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A Generalized Subclass of Close-to-Convex Functions

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Abstract: The purpose of the present paper is to study a new generalized subclass of close-to-convex functions defined by subordination and we obtain coefficient estimates and also prove distortion and radii theorems for this class. Relevant connections of the results presented here with various known results are briefly indicated.

MSC: 30C45, 30C50.

Keywords: Subordination, Univalent functions, Analytic functions, Convex functions, Close-to-convex functions.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$. Let U be the class of bounded functions

$$w(z) = \sum_{n=1}^{\infty} c_n z^n$$

which are regular in the unit disc and satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ in E. For functions f and g analytic in E, we say that f is subordinate to g, denoted by $f \prec g$, if there exists a Schwarz function $w(z) \in U$, $w(z)$ analytic in E with $w(0) = 0$ and $|w(z)| < 1$ in E, such that $f(z) = g(w(z))$. By S, S^* and C we denote subclass of A, consisting of functions which are respectively univalent, starlike and convex in E. Let C' denote the class of functions $f(z) \in A$ of the form (1) and satisfying the conditions $Re\left(\frac{zf'(z)}{g(z)}\right) > 0$, where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is convex in E. The class C' was introduced by Selvaraj [4] and studied further by Abdel-Gawad and Thomas [1]. Let $C'(A, B)$ denote the class of functions $f(z) \in A$ of the form (1) and satisfying the conditions

$$Re\left(\frac{zf'(z)}{g(z)}\right) \prec \frac{1+Az}{1+Bz}, \quad -1 \le B < A \le 1,$$

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where $3b_3 = p_2 + p_1 d_2 + d_3$ is convex in E. The class $C'(A,B)$ was introduced and studied by Mehrok and Singh [3]. In particular $C'(1,-1) \equiv C'$. Let $K'_C(\alpha)$ denote the class of functions $f(z) \in A$ of the form (1) and satisfying the conditions

$$Re\left(\frac{f'(z)}{g'(z)}\right) > \alpha, \quad z \in E$$

where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ belongs to C'. The class $K'_C(\alpha)$ was introduced and studied by Stelin and Selvaraj [6]. Let $K'_C(A,B)$ denote the class of functions $f(z) \in A$ of the form (1) and satisfying the conditions

$$\frac{f'(z)}{g'(z)} \prec \frac{1+Az}{1+Bz}, \quad -1 \leq B < A \leq 1, \quad z \in E$$

where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ belongs to C'. The class $K'_C(A,B)$ was introduced and studied by Singh and Singh [5]. In particular, $K'_C(1-2\alpha,-1) \equiv K'_C(\alpha)$, the class studied by Stelin and Selvaraj [6]. Also $K'_C(1,-1) \equiv K'_C$. Motivated by above defined classes, we introduce the following subclass of analytic functions. Let $K'_C(A,B;C,D)$ denote the class of functions $f(z) \in A$ of the form (1) and satisfying the conditions

$$\frac{f'(z)}{g'(z)} \prec \frac{1+Cz}{1+Dz}, \quad z \in E$$

where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ belongs to $C'(A,B)$ and $-1 \leq D \leq B < A \leq C \leq 1$. The following observations are obvious:

(1). $K'_C(1,-1;A,B) \equiv K'_C(A,B)$.

(2). $K'_C(1,-1;1-2\alpha,-1) \equiv K'_C(\alpha)$.

By definition of subordination it follows that $f(z) \in K'_C(A,B;C,D)$ if and only if $f(z)$ can be represented in the form

$$\frac{f'(z)}{g'(z)} = \frac{1+Cw(z)}{1+Dw(z)} = P(z), w(z) \in U, -1 \leq D \leq B < A \leq C \leq 1, z \in E. \tag{2}$$

To avoid repetition, we lay down once for all that $-1 \leq D \leq B < A \leq C \leq 1$, $z \in E$. We study the class $K'_C(A,B;C,D)$ and obtain coefficient estimates, distortion theorems and radius of convexity.

2. Preliminary Results

We need the following lemmas:

Lemma 2.1 ([2]). *Let*

$$\frac{f'(z)}{g'(z)} = P(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{3}$$

then

$$|p_n| \leq (C-D), \quad n \geq 1. \tag{4}$$

The bounds are sharp, being attained for the functions

$$P_n(z) = \frac{1+C\delta z^n}{1+D\delta z^n}, \ |\delta| = 1.$$

Lemma 2.2 ([3]). *If* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in C'(A,B)$, *then*

$$|b_n| \leq \frac{(n-1)(A-B)}{n} + \frac{1}{n}, \ n \geq 2.$$

The bounds are sharp being attained for the function

$$g'(z) = \frac{1}{(1-\delta_1 z)^2} \frac{1+A\delta_2 z^{n-1}}{1+B\delta_2 z^{n-1}}, \ |\delta_1| = 1, \ |\delta_2| = 1.$$

3. Coefficient Estimates

Theorem 3.1. *If $f(z) \in K_C'(A,B;C,D)$, then*

$$|a_n| \leq \frac{(n-1)}{n}\left[(C-D)+(A-B)+\frac{(C-D)(A-B)(n-2)}{2}\right]+\frac{1}{n}, \quad n \geq 2. \tag{5}$$

The bounds are sharp.

Proof. As $f(z) \in K_C'(A,B;C,D)$, therefore (2) and (3) gives

$$1+\sum_{n=2}^{\infty} n a_n z^{n-1} = \left(1+\sum_{n=2}^{\infty} n b_n z^{n-1}\right)\left(1+\sum_{n=1}^{\infty} p_n z^n\right). \tag{6}$$

Equating the coefficients of z^{n-1} in (6), we have

$$n a_n = n b_n + (n-1) b_{n-1} p_1 + (n-2) b_{n-2} p_2 + \ldots + 2 b_2 p_{n-2} + p_{n-1}. \tag{7}$$

Therefore using (4),

$$n|a_n| \leq n|b_n| + (C-D)\left[(n-1)|b_{n-1}| + (n-2)|b_{n-2}| + \ldots + 2|b_2| + 1\right]. \tag{8}$$

Using Lemma 2.2 in (8), it yields

$$|a_n| \leq \frac{(n-1)}{n}\left[(C-D)+(A-B)+\frac{(C-D)(A-B)(n-2)}{2}\right]+\frac{1}{n}.$$

For $n=2$, equality signs in (5) hold for the functions $f_n(z)$ defined by

$$f_n'(z) = \frac{1}{(1-\delta_1 z)^2}\frac{1+A\delta_2 z^{n-1}}{1+B\delta_2 z^{n-1}}\left(\frac{1+C\delta_3 z^n}{1+D\delta_3 z^n}\right), |\delta_1|=1, |\delta_2|=1, |\delta_3|=1. \tag{9}$$

□

On replacing A by 1, B by -1, C by A and D by B, Theorem 3.1 gives the following result proved by Singh and Singh [5]:

Corollary 3.2. *If $f(z) \in K_C'(A,B)$, then*

$$|a_n| \leq \frac{(2n-1)}{n} + \frac{(A-B)(n-1)^2}{n}, \ n \geq 2.$$

On replacing A by 1, B by -1, C by $1-2\alpha$ and D by -1 in Theorem 3.1, the following result due to Stelin and Selvaraj [6] is obvious:

Corollary 3.3. *If $f(z) \in K_C'(\alpha)$, then*

$$|a_n| \leq \frac{2}{n}(1-\alpha)(n-1)^2 + \frac{(2n-1)}{n}, \ n \geq 2.$$

On replacing A by 1, B by -1, C by 1 and D by -1, Theorem 3.1 gives the following result:

Corollary 3.4. *If $f(z) \in K_C'$, then*

$$|a_n| \leq \frac{2n^2-2n+1}{n}, \ n \geq 2.$$

Theorem 3.5. *If $f(z) \in K'_C(A,B;C,D)$, then*

$$|a_2| \le \frac{1}{2}\left[(C-D)+(A-B)+1\right], \tag{10}$$

$$|a_3| \le \frac{1}{3}\left[(C-D)(A-B+2)+2(A-B)+1\right]. \tag{11}$$

Proof. Since $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n \in C'(A,B)$, it follows that

$$zg'(z) = q(z)P(z), q(z) \in C, \tag{12}$$

where

$$q(z) = z + \sum_{n=2}^{\infty} d_n z^n.$$

Also $P(z) = 1 + \sum\limits_{n=1}^{\infty} p_n z^n$, so (12) yields

$$2b_2 = p_1 + d_2,$$

$$3b_3 = p_2 + p_1 d_2 + d_3.$$

As $f(z) \in K'_C(A,B;C,D)$, we have

$$f'(z) = g'(z)Q(z), \tag{13}$$

where $Q(z) = 1 + \sum\limits_{n=1}^{\infty} q_n z^n$. Equating coefficients in (13), we get

$$2a_2 = q_1 + 2b_2, \tag{14}$$

$$3a_3 = q_2 + 2b_2 q_1 + 3b_3. \tag{15}$$

The results (10) and (11) follows on using classical inequalities $|p_1| \le (A-B), |p_2| \le (A-B)$, $|q_1| \le (C-D), |q_2| \le (C-D)$, $|d_1| \le 1$ and $|d_2| \le 1$ in (14) and (15). □

On replacing A by 1, B by -1, C by A and D by B, Theorem 3.2 gives the following result:

Corollary 3.6. *If $f(z) \in K'_C(A,B)$, then*

$$|a_2| \le \frac{1}{2}\left[(A-B)+3\right],$$

$$|a_3| \le \frac{1}{3}\left[4(A-B)+5\right].$$

On replacing A by 1, B by -1, C by $1-2\alpha$ and D by -1 in Theorem 3.1, the following result due to Stelin and Selvaraj [6] is obvious:

Corollary 3.7. *If $f(z) \in K'_C(\alpha)$, then*

$$|a_2| \le \frac{5-2\alpha}{2},$$

$$|a_3| \le \frac{13-8\alpha}{3}.$$

On replacing A by 1, B by -1, C by 1 and D by -1, Theorem 3.1 gives the following result:

Corollary 3.8. *If $f(z) \in K'_C$, then*

$$|a_2| \le \frac{5}{2},$$

$$|a_3| \le \frac{13}{3}.$$

4. Distortion Theorems

Theorem 4.1. *If $f(z) \in K'_C(A,B;C,D)$, then for $|z| = r$, $0 < r < 1$, we have*

$$\frac{(1-Cr)(1-Ar)}{(1-Dr)(1-Br)(1+r)} \leq |f'(z)| \leq \frac{(1+Cr)(1+Ar)}{(1+Dr)(1+Br)(1-r)} \tag{16}$$

and

$$\int_0^r \frac{(1-Ct)(1-At)}{(1-Dt)(1-Bt)(1+t)} dt \leq |f(z)| \leq \int_0^r \frac{(1+Ct)(1+At)}{(1+D)(1+Bt)(1-t)} dt. \tag{17}$$

Estimates are sharp.

Proof. From (2), we have

$$|f'(z)| = |g'(z)| \left| \frac{1+Cw(z)}{1+Dw(z)} \right|, w(z) \in U. \tag{18}$$

It is easy to show that the transformation

$$\frac{f'(z)}{g'(z)} = \frac{1+Cw(z)}{1+Dw(z)}$$

maps $|w(z)| \leq r$ onto the circle

$$\left| \frac{f'(z)}{g'(z)} - \frac{1-CDr^2}{1-D^2r^2} \right| \leq \frac{(C-D)r}{(1-D^2r^2)}, |z| = r.$$

This implies that

$$\frac{1-Cr}{1-Dr} \leq \left| \frac{1+Cw(z)}{1+Dw(z)} \right| \leq \frac{1+Cr}{1+Dr}. \tag{19}$$

Also it was proved by Mehrok and Singh [3] that for $g(z) \in C'(A,B)$,

$$\frac{(1-Ar)}{(1-Br)(1+r)} \leq |g'(z)| \leq \frac{(1+Ar)}{(1+Br)(1-r)}. \tag{20}$$

(18) together with (19) and (20) yields (16). On integrating (16), (17) follows. For $n = 2$, the function $f_n(z)$ defined by (9), gives sharp estimates. □

On replacing A by 1, B by -1, C by A and D by B, Theorem 3.1 gives the following result proved by Singh and Singh [5]:

Corollary 4.2. *If $f(z) \in K'_C(A,B)$, then for $|z| = r$, $0 < r < 1$, we have*

$$\frac{(1-r)(1-Ar)}{(1-Br)(1+r)^2} \leq |f'(z)| \leq \frac{(1+r)(1+Ar)}{(1+Br)(1-r)^2}$$

and

$$\int_0^r \frac{(1-t)(1-At)}{(1-Bt)(1+t)^2} dt \leq |f(z)| \leq \int_0^r \frac{(1+t)(1+At)}{(1+Bt)(1-t)^2} dt.$$

On replacing A by 1, B by -1, C by $1-2\alpha$ and D by -1, Theorem 4.1 gives the following result due to Stelin and Selvaraj [6]:

Corollary 4.3. *If $f(z) \in K'_C(\alpha)$, then*

$$\frac{(1-r)(1-(1-2\alpha)r)}{(1+r)^3} \leq |f'(z)| \leq \frac{(1+r)(1+(1-2\alpha)r)}{(1-r)^3}$$

and

$$(1-2\alpha)\log(1+r) - \frac{(6\alpha-4)}{1+r} - \frac{(2-2\alpha)}{(1+r)^2} - (2-4\alpha) \leq |f(z)| \leq -(1-2\alpha)\log(1-r) + \frac{(6\alpha-4)}{1-r} + \frac{(2-2\alpha)}{(1-r)^2} + (2-4\alpha)$$

for $|z| = r$, $0 < r < 1$, $z \in E$.

For $C = 1,\ D = -1,\ A = 1,\ B = -1$, Theorem 4.1 gives the following result:

Corollary 4.4. *If $f(z) \in K'_C$, then*

$$\frac{(1-r)^2}{(1+r)^3} \leq |f'(z)| \leq \frac{(1+r)^2}{(1-r)^3}.$$

5. Radius of Convexity

Theorem 5.1. *Let $f(z) \in K'_C(A,B;C,D)$, then $f(z)$ is convex in $|z| < r_0$, where r_0 is the smallest positive root in (0,1) of*

$$\begin{aligned}&[(CD - C + D)AB + (B-A)CD]r^4 + 2[ABD - ACD + BC - AD]r^3 \\ &\qquad + [CD - AC - 3AD + AB + BC - BD - A + B - C + D]r^2 + (2D-2A)r + 1 = 0.\end{aligned} \tag{21}$$

Results are sharp.

Proof. As $f(z) \in K'_C(A,B;C,D)$, we have

$$f'(z) = g'(z)P(z). \tag{22}$$

After differentiating (22) logarithmically, we get

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{zg''(z)}{g'(z)} + \frac{zP'(z)}{P(z)}. \tag{23}$$

It was proved by Mehrok and Singh [3] that for $g \in C'(A,B)$, we have

$$Re\left(1 + \frac{zg''(z)}{g'(z)}\right) \geq \frac{1 - 2Ar + (AB - A + B)r^2}{(1+r)(1-Ar)(1-Br)}.$$

Also from (19), we have

$$\left|\frac{1 + Cw(z)}{1 + Dw(z)}\right| = |P(z)| \leq \frac{1+Cr}{1+Dr}.$$

So

$$\left|\frac{zP'(z)}{P(z)}\right| \leq \frac{r(C-D)}{(1+Cr)(1+Dr)}.$$

Therefore (23) yields,

$$\begin{aligned}Re\left(1 + \frac{zf''(z)}{f'(z)}\right) &\geq Re\left(1 + \frac{zg''(z)}{g'(z)}\right) - \left|\frac{zp'(z)}{p(z)}\right| \\ &\geq \frac{1 - 2Ar + (AB - A + B)r^2}{(1+r)(1-Ar)(1-Br)} - \frac{r(C-D)}{(1+Cr)(1+Dr)} \\ &= \frac{\begin{array}{l}[(CD - C + D)AB + (B-A)CD]r^4 + 2[ABD - ACD + BC - AD]r^3 \\ + [CD - AC - 3AD + AB + BC - BD - A + B - C + D]r^2 + (2D-2A)r + 1\end{array}}{(1+r)(1-Ar)(1-Br)(1+Cr)(1+Dr)}\end{aligned}$$

Hence $f(z)$ is convex in $|z| < r_0$, where r_0 is the smallest positive root in $(0,1)$ of

$$\begin{aligned}&[(CD - C + D)AB + (B-A)CD]r^4 + 2[ABD - ACD + BC - AD]r^3 \\ &\qquad + [CD - AC - 3AD + AB + BC - BD - A + B - C + D]r^2 + (2D-2A)r + 1 = 0.\end{aligned}$$

$\square$

On replacing A by 1, B by -1, C by A and D by B, Theorem 5.1 gives the following result due to Singh and Singh [5]:

Corollary 5.2. *Let $f(z) \in K'_C(A,B)$, then $f(z)$ is convex in $|z| < r_1$, where r_1 is the smallest positive root in (0,1) of*

$$(A - B - 3AB)\, r^3 + (AB - 3A - 3B)\, r^2 + (2B - 3)\, r + 1 = 0.$$

On replacing A by 1, B by -1, C by $1 - 2\alpha$ and D by -1, Theorem 5.1 gives the following result due to Stelin and Selvaraj [6]:

Corollary 5.3. *If $f(z) \in K'_C(\alpha)$, then $f(z)$ is convex in $|z| < r_2$, where r_2 is the smallest positive root in (0,1) of the equation*

$$(8\alpha - 5)\, r^2 - 4r + 1 = 0.$$

For $A = 1$, $B = -1$, $C = 1$, $D = -1$, Theorem 5.1 gives the following result:

Corollary 5.4. *Let $f(z) \in K'_C$, then $f(z)$ is convex in $|z| < \frac{1}{5}$.*

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