International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **11(3)**(2023), 93–101
Available Online: http://ijmaa.in

gl-subgroups and Isomorphism Theorems of gl-groups

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Abstract

A generalised lattice ordered group (gl-group) is a partially ordered group (po-group) in which the underlying poset is a generalised lattice. This paper deals with the concept of gl-subgroup of a gl-group. Introduced the concept of gl-subgroup and proved that the quotient of a gl-group by its normal convex gl-subgroup is again a gl-group. Later, introduced the concept of gl-homomorphism and obtained the isomorphism theorems of gl-groups.

Keywords: group; subgroup; poset; lattice; homomorphism; isomorphism.
2020 Mathematics Subject Classification: 06-XX.

1. Introduction

The theory of lattice ordered groups (l-groups) is well known from the books [1, 7, 9]. The concept of generalised lattice introduced by Murty and Swamy in [8] and the theory of generalised lattices developed by the author Kishore in [2, 3, 4] that can play an intermediate role between posets and lattices. The concept of generalised lattice ordered group (gl-group) introduced and developed by the author Kishore in [5, 6]. In this paper section 2 contains some preliminaries from the references those are useful in the next sections. In section 3 introduced the concepts positive part of a finite subset of a gl-group, gl-subgroup of a gl-group, obtained an equivalent condition for a subgroup of a gl-group to be a gl-subgroup, observed that the class of all gl-subgroups (or convex gl-subgroups) is a complete lattice and finally proved that the quotient of a gl-group by its normal convex gl-subgroup is again a gl-group. In section 4 introduced the concepts gl-homomorphism, gl-isomorphism, proved that Kernel and Image of a gl-homomorphism are gl-subgroups, obtained an equivalent condition for a group homomorphism of gl-groups to be a gl-homomorphism, discussed about gl-homomorphic images and pre-images of gl-subgroups and finally obtained the isomorphism theorems of gl-groups.

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2. Preliminaries

This section contains some preliminaries of this paper which are taken from the references those are useful in the next sections.

Definition 2.1 ([8]). *Let $(P, \leq)$ be a poset. P is said to be a generalised meet semilattice if for every non empty finite subset A of P, there exist a non empty finite subset B of P such that, $x \in L(A)$ if and only if $x \leq b$ for some $b \in B$. P is said to be a generalised join semilattice if for every non empty finite subset A of P, there exist a non empty finite subset B of P such that, $x \in U(A)$ if and only if $b \leq x$ for some $b \in B$. P is said to be a generalised lattice if it is both generalised meet and join semilattice.*

It is observed that if P is a generalised meet (join) semilattice, then for any $L(A) \in \mathcal{L}(P)$ $(U(A) \in \mathcal{U}(P))$ there exists a unique finite subset B of P such that $L(A) = \bigcup_{b \in B} L(b)$ $(U(A) = \bigcup_{b \in B} U(b))$ and the elements of B are mutually incomparable and the set is denoted by $ML(A)$ $(mu(A))$.

Definition 2.2 ([2]). *Let P be a generalised lattice and S be a non-empty subset of P. Then S is said to be a subgeneralised lattice of P if for any finite subset A of S we have $ML(A) \subseteq S$ and $mu(A) \subseteq S$.*

Definition 2.3 ([2]). *Let P_1, P_2 be generalised lattices. A map $f : P_1 \to P_2$ is said to be a homomorphism (or homomorphism of generalised lattices) if $f(ML(A)) = ML(f(A))$ and $f(mu(A)) = mu(f(A))$ for any finite subset A of P_1.*

The definitions of partially ordered group (po-group) and lattice ordered group (l-group) are well known from the books [1, 7, 9]. The additive identity element of a po-group is denoted by 0. The set $G^+ = \{x \in G \mid x \geq 0\}$ is called positive cone of a po-group G.

Theorem 2.4 ([9]). *Let G be a po-group and S be a subgroup of G. The set of all left cosets of S in G, $G/S = \{x + S \mid x \in G\}$. Define a relation $\leq$ on G/S by $x + S \leq y + S$ if and only if $x \leq y + s$ for some $s \in S$. Then (i) the relation $\leq$ is reflexive and transitive (ii) $(G/S, \leq)$ is a poset if and only if S is convex.*

Theorem 2.5 ([9]). *Let G be a po-group and S be a convex normal subgroup of G. Then G/S is a po-group.*

Theorem 2.6 ([9]). *Let G, H be a po-groups and $f : G \to H$ be a group homomorphism. Then f is a po-homomorphism if and only if $f(G^+) \subseteq H^+$.*

Definition 2.7 ([5]). *A system $(G, +, \leq)$ is called a generalised lattice ordered group (gl-group) if (i) $(G, \leq)$ is a generalised lattice, (ii) $(G, +)$ is a group and (iii) every group translation $x \to a + x + b$ on G is isotone. That is $x \leq y \Rightarrow a + x + b \leq a + y + b$ for all $a, b \in G$.*

Example 2.8 ([5]). *Let $G = \{na \mid n \in \mathbb{Z}\}$ be an infinite cyclic additive group generated by an element a in G. Define a relation $\leq$ on G such that for each $n \in \mathbb{Z}$, the element na is incomparable to $(n+1)a$ and it is covered by the elements $(n+2)a$ and $(n+3)a$. Then $(G, +, \leq)$ is a gl-group.*

Definition 2.9 ([6]). *Let $(G, +, \leq)$ be a gl-group. For any $x \in G$, define the positive part, negative part and modulus of x respectively by $x^+ = mu\{x, 0\}$, $x^- = mu\{-x, 0\}$ and $|x| = mu\{x, -x\}$.*

3. gl-subgroup and Quotient gl-group

In this section introduced the concepts positive part of a finite subset of a gl-group, gl-subgroup of a gl-group, observed that the class of all gl-subgroups (or convex gl-subgroups) is a complete lattice and finally discussed under what conditions the quotient of a gl-group by a gl-subgroup is a gl-group. Throughout this section, we shall denote by G a gl-group.

Definition 3.1. *Let X be a finite subset of G. Define the positive part of X by $X^+ = mu(ML(X) \cup \{0\})$.*

Recall that any non-trivial po-group is neither bounded below nor bounded above and hence not bounded. In Definition 3.1, if G is finite then it must be trivial, that is $G = \{0\}$, and this definition coincides with the usual definition of positive cone.

Example 3.2. *Consider the gl-group as in the Example 2.8. Let $X = \{2a, 3a\}, Y = \{0, a\}$ and $Z = \{-2a, -a\}$. Then $X^+ = X$ and $Y^+ = \{0\} = Z^+$.*

Theorem 3.3. *Let X be a finite subset of G.*

(i) If $X \geq 0$ (that is $x \geq 0$ for all $x \in X$) then $ML(X^+) = ML(X)$.

(ii) If $X \leq 0$ (that is $x \leq 0$ for all $x \in X$) then $ML(X^+) = \{0\}$.

Definition 3.4. *A subgroup S of G is said to be a gl-subgroup (or sub gl-group) of G if S is a subgeneralised lattice of G.*

In the following theorem obtained an equivalent condition for a subgroup of a gl-group to be a gl-subgroup using positive parts of finite subsets of it.

Theorem 3.5. *A subgroup S of G is a gl-subgroup of G if and only if $X^+ \subseteq S$ for any finite subset X of S.*

Proof. Suppose S is a gl-subgroup of G and X is a finite subset of S. Since S is a subgroup of G, we have $0 \in S$. Since S is a subgeneralised lattice of G, we have $ML(X) \subseteq S$ and then $X^+ = mu(ML(X) \cup \{0\}) \subseteq S$. Conversely suppose $X^+ \subseteq S$ for any finite subset X of S. Let A be a finite subset of S and say $|A| = n$. To show that $mu(A) \subseteq S$. We prove this by induction on n. If $n = 1$ and $A = \{a\}$, then $mu(A) = mu(\{a\}) = \{a\} \subseteq S$. Suppose $n > 1$ and assume that the result is true for $n = k$. Now we prove the result for $n = k + 1$. Say $A = \{a_1, a_2, \cdots, a_k, a_{k+1}\} = B \cup \{a_{k+1}\}$ where $B = \{a_1, a_2, \cdots, a_k\}$. Since B is a finite subset of S and $|B| = k$, by induction hypothesis we have $mu(B) \subseteq S$. By the given condition and the subgroupness of S, we have $mu(A) = (mu(B) - a_{k+1})^+ + a_{k+1} \subseteq S$. Thus $mu(A) \subseteq S$ for any finite subset A of S. Similarly we can prove $ML(A) \subseteq S$ for any finite subset A of S. Therefore S is a subgeneralised lattice of G and hence S is a gl-subgroup of G. □

The intersection of any family of gl-subgroups of G is again a gl-subgroup of G. But the union of gl-subgroups of G need not be a gl-subgroup of G.

Example 3.6. *Consider the gl-group as in the Example 2.8. Let* $S = \{2na \mid n \in \mathbb{Z}\}, T = \{4na \mid n \in \mathbb{Z}\}$ *and* $U = \{6na \mid n \in \mathbb{Z}\}$. *Then* S, T *and* U *are gl-subgroups of* G. *But* $T \cup U$ *is not a gl-subgroup of* G.

Definition 3.7. *A gl-subgroup* S *of* G *is said to be convex gl-subgroup of* G *if* S *is a convex subset of* G.

In the Example 3.6, the gl-subgoups S, T and U are not convex. Clearly $\{0\}$ and G are convex gl-subgroups of G. The intersection of any two convex gl-subgroups of G is again a convex gl-subgroup of G. But in general the union of two convex gl-subgroups of G need not be a convex gl-subgroup of G.

Definition 3.8. *Let* X *be a subset of* G. *Then the intersection of all gl-subgroups of* G *that contain* X *is called the gl-subgroup of* G *generated by* X *and it is denoted by* $< X >$. *Also the intersection of all convex gl-subgroups of* G *that contain* X *is called the convex gl-subgroup of* G *generated by* X *and it is denoted by* $C(X)$.

Note 3.9. *The set of all gl-subgroups of* G *is denoted by* $\mathcal{S}(G)$ *and the set of all convex gl-subgroups of* G *is denoted by* $\mathcal{C}(G)$.

Theorem 3.10.

(i) $\mathcal{S}(G)$ *is a complete lattice, in which* $Inf\{A, B\} = A \cap B$ *and* $Sup\{A, B\} =< A \cup B >$ *for any* $A, B \in \mathcal{S}(G)$.

(ii) $\mathcal{C}(G)$ *is a complete lattice, in which* $Inf\{A, B\} = A \cap B$ *and* $Sup\{A, B\} = C(A \cup B)$ *for any* $A, B \in \mathcal{C}(G)$.

In Theorem 2.4, we have the quotient $(G/S, \leq)$ is a poset under some conditions. Now in the following theorem obtained that $(G/S, \leq)$ is a generalised lattice under some conditions.

Theorem 3.11. *Let* S *be a convex subgroup of* G. *If* S *is a gl-subgroup of* G, *then* G/S *is a generalised lattice and the map* $\phi : G \to G/S$ *defined by* $\phi(a) = a + S$ *is a homomorphism of generalised lattices.*

Proof. By Theorem 2.4, we have $(G/S, \leq)$ is a poset. To show that $(G/S, \leq)$ is a generalised lattice. Let $x + S, y + S \in G/S$ where $x, y \in G$. Since $t + S \in U(\{x + S, y + S\})$ for any $t \in mu(\{x, y\})$, we have $\bigcup_{t \in mu\{x,y\}} U(t + S) \subseteq U(\{x + S, y + S\})$. Let $p + S \in U(\{x + S, y + S\})$. Then there exists $s_1, s_2 \in S$ such that $x \leq p + s_1$ and $y \leq p + s_2$. Since S is a subgeneralised lattice of G, we have $mu(\{s_1, s_2\}) \subseteq S$. Thus for any $s \in mu(\{s_1, s_2\})$, we get $x, y \leq p + s$ and that is $p + s \in U(\{x, y\})$. Then there exists $t_1 \in mu(\{x, y\})$ such that $t_1 \leq p + s$. This implies $t_1 + S \leq p + S$ and that is $p + S \in U(t_1 + S)$. Thus $p + S \in \bigcup_{t \in mu(\{x,y\})} U(t + S)$ and then $U(\{x + S, y + S\}) \subseteq \bigcup_{t \in mu\{x,y\}} U(t + S)$. Therefore $U(\{x + S, y + S\}) = \bigcup_{t \in mu\{x,y\}} U(t + S)$. Thus $mu(\{x + S, y + S\}) = \{t + S \mid t \in mu(\{x, y\})\}$. Similarly we can prove $ML(\{x + S, y + S\}) = \{t + S \mid t \in ML(\{x, y\})\}$. In the same manner we can prove $mu(A + S) = mu(\{a + S \mid a \in A\}) = \{t + S \mid t \in mu(A)\} = mu(A) + S$ and $ML(A + S) = ML(A) + S$ for any finite subset A of G. Hence $(G/S, \leq)$ is a generalised lattice. Now to show that ϕ is a homomorphism of generalised lattices. Let A be a finite subset of G. Consider $\phi(mu(A)) =$

$\{\phi(t) \mid t \in mu(A)\} = \{t + S \mid t \in mu(A)\} = mu(A) + S = mu(A + S) = mu(\{a + S \mid a \in A\}) = mu(\{\phi(a) \mid a \in A\}) = mu(\phi(A))$. Similarly we can prove $\phi(ML(A)) = ML(\phi(A))$. Therefore ψ is a homomorphism of generalised lattices. □

In the following theorem observed about that the converse of Theorem 3.11.

Theorem 3.12. *Let S be a convex subgroup of G. If the poset $(G/S, \leq)$ is a generalised lattice and the map $\phi : G \to G/S$ defined by $\phi(a) = a + S$ is a homomorphism of generalised lattices then S is a gl-subgroup of G.*

Proof. Let X be a finite subset of S. Then clearly X is a finite subset of G and $\phi(ML(X)) = ML(\phi(X))$. This implies $\{t + S \mid t \in ML(X)\} = \phi(ML(X)) = ML(\phi(X)) = ML(\{x + S \mid x \in X\}) = ML(\{S\}) = \{S\}$. Thus $t + S = S$ for all $t \in ML(X)$ and therefore $ML(X) \subseteq S$. Since S is a subgroup of G, we have $ML(X) \cup \{0\} \subseteq S$. Now consider $\{a + S \mid a \in mu(ML(X) \cup \{0\})\} = \phi(mu(ML(X) \cup \{0\})) = mu(\phi(ML(X) \cup \{0\})) = mu(\{a + S \mid a \in ML(X) \cup \{0\}) = mu(\{S\}) = \{S\}$. Then $a + S = S$ for all $a \in mu(ML(X) \cup \{0\})$. Thus $X^+ = mu(ML(X) \cup \{0\}) \subseteq S$. Therefore by theorem 3.5, we have S is a gl-subgroup of G. □

In Theorem 2.5, we have the quotient G/S is a po-group under some conditions. Now in the following theorem obtained that G/S is a gl-group under some conditions.

Theorem 3.13. *Let S be a normal convex gl-subgroup of G. Then G/S is a gl-group.*

Proof. By Theorem 2.5, we have the quotient G/S is a po-group and by Theorem 3.11, we have it is a generalised lattice. Therefore G/S is a gl-group. □

4. Homomorphism and Isomorphism Theorems

In this section introduced the concept of gl-homomorphism, discussed about gl-homomorphic images and pre-images of gl-subgroups and finally obtained the isomorphism theorems of gl-groups.

Definition 4.1. *Let G, H be gl-groups. A group homomorphism $f : G \to H$ is said to be a gl-homomorphism if f is a homomorphism of generalised lattices.*

Definition 4.2. *Let G, H be gl-groups. A map $f : G \to H$ is said to be a gl-isomorphism of G onto H if f is a bijection and homomorphism of generalised lattices. G, H are said to be isomorphic denoted by $G \cong H$ if there is a gl-isomorphism of G onto H.*

Theorem 4.3. *Let G, H be gl-groups and $f : G \to H$ be a gl-homomorphism. Then*

(i) $Kerf = \{x \in G \mid f(x) = 0\}$ is a normal convex gl-subgroup of G and

(ii) $Imf = f(G) = \{f(x) \mid x \in G\}$ is a gl-subgroup of H.

In the following theorem discussed about gl-homomorphic images and pre-images of gl-subgroups.

Theorem 4.4. *Let G, H be gl-groups and $f : G \to H$ be a gl-homomorphism. Then*

(i) S is a gl-subgroup of G implies $f(S) = \{f(x) \mid x \in S\}$ is a gl-subgroup of H.

(ii) T is a gl-subgroup of H implies $f^{-1}(T) = \{x \in G \mid f(x) \in T\}$ is a gl-subgroup of G.

(iii) T is a convex gl-subgroup of H implies $f^{-1}(T) = \{x \in G \mid f(x) \in T\}$ is a convex gl-subgroup of G.

(iv) f is bijection and S is a convex gl-subgroup of G implies $f(S) = \{f(x) \mid x \in S\}$ is a convex gl-subgroup of H.

In the following theorem obtained an equivalent condition for a group homomorphism of gl-groups to be a gl-homomorphism using positive parts of finite subsets of it.

Theorem 4.5. *Let G, H be gl-groups and $f : G \to H$ be a group homomorphism. Then f is a gl-homomorphism if and only if $f(X^+) = (f(X))^+$ for any finite subset X of G.*

Proof. Suppose f is a gl-homomorphism. Let X be a finite subset of G. Consider $f(X^+) = f(mu(ML(X) \cup \{0\})) = mu(f(ML(X) \cup \{0\})) = mu(f(ML(X)) \cup \{f(0)\}) = mu(ML(f(X)) \cup \{f(0)\}) = (f(X))^+$. Conversely suppose $f(X^+) = (f(X))^+$ for any finite subset X of G. To show that f is a homomorphism of generalised lattices. Let A be a finite subset of G and say $|A| = n$. To show that $mu(f(A)) = f(mu(A))$. We prove this by induction on n. If $n = 1$ and say $A = \{a\}$, then $f(mu(A)) = f(mu(\{a\})) = f(a) = mu(\{f(a)\}) = mu(f(A))$. Therefore the result is true for $n = 1$. Suppose $n > 1$ and assume that the result is true for $n = k$. Now we prove that the result is true for $n = k + 1$. Say $A = \{a_1, a_2, \cdots, a_k, a_{k+1}\} = B \cup \{a_{k+1}\}$, where, $B = \{a_1, a_2, \cdots, a_k\}$. Since B is a finite subset of S and $|B| = k$, by induction hypothesis we have $f(mu(B)) = mu(f(B))$. Consider

$$\begin{aligned}
f(mu(A)) &= f((mu(B) - a_{k+1})^+ + a_{k+1}) \\
&= f((mu(B) - a_{k+1})^+) + f(a_{k+1}) \\
&= (f(mu(B) - a_{k+1}))^+ + f(a_{k+1}) \\
&= mu(ML(f(mu(B))) \cup \{f(a_{k+1})\}) \\
&= mu(ML(mu(f(B))) \cup \{f(a_{k+1})\}) \\
&= mu(f(B) \cup \{f(a_{k+1})\}) \\
&= mu(f(B \cup \{a_{k+1}\})) \\
&= mu(f(A))
\end{aligned}$$

Therefore $mu(f(A)) = f(mu(A))$ for any finite subset A of G. Similarly we can prove $ML(f(A)) = f(ML(A))$ for any finite subset A of G. Thus f is a homomorphism of generalised lattices. Hence f is a gl-homomorphism. □

In the following we prove first isomorphism theorem, correspondence theorem, second isomorphism theorem and third isomorphism theorem of gl-groups.

Theorem 4.6 (First isomorphism theorem). *Let G, H be gl-groups and $f : G \to H$ be a gl-homomorphism. Then $G/Kerf \cong f(G)$.*

Proof. Let $N = Kerf$. By Theorem 4.3, we have N is a normal convex gl-subgroup of G and $f(G)$ is a gl-subgroup of H. Then by theorem 3.13 we have G/N is a gl-group. Define a map $\psi : G/N \to f(G)$ by $\psi(x + N) = f(x)$. Clearly ψ is a group homomorphism and bijection. To show that ψ is a gl-homomorphism. Let T be a finite subset of G/N and say $T = X + N = \{x + N \mid x \in X\}$ where X is a finite subset of G. Consider $T^+ = (X + N)^+ = mu(ML(X + N) \cup \{N\}) = mu((ML(X) + N) \cup (0 + N)) = mu((ML(X) \cup \{0\}) + N) = mu(ML(X) \cup \{0\}) + N = X^+ + N$. Then $\psi(T^+) = f(X^+ + N) = f(X^+) = (f(X))^+ = (\psi(X + N))^+ = (\psi(T))^+$. Thus by theorem 4.5 ψ is a gl-homomorphism. Therefore ψ is a gl-isomorphism of G/N onto $f(G)$. Hence $G/N \cong f(G)$. □

Theorem 4.7 (Correspondence theorem). *Let G be a gl-group and N be a normal convex gl-subgroup of G. Let $\mathcal{A}$ be the set of all convex gl-subgroups of G that contain N and $\mathcal{B} = \mathcal{C}(G/N)$. Define a map $\psi : \mathcal{A} \to \mathcal{B}$ by $\psi(A) = A/N$. Then ψ is a lattice isomorphism of $\mathcal{A}$ onto $\mathcal{B}$.*

Proof. By Theorem 3.10, we have $\mathcal{A}$ is a sublattice of $\mathcal{C}(G)$ and $\mathcal{C}(G/N)$ is a lattice. Clearly ψ is well defined and one-one map. To show that ψ is a lattice homomorphism. Let $A_1, A_2 \in \mathcal{A}$. Then $\psi(A_1 \wedge A_2) = \psi(A_1 \cap A_2) = (A_1 \cap A_2)/N = (A_1/N) \cap (A_2/N) = \psi(A_1) \wedge \psi(A_2)$ and $\psi(A_1 \vee A_2) = \psi(C(A_1 \cup A_2)) = C(A_1 \cup A_2)/N = C((A_1/N) \cup (A_2/N)) = \psi(A_1) \vee \psi(A_2)$. Therefore ψ is a lattice homomorphism. To show that ψ is onto map. Let $B \in \mathcal{B} = \mathcal{C}(G/N)$. Since B is a subgroup of G/N, there exists a subgroup A of G that contain N such that $B = A/N$. Since A/N is a convex subset of G/N, we have A is a convex subset of G. Let X be a finite subset of A. Then X/N is a finite subset of $A/N = B$. Since B is a subgeneralised lattice of G/N, we have $ML(X)/N = ML(X/N) \subseteq A/N$. Then $ML(X) \subseteq A$. Similarly we can prove $mu(X) \subseteq A$. Thus A is a subgeneralised lattice of G. Then A is a convex gl-subgroup of G that contain N, that is $A \in \mathcal{A}$ and $\psi(A) = A/N = B$. Therefore ψ is an onto map. Hence ψ is a lattice isomorphism of $\mathcal{A}$ onto $\mathcal{B}$. □

Theorem 4.8 (Second isomorphism theorem). *Let G be a gl-group. Let A be a convex gl-subgroup of G and N be a normal convex gl-subgroup of G. Then*

(i) $A + N = \{a + n \mid a \in A, n \in N\}$ *is a gl-subgroup of G*

(ii) $A \cap N$ *is a normal convex gl-subgroup of A*

(iii) $A + N/N \cong A/A \cap N$.

Proof.

(i) Since A, N are subgroups of G, clearly $A+N = \{a+n \mid a \in A, n \in N\}$. To show that $A+N$ is a gl-subgroup of G. Let Z be a finite subset of $A+N$. Then write $Z = X+M$ for some finite subset X of A and a finite subset M of N. Observe that $Z^+ + N = X^+ + N$ and then $Z^+ \subseteq A+N$. Therefore by theorem 3.5 we have $A+N$ is a gl-subgroup of G.

(ii) Clearly can prove.

(iii) By (i) and Theorem 3.13, we have $(A+N)/N$ is a gl-group. Define a map $\psi : A \to (A+N)/N$ by $\psi(a) = a+N$. Then clearly ψ is a group homomorphism and onto map. Let X be a finite subset of A. Consider $ML(\psi(X)) = ML(X+N) = ML(X)+N = \psi(ML(X))$. Similarly we can prove $mu(\psi(X)) = \psi(mu(X))$. Then ψ is a homomorphism of generalised lattices. Thus ψ is a gl-homomorphism and onto map. Now $Ker\psi = \{a \in A \mid \psi(a) = N\} = \{a \in A \mid a+N = N\} = \{a \in A \mid a \in N\} = A \cap N$. Therefore by first isomorphism theorem we get $A/A\cap N \cong A+N/N$.

□

Theorem 4.9 (Third isomorphism theorem)**.** *Let G be a gl-group and N, K be normal convex gl-subgroups of G with $N \subseteq K$. Then $(G/N)/(K/N) \cong G/K$.*

Proof. By Theorem 3.13, we have G/N and G/K are gl-groups. Since K be normal convex gl-subgroup of G containing N, by theorem 4.7 we have K/N is a normal convex gl-subgroup of G/N. Define a map $\psi : G/N \to G/K$ by $\psi(a+N) = a+K$. Clearly ψ is a group homomorphism and onto map. Let Z be a finite subset of G/N and write $Z = A/N$ for some A is a finite subset of G. Consider $\psi(ML(Z)) = \psi(ML(A)/N) = ML(A)+K = ML(A+K) = ML(A/K) = ML(\psi(A/N)) = ML(\psi(Z))$. Similarly we can prove $\psi(mu(Z)) = mu(\psi(Z))$. Then ψ is a homomorphism of generalised lattices. Thus ψ is a gl-homomorphism. Now $Ker\psi = \{x+N \in G/N \mid \psi(x+N) = K\} = \{x+N \in G/N \mid x+K = K\} = \{x+N \in G/N \mid x \in K\} = K/N$. Therefore by theorem 4.6 we have $(G/N)/(K/N) \cong G/K$. □

Acknowledgment

The author would like to thank (late) Prof. Mellacheruvu Krishna Murty, Professor, Department of Mathematics, Andhra University, Visakhaptnam, India (and M.Phil & Ph.D - Research Director of the author Parimi Radhakrishna Kishore) for their valuable suggestions during the preparation of the paper.

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