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# Population of Fish: Mathematical Modeling of the Optimal Fishing Rate

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**Abstract:** We consider a discrete mathematical model of the population size of fish in an isolated environment. This model includes the effect of fishing at a constant rate. We analyze the model and obtain the optimal fishing rate. This rate is optimal in the sense that the population maintains a certain size at this rate, but any larger fishing rate leads to the extinction of the fish population.

**Keywords:** Population of Fish, Mathematical Modeling, Optimal Fishing Rate.


## 1. Introduction

By a population we refer to a group of living organisms in a certain environment. For example, we could be referring to a population of bears in a certain geographic region (like a forest), bacteria (maybe in an experiment), humans (in a city or country), etc. The use of mathematical models to predict the how the size (number of individuals) of the population changes as a function of time is common practice and it is an important field of mathematical biology [1–6].

In this article we consider a population of fish. This population is in a certain environment like a lake or the coast of a region. We assume that the only factors that affect the change in population size is the current size of the population and the harvesting rate. The model we study is discrete in time.

This article is organized as follows. The model is explained in Section 2. We analyze the model by means of numerical simulations in Section 3. In Section 4, we study the model analytically. We summarize our findings and conclusions in Section 5.

## 2. Mathematical Model

We denote by $x_n$ the population size at time $n$. For concreteness we assume that the time is measured in years. Thus, $x_n$ is the population size at year $n$, where year 0 was arbitrarily chosen. Every year we want to harvest the same number of fish, we call this number $h$. In other words, we want to removed $h$ fish from the environment for consumption. We, in fact, harvest $h$ fish per year if there are at least $h$ fish. If there are less than $h$ fish, we harvest all of them and the population extinguishes. We assume that no other factor affects the size of the population. If the population is $x$, we denote by $f(x)$

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be the population one year in the absence of harvesting. The discussion in the paragraph leads to the model

$$x_{n+1} = \begin{cases} f(x_n) - h & \text{if } f(x_n) - h > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

In the absence of harvesting, we assume that, if $x$ is small, the population grows by a factor $R > 1$, where $R$ is a parameter that depends on the fish and environment conditions and could, in principle, be measured experimentally. Note that we require $R > 1$ because, if $R < 1$, the population will decrease with time and go to zero even without harvesting.

On the other hand, as $x$ increases, we assume that there is competition among the fish for natural resources, and the growth after a year (in the absence of harvesting) is no longer proportional to $R$. In fact, we assume that the population size has a limiting capacity $K$, where $K$ is also a parameter of the system, that could also be in principle measured experimentally. In other words, the population is never greater than $K$.

A third natural assumption is that $f(x)$ is an increasing function of $x$. The simplest function that satisfies all these three conditions is

$$f(x) = \frac{Rx}{1 + \frac{R}{K}x}. \tag{2}$$

A plot of the graph of the function $f(x)/K$ as a function of $x/K$ in the case when $R = 2$ is displayed in Figure 1. The horizontal axis is $x/K$.

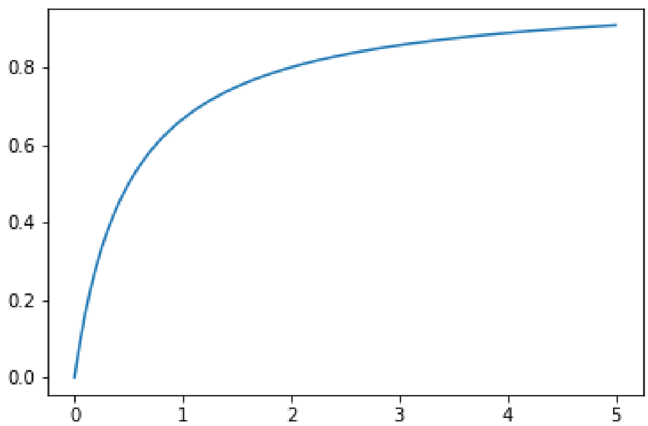


**Figure 1.** **Plot of $f(x)/K$ as a function of $x/K$ (see Equation (2)).**

Our task, is to understand the population dynamics so that we can select $h$ as large as possible leading to a sustainable situation. In other words, we do not want the harvesting to be so large that it extinguishes the population.

# 3. No Fishing, $h = 0$

In this section, we assume that there is no fishing. In other words, $h = 0$ in Equation (1). Figure 2 shows the results of numerical simulations with different initial conditions, i.e. different values of $x_0$, in the case $R = 2$.

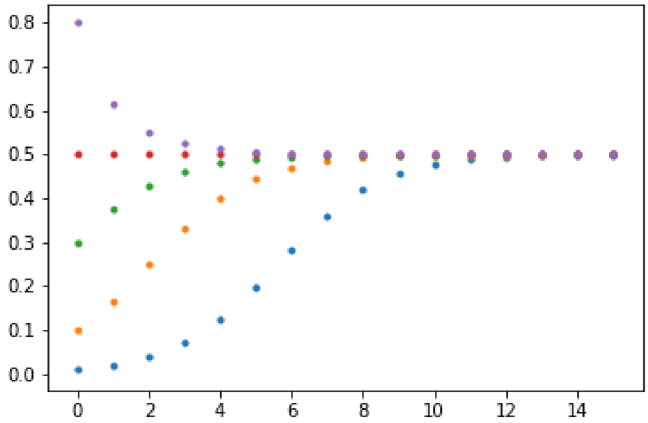


**Figure 2.** **Plot of $x_n/K$ as a function of $n$ with $h = 0$.**

We observe, that no matter the initial condition, the population size approaches $K/2$ as time $t = n$ increases. In mathematical language, we say that the limit of $x_n$ as $n$ tends to infinity is $K/2$, and we write

$$\lim_{n\to\infty} x_n = \frac{K}{2}. \tag{3}$$

In Figure 3, we plot the graph $y = f(x)$ and the line $y = x$. We also display segments connecting points that are labeled. We start with the point $(x_0, x_0)$. This point is on the line $y = x$. We draw a vertical segment starting from the point $(x_0, x_0)$ and ending in the graph $y = f(x)$. Since the segment is vertical, the value of $x$ does not change and thus, the end of this segment that is in the graph of $y = f(x)$ is the point $(x_0, f(x_0)) = (x_0, x_1)$ (because $x_1 = f(x_0)$), as indicated in the figure. Next, we draw a horizontal segment from the point $(x_0, x_1)$ to the line $y = x$. Since this segment is horizontal, the value of $y$ remains the same thought the segment, and this value is $x_1$, because one of the end points of the segment is $(x_0, x_1)$. Thus, the end point of the segment that is in the line $y = x$ is $(x_1, x_1)$. This procedure of drawing a sequence of segments, starting with a vertical one from the line $y = x$ to the curve $y = f(x)$, then a horizontal one from the end point of the previous segment that is in the curve $y = f(x)$ to the line $y = x$, then a vertical one again from the end point of the previous segment that is in the line $y = x$ to the curve $y = f(x)$, and so on, is shown in Figure 3. This sequence of segments clearly shows that

$$\lim_{n\to\infty} x_n = x_\star, \tag{4}$$

where $(x_\star, x_\star)$ is the point that is both in the curve $y = f(x)$ and the line $y = x$. Thus $x_\star$ can be found by solving the equation

$$x_\star = f(x_\star). \tag{5}$$

This equation can be easily solve to get

$$x_\star = \frac{(R-1)}{R} K. \tag{6}$$

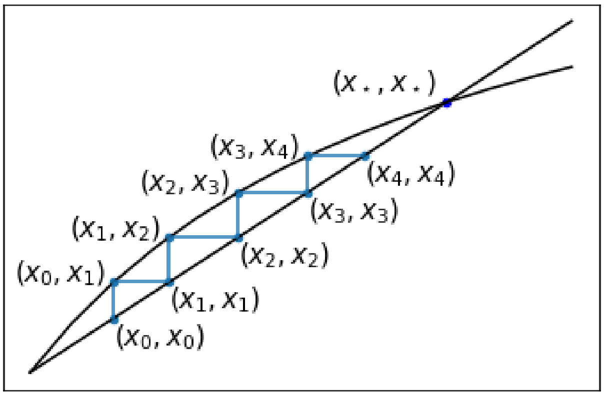


**Figure 3.** **Plot of the curve $y = f(x)$, plot of the line $y = x$, and visualization of the evolution of the sequence $x_n$.**

Note that, in Figure 3, the initial condition $x_0$ was smaller than $x_\star$. Nevertheless, the conclusion that $\lim_{n\to\infty} x_n = x_\star$ remains valid even if $x_0 > x_\star$. This is illustrated in Figure 4.

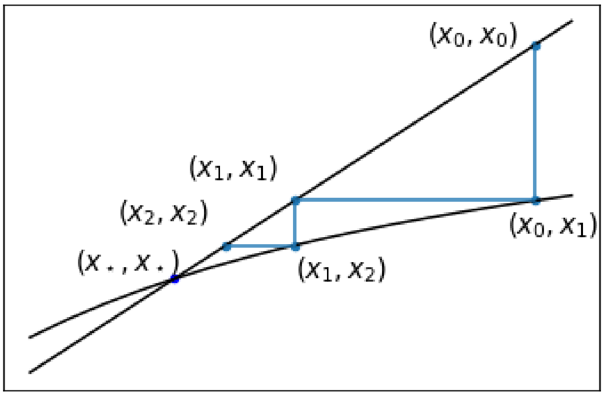


**Figure 4.** **Similar to Figure 4, but with $x_0 > x_\star$.**

# 4. Fishing $h > 0$ Fish per Unit Time

Assume now that $h > 0$, i.e. $h$ fish per year are captured and removed from the population (unless there are less than $h$ fish, in which case all fish are removed from the population and the population is extinguished). Assume that the initial size of the population is $x_0 = (R-1)K/R$. This corresponds to having had many years without fishing, so that the population size became very close to a $(R-1)K/R$, as we have shown in the last section, and then, at $t = 0$, we start harvesting $h$ fish per year. In Figure 5 we show the result of numerical simulations for several values of $h$. The values of $h$ used were: 0.05 (red dots), 0.08 (green dots), 0.085 (orange dots), 0.086 (blue dots). The larger the value of $h$, the smaller the value of $x_n$. Note that there is a value $\bar{h} > 0$ and a value $\bar{x} > 0$ such that $x_n > \bar{x}$ for all $n$ if $h \le \bar{h}$. On the other hand, if $h > \bar{h}$, $x_n$ eventually becomes 0 (the population becomes extinct).

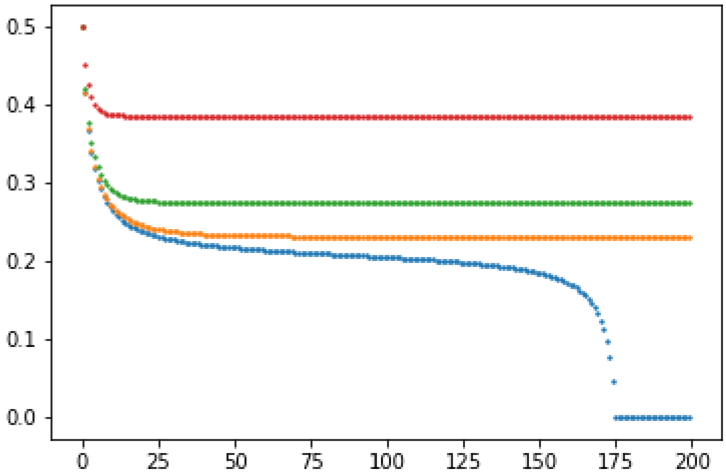


**Figure 5.** **Plot of $x_n/K$ as a function of $n$ for several values of $h$.**

Figures 6 and 7 are analogous to Figures 3 and 4, but with the values $h = 0.6K$ (Figure 6) and $h = 0.13K$ (Figure 7) instead of $h = 0$ (Figures 3 and 4). Note that, when $h = 0.06K$, the curve $y = f(x)$ intersects the line $y = x$ at two positive values of $x$ and $\lim_{n\to\infty} x_n = x_\star$, where $x_\star$ is the largest of these two values. As in the case of the last section where $h = 0$, finding $x_\star$ (with $h > 0$ now) is simple, we have to find the largest solution $x = f(x)$. This solution can be explicitly found. More precisely, $x = f(x)$ if and only if $x = Rx/(1 + Rx/K) - h$. Multiplying both sides of the last equation by $1 + Rx/K$ we get that $x = f(x)$ if and only if $x(1 + Rx/K) = Rx - h(1 + Rx/K)$. After manipulation, this equation becomes

$$x^2 + \left(\frac{K}{R} - K + h\right)x + \frac{Kh}{R} = 0 \tag{7}$$

This is a quadratic equation that sometimes has two positive solutions, one positive solution or no solutions. More precisely, simple algebraic manipulation leads to

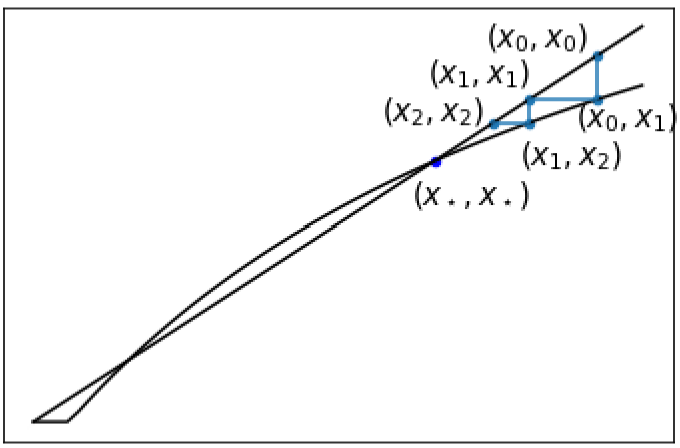


Figure 6. Plot of the curve $y = f(x) - h$, plot of the line $y = x$, and visualization of the evolution of the sequence $x_n$ in the case $h = 0.06$.

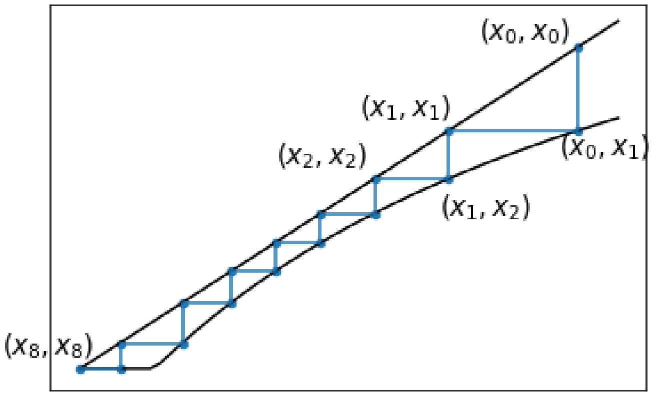


Figure 7. Plot of the curve $y = f(x)$, plot of the line $y = x$, and visualization of the evolution of the sequence $x_n$ in the case $h = 0.13$.

**Observation 4.1.** *Let* $\bar{h} = \left(1 - \frac{1}{\sqrt{R}}\right)^2 K$. *Then*

1. *If* $h < \bar{h}$, $x = f(x)$ *has two positive solutions, and the largest of both is (see Figure 6)*

$$x_\star = \frac{K}{2}\left(1 - \frac{h}{K} - \frac{1}{R} + \sqrt{\left(1 - \frac{h}{K} - \frac{1}{R}\right)^2 - \frac{4h}{KR}}\right). \tag{8}$$

2. *If* $h = \bar{h}$, $x = f(x)$ *has one positive solutions, and that solution is (see Figure 8).*

$$x_\star = \frac{K}{2}\left(1 - \frac{h}{K} - \frac{1}{R}\right) \tag{9}$$

3. *If* $h > \bar{h}$, $x = f(x)$ *has no solutions (see Figure 7).*

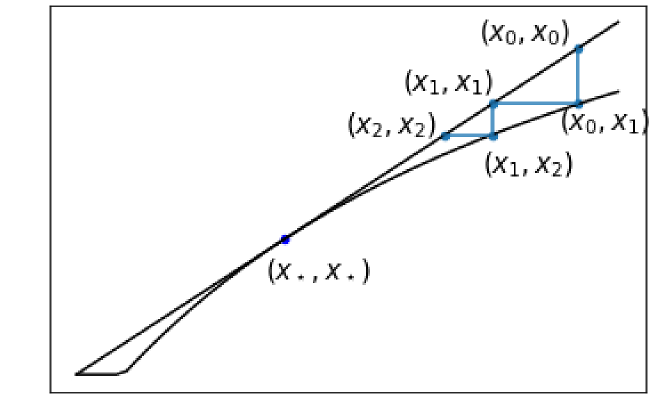


Figure 8. Plot of the curve $y = f(x)$, plot of the line $y = x$, and visualization of the evolution of the sequence $x_n$ in the case $h = \bar{h}$.

The last observation and Figures 6, 7 and 8 make the dynamics of our system clear. Our findings are summarized in the following observation

**Observation 4.2.** *Let* $\bar{h} = \left(1 - \frac{1}{\sqrt{R}}\right)^2 K$*. Then*

1. *If* $h < \bar{h}$*,*

$$\lim_{n\to\infty} x_n = x_\star = \frac{K}{2}\left(1 - \frac{h}{K} - \frac{1}{R} + \sqrt{\left(1 - \frac{h}{K} - \frac{1}{R}\right)^2 - \frac{4h}{KR}}\right). \tag{10}$$

2. *If* $h = \bar{h}$*,*

$$\lim_{n\to\infty} x_n = \frac{K}{2}\left(1 - \frac{h}{K} - \frac{1}{R}\right) \tag{11}$$

3. *If* $h > \bar{h}$*,*

$$\lim_{n\to\infty} x_n = 0. \tag{12}$$

We can now answer our original question, the optimal fishing rate $h$, i.e. the largest rate $h$ at which we can fish every year, with the population size remaining always larger than $h$ so there are enough fish to fish is

$$h = \bar{h} = \left(1 - \frac{1}{\sqrt{R}}\right)^2 K \tag{13}$$

# 5. Discussion

In this short article, we illustrate techniques common in the modeling of the dynamics of populations, and in the analysis and simulations of first order autonomous dynamical systems. We also illustrate how this type of simple analysis can lead to guidelines of policies, in the particular, the maximum number of fish allowed to be harvested per year to not extinguish the fish population.

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