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Comparison in Between Differential Transformation Method and Power Series Method for 13^{th} Order Differential Equation with Boundary Value Problems

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Abstract: Differential transformation method is proposed to discover the solution of the higher order differential equations with the boundary value problem. The estimated solution of the problem is represented in the form of a rapid convergent series. A numerical example has been considered to demonstrate the effectiveness, exactness and implementation of the method and the results are compared with the exact solution. Afterward in the form of results are revealed graphically. The numerical result find by DTM are compared with the solution which are find by Power series Method and other presented method for instance the Variational Iteration Method (VIM) presented in this paper.

Keywords: Higher order Differential Equations, Differential Transformation Method, Power series Method, Taylor Series, Recurrence Relation and Variational Iteration Method.

1. Introduction

The standard Taylor series is one of the earliest analytic ways to solve many problems, like ordinary differential equations, Partial differential equations, Linear Ordinary Differential Equations of higher order and integral equations. However, since it requires a lot of calculation for the derivatives of functions. It takes a lot of computational time to higher order derivatives. We have introduce Modification in Taylor series method which is called the differential transform method (DTM) in this method to discover the coefficients of the Taylor series of the given function by solving the induced recursive equation from the given differential equation. Since proposed in (Zhou, 1986), there are excellent curiosity on the applications of DTM to solve various scientific problems. The DTM is an approximation to exact solution of the functions which are differentiable in the form of polynomials. This method is dissimilar with the usual higher order Taylor series since the series needs extra time in calculation and requires the computation of the essential derivatives. The DTM is an alternative method for receiving Taylor series solution of the differential equations. This method reduces the size of computational domain and is effortlessly relevant to various problems. Enlarge list of methods, exact, approximate and purely numerical are offered for the solution of differential equations. Nearly all of these methods are computationally exhaustive, because they are trial-and error in nature, or require complicated symbolic computations. The differential transformation technique is one of the numerical methods for ordinary differential equations. This method establishes a semi-analytical numerical technique that uses Taylor

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series for the solution of differential equations in the form of a polynomial. Which is dissimilar from the high-order Taylor series method which needs symbolic computation of the essential derivatives of the data functions? The Taylor series method is computationally lengthy mainly for high order equations. The differential transform is an iterative procedure for obtaining analytic Taylor series solutions of differential equations. The main benefit of this method is that it can be applied directly to nonlinear ODEs with no requiring linearization, perturbation. This method will not utilize too much computer time when applying to nonlinear or parameter varying systems. This method gives an analytical solution in the form of a polynomial. But, it is different from Taylor series method that requires computation of the high order derivatives. The differential transform method is an iterative procedure that is described by the transformed equations of original functions for solution of differential equations. Chen and Liu have applied this method to solve two-boundary-value problems [6]. Jang, Chen and Liu apply the two-dimensional differential transform method to solve partial differential equations. Yu and Chen apply the differential transformation method to the optimization of the rectangular fins with variable thermal parameters [7]. Contrasting the traditional high order Taylor series method which requires a lot of symbolic computations, the differential transform method is an iterative procedure for obtaining Taylor series solutions. The power series method, developed in [11]. The analytical results of the boundary value problems have been obtained in terms of a convergent series with easily computable components.

2. Differential Transformation Method

In this section, we introduce the differential transform method used in this paper to obtain approximate analytical solutions for the linear ordinary order differential equations. This method has been developed in [1] as follows:
The differential transformation of the k^{th} derivative of a function $f(x)$ is defined by

$$F(k) = \frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0} \tag{1}$$

$$f(x) = \sum_{k=0}^{\infty} F(k)(x - x_0)^k \tag{2}$$

and the inverse differential transformation of F (k) is defined by

$$y(t) = \sum_{k=0}^{\infty} F(k)t^k \tag{3}$$

The combination results of above equations are called app. solution of the functions y (t).

3. Properties of Differential Transform Method (DTM)

Serial No's.	The operation Properties of Differential Transform Method (DTM)	
1	$z(t) = u(t) \pm v(t)$	$Z(k) = U(k) \pm V(k)$
2	$z(t) = \alpha u(t)$	$Z(k) = \alpha U(k)$
3	$z(t) = \frac{du(t)}{dt}$	$Z(k) = (k+1)U(k+1)$
4	$z(t) = \frac{d^2u(t)}{dt^2}$	$Z(k) = (k+1)(k+2)U(k+2)$
5	$z(t) = \frac{d^m u(t)}{dt^m}$	$Z(k) = (k+1)(k+2).....(k+m)U(k+m)$
6	$z(t) = u(t)v(t)$	$Z(k) = \sum_{n=0}^{k} V(n)U(k-n)$
7	$z(t) = t^m$	$Z(k) = \delta(k-m), \delta(k-m) = \begin{cases} 1, if \\ k = m \\ 0, if \\ k \neq m \end{cases}$

Serial No's.	The operation Properties of Differential Transform Method (DTM)	
8	$z(t) = \exp(\omega t)$	$Z(k) = \frac{\omega^k}{k!}$
9	$z(t) = (1+t)^m$	$Z(k) = \frac{m(m-1)....(m-k+1)}{k!}$
10	$z(t) = \sin(\omega t + \lambda)$ *and* $z(t) = \cos(\omega t + \lambda)$	$Z(k) = \frac{\omega^k}{k!} \sin\left(\frac{\prod k}{2} + \lambda\right)$ *and* $Z(k) = \frac{\omega^k}{k!} \cos\left(\frac{\prod k}{2} + \lambda\right)$

4. Power Series Method

The power series method, developed in [11], is use to search for a power series solution to certain differential Equations. We consider the nth order BVP of the form

$$y^n(x) + f(x)y(x) = g(x),\ \lambda_0 < x < \lambda_1 \tag{4}$$

With the boundary conditions

$$y^{2n}(\lambda_0) = \alpha_{2n},\ n = 0, 1, 2, 3, \ldots, (k-1) \tag{5}$$

$$y^{2n}(\lambda_1) = \beta_{2n},\ n = 0, 1, 2, 3, \ldots, (k-1) \tag{6}$$

Where $f(x)$, $g(x)$, and $y(x)$ are assumed real and continuous on $\lambda_0 < x < \lambda_1$, α_{2n}, β_{2n}, are finite real constants. The given n $^{\text{th}}$ order BVP (4), (5) and (6) are transformed to systems of ODEs such that we have

$$\frac{dy}{dx} = y_1,\ \frac{dy_1}{dx} = y_2,\ \frac{dy_2}{dx} = y_3,\ \frac{dy_3}{dx} = y_4, \ldots, \frac{dy_n}{dx} = g(x) - f(x)y(x) \tag{7}$$

With the boundary conditions

$$y_1(\lambda_0) = \alpha_0, y_2(\lambda_0) = \alpha_1, y_3(\lambda_0) = \alpha_2, \ldots, y_{2k}(\lambda_0) = \alpha_{2k-1},\ and \tag{8}$$

$$y_1(\lambda_1) = \beta_0, y_2(\lambda_1) = \beta_1, y_3(\lambda_1) = \beta_2, \ldots, y_{2k}(\lambda_1) = \beta_{2k-1}, \tag{9}$$

Let the series approximation of (1), (2) and (3) be given as

$$y_n(x) = \sum_{i=0}^{N} a_i x^i, \quad N < \infty \tag{10}$$

Where a_i, $i = [0, N]$ are unknown constants to be determined and $x \in [\lambda_0, \lambda_1]$. Now, we estimate the unknown constants a_i; $i = [0, N]$ at $x = \lambda_0$ by substituting (10) in (7) successively, which is as follows:

We consider the first derivative of y_N w.r.t to x as y_1, i.e.

$$\frac{dy_N}{dx} = y_1 \Rightarrow \frac{d}{dx} i \sum_{i=1}^{N} a_i x^{i-1} = y_1 \Rightarrow a_1 + i \sum_{i=2}^{N} a_i x^{i-1} = y_1 \tag{11}$$

At $y_1(\lambda_0) = \alpha_0$

$$a_1 + i \sum_{i=2}^{N} a_i \lambda_0^{i-1} = \alpha_0 \Rightarrow a_1 = \alpha_0 - i \sum_{i=2}^{N} a_i \lambda_0^{i-1}, \tag{12}$$

Now equation (11) will become

$$y_1 = \alpha_0 - \frac{d}{dx} i(i-1) \sum_{i=0}^{N} a_i \lambda_0^{i-1} + i \sum_{i=0}^{N} a_i x^{i-1} \tag{13}$$

$$\frac{dy_1}{dx} = y_2 \Rightarrow i(i-1)\sum_{i=2}^{N} a_i x^{i-2} = y_2 \Rightarrow 2a_2 + i(i-1)\sum_{i=3}^{N} a_i x^{i-2} = y_2 \tag{14}$$

Next at $y_2(\lambda_0) = \alpha_1$

$$2a_2 + i(i-1)\sum_{i=3}^{N} a_i \lambda_0^{i-2} = \alpha_1 \Rightarrow a_2 = \frac{1}{2}\left[\alpha_1 - i(i-1)\sum_{i=3}^{N} a_i \lambda_0^{i-2}\right] \tag{15}$$

$$y_2 \Rightarrow \alpha_1 - i(i-1)\sum_{i=3}^{N} a_i \lambda_0^{i-2} + i(i-1)\sum_{i=3}^{N} a_i x^{i-2} \tag{16}$$

Carrying on the above sequential approach to the n $^{\text{th}}$ order we obtain the following recursive formulae at $x = \lambda_0$

$$a_k = \frac{1}{k!}\left[\alpha_{k-1} - k!\sum_{i=k+1}^{N} a_i \lambda_0^{i-k}\right],\ k \geq 0 \tag{17}$$

$$y_k \Rightarrow \alpha_{k-1} - k!\sum_{i=k+1}^{N} a_i \lambda_0^{i-k} + k!\sum_{i=k+1}^{N} a_i x^{i-k} \tag{18}$$

Here, the choice of N is equivalent to the order of the BVP considered.

5. Illustrated Examples

In this section, the DTM has been successfully used to study the higher order linear ordinary differential equations. Choosing examples with known solutions allows for a more complete error analysis as shown in example.

Example 5.1. *Solve the 13th order differential equation*

$$\frac{d^{13}y}{dx^{13}} = \cos x - \sin x \tag{19}$$

With the boundary conditions

$$\begin{aligned} &y^{0}(0) = 1,\ y^{1}(0) = 1,\ y^{2}(0) = -1,\ y^{3}(0) = -1,\ y^{4}(0) = 1,\ y^{5}(0) = 1,\ y^{6}(0) = -1,\\ &y^{0}(1) = 1,\ y^{1}(1) = -1,\ y^{2}(1) = -1,\ y^{3}(1) = 1,\ y^{4}(1) = 1,\ y^{5}(1) = -1 \end{aligned} \tag{20}$$

The exact solution is $y(x) = \sin x + \cos x$. Now, equation (19) transferred to system of ODEs such that we have

$$\frac{dy}{dx} = y_1,\ \frac{dy_1}{dx} = y_2,\ \frac{dy_2}{dx} = y_3,\ \frac{dy_3}{dx} = y_4, \ldots, \frac{dy_n}{dx} = \sin x + \cos x$$

With the boundary conditions at $x = \lambda_0 = 0$, $y_1(0) = 1$, $y_2(0) = 1$, $y_3(0) = -1$, $y_4(0) = -1$, $y_5(0) = 1$, $y_6(0) = 1$, $y_7(0) = -1$, $y_8(0) = a$, $y_9(0) = b$, $y_{11}(0) = d$, $y_{12}(0) = e$, $y_{13}(0) = f$. The series approximation of (19) is given as (10) were the unknown constant a_i, $i = [0, N]$ is uniquely determined by equation (17):

Since, $\lambda_0 = 0$ we have equation (17) as

$$a_k = \frac{\alpha_{k-1}}{k!},\ k \geq 0 \tag{21}$$

Using Equation (21) for $k = [0, 11]$, we have the following:

All values will change by above result (21)

$$\begin{aligned} &a_0 = 1,\ a_1 = 1,\ a_2 = -\frac{1}{2},\ a_3 = -\frac{1}{6},\ a_4 = \frac{1}{24},\ a_5 = \frac{1}{120},\ a_6 = \frac{1}{720},\ a_7 = \frac{a}{5040},\\ &a_8 = \frac{b}{40320},\ a_9 = \frac{c}{362880},\ a_{10} = \frac{d}{3628800},\ a_{11} = \frac{e}{39916800}. \end{aligned} \tag{22}$$

Putting (22) into (10) for $N = [0, 11]$, we obtain

$$y(x) = \frac{1}{120}x^5 + \frac{1}{24}x^4 - \frac{1}{6}x^3 - \frac{1}{2}x^2 + x + 1\frac{a}{5040}x^7 + \frac{b}{40320}x^8 + \frac{c}{362880}x^9 + \frac{d}{3628800}x^{10} - \frac{1}{6}x^3 - \frac{1}{720}x^6 + \frac{e}{39916800}x^{11}. \quad (23)$$

Using the boundary condition at $x = \lambda_1 = 1$ in equation (23) we obtain the value of a, b, c, d, e at $a = 1$, $b = 1$, $c = -1$, $d = 0.99$, $e = -1$ put all the values in (23). Same problem is evaluated by DTM as follows:

Taking differential transformation using properties of DTM and the following recurrence relation is obtained as follows:

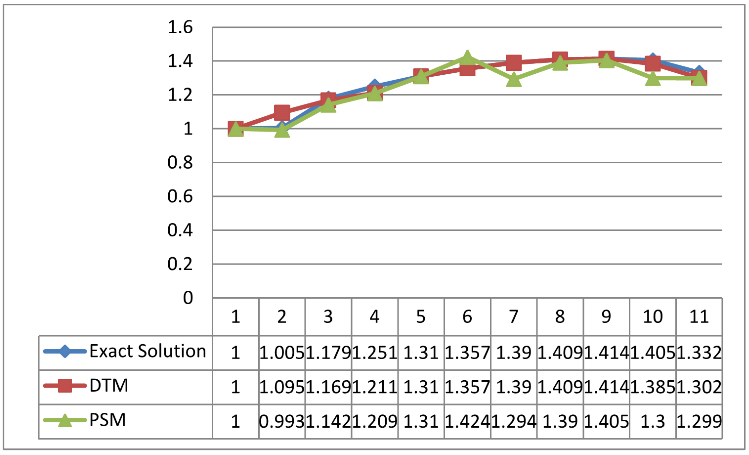
$$Y(k+13) = \frac{k!}{(k+13)}\left[\frac{1}{k!}\cos\left(\frac{\pi k}{2}\right) - \frac{1}{k!}\sin\left(\frac{\pi k}{2}\right)\right] \quad (24)$$

Using the boundary conditions at 0 and 1, we get

$$y(x) = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} - \frac{x^6}{720} + Ax^7 + Bx^8 + Cx^9 + Dx^{10} + Ex^{11} + Fx^{12} + \frac{x^{13}}{62270208000} + O(x^{14})$$

Using inverse transformation, we get Values of A, B, C, D, E, F as follows

$$y(x) = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} - \frac{x^6}{720} - 0.00019x^7 + 0.0000248x^8 + 2.756 \times 10^{-6}x^9 - 2.7654 \times 10^{-7}x^{10}$$
$$- 2.411 \times 10^{-8}x^{11} + 1.81058 \times 10^{-9}x^{12} + \frac{x^{13}}{62270208000} + O(x^{14})$$



X	Exact Solution	DTM	PSM	VIM
0	1.000000	1.000000	1.000000	1
0.1	1.094837	1.09484	1.09234	0.994
0.2	1.178735	1.17874	1.17118	0.931
0.3	1.250856	1.25086	1.24950	0.769
0.4	1.310479	1.31048	1.31011	0.784
0.5	1.357008	1.35701	1.35387	0.657
0.6	1.389978	1.38998	1.37402	0.537
0.7	1.409059	1.40906	1.39033	0.384
0.8	1.414062	1.41406	1.40472	0.240
0.9	1.404936	1.40494	1.40001	0.129
1.0	1.381773	1.38177	1.36730	0

6. Conclusion

The comparison in between the exact solution and its approximate solutions in example obtained with the help of PSM and DTM. From the numerical results, it is clear that the DTM is efficient and accurate as compared to PSM. The results are also expressed graphically in Figure. The Blue line represents the curve corresponding to the exact solution whereas the red line (DTM) and green line (PSM) corresponds to the approximate solution.

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