get

$$\{y(1+x)]^{n+1} - y^{n+1}(1+x)^{n+1}\} \text{ commutes with y} \tag{34}$$

$$\text{and } \{(1+x)y]^{n+1} - (1+x)^{n+1}y^{n+1}\} \text{ commutes with y} \tag{35}$$

According that $A^2 \in Z(R)$ and subtracting (34) from (35), we get

$$n[y^{n+1}, x] \text{ commutes with y, } \forall\ x \in A, y \in R/A \tag{36}$$

Since A is commutative, R is $(n+1)n$-torsion free and by (36) implies,

$$[x, y^{n+1}] \text{ commutes with y, for all } x \in A, y \in R \tag{37}$$

by (29), $[a^n, b^n] = 0\ \forall\ a, b \in R$, $1 \in R$ (case (i)) and R is n-torsion free. Hence

$$[x, y^n] = 0\ \forall\ x \in A, y \in R \tag{38}$$

Now, (37) implies that $xy^{n+2} - y^{n+1}xy = yxy^{n+1} - y^{n+2}x$ and by (38), we obtain $y^n xy^2 - y^{n+1}xy = y^{n+1}xy - y^{n+2}x$. Hence $y^n\{[x, y]y\} = y^n[y[x, y]]$. Thus

$$y^n[(x_1 y)y] = 0\ \forall\ x \in A, y \in R \tag{39}$$

write y by $1+y$ in (39), we get

$$(1+y)^n[[x, y], y] = 0\ \forall\ x \in N, y \in R \tag{40}$$

by (7), (39) and (40) gives that

$$[[x, y], y] = 0\ \forall\ x \in A, y \in R \tag{41}$$

Since A is commutative, R is periodic and by Lemma 2.5 and (41), R is commutative. □

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