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Total Coloring of Splitting Graph of Path, Cycle and Star Graphs

D. Muthuramakrishnan[1] and G. Jayaraman[2,*]

1 Department of Mathematics, National College(Autonomous), Tiruchirappallai, Tamil Nadu, India.

2 Department of Mathematics, Vels Institute of Science, Technology and Advanced Studies(VISTAS), Chennai, Tamil Nadu, India.

Abstract: In 1965, the concept of total coloring was introduced by Behzad [1] and in 1967 he [2] came out new ideology that, the total chromatic number of complete graph and complete bi-partite graph. A total coloring of a graph G is an assignment of colors to both the vertices and edges of G, such that no two adjacent or incident vertices and edges of G are assigned the same colors. In this paper, we have discussed the total coloring of splitting graph of Path, Cycle and Star graph. Also, we determine the total chromatic number of splitting graph of Path, Cycle and Star graphs.

MSC: 05C15.

Keywords: Path graph, Cycle graph, Star graph, Splitting graph, Total coloring and Total chromatic number.

1. Introduction

In this paper, we have chosen finite, simple and undirected graphs. Let $G = (V(G), E(G))$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$, respectively. A total coloring of G is a function $f : S \to C$, where $S = V(G) \cup E(G)$ and C is a set of colors to satisfies the given conditions.

(1). no two adjacent vertices receive the same color.

(2). no two adjacent edges receive the same color.

(3). no edges and its end vertices receive the same color.

The *total chromatic number* $\chi''(G)$ of a graph G is the minimum cardinality k such that G may have a total coloring by $k$colors. Behzad [1] and Vizing [13] Conjectured that for every simple graph G has $\Delta(G)+1 \leq \chi''(G) \leq \Delta(G)+2$, where $\Delta(G)$ is the maximum degree of the graph G. This conjecture is called as the Total Coloring Conjecture(TCC). Rosenfeld [9] and Vijayaditya [12] verified the TCC, for every graph G with maximum degree ≤ 3 and Kostochka [6] for maximum degree ≤ 5. In Borodin [3] verified the total coloring conjecture(TCC) for the maximaum degree ≥ 9 in planar graphs. In 1989, Yap et.al [14] proved that the TCC(Total Coloring Conjecture) is true for any graph G of order n having maximum degree at least $n-4$. In 1993, Hilton et.al [5] proved that any graph G has a total coloring with at most $\Delta(G)+2$ colors if $\Delta(G) \geq \frac{3}{4}|V(G)|$. In recent era, total coloring have been extensively studied in different families of graph. Geetha et.al [4] given the tight bound

* *E-mail: jayaram07maths@gmail.com*

of the Behzad and Vizing conjecture on total coloring for the generalized Sierpinski graphs of cycle graphs and Hypercube graphs. Also they give a total coloring for the Wk-recursive topology of some graphs. Mohan et.al [7] given the tight bound for the Behzad and Vizing conjecture in Corona product of certain classes of graph. Muthuramakrishnan et.al [8] prove that total chromatic number of middle, total, line graph of star graph and square graph of bistar graph. Sudha et.al [11] proved that total chromatic number of sudha grid graphs, gear and crown graphs.

Definition 1.1. *The path graph P_n, is a graph with n vertices that can be enumerated such that two vertices are adjacent if and only if they are consecutive in the enumeration.*

Definition 1.2. *The cycle graph C_n, is a graph with n vertices that can be enumerated such that two vertices are adjacent if and only if they are consecutive in the enumeration or are the first and last vertex in the enumeration.*

Definition 1.3. *A tree containing exactly one vertex that is not a pendent vertex is called a star graph $K_{1,n}$.*

Definition 1.4. *For a graph G, the splitting graph $S'(G)$ [10] of a graph G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$.*

2. Main Results

Theorem 2.1. *Let $S'(P_n)$ be the splitting graph of path graph of order n. Then $\chi''(S'(P_n)) = \Delta(S'(P_n)) + 1,\ n \geq 2$.*

Proof. Let $\{v_1, v_2, \dots, v_n\}$ be the vertices of path P_n and $\{e_1, e_2, \dots, e_n\}$, where $\{e_i = v_iv_{i+1} : 1 \leq i \leq n-1\}$ be the edges of path P_n. By the definition of splitting graph, adding the new vertices $\{{v_i}' : 1 \leq i \leq n\}$ corresponding to the vertices $\{v_i : 1 \leq i \leq n\}$ of P_n, which are added to obtain $S'(P_n)$. In $S'(P_n)$, the vertex set and the edge set is given by $V(S'(P_n)) = \{v_i : 1 \leq i \leq n\} \cup \{{v_i}' : 1 \leq i \leq n\}$ and $E(S'(P_n)) = \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{v_i{v_{i+1}}' : 1 \leq i \leq n-1\} \cup \{{v_i}'v_{i+1} : 1 \leq i \leq n\}$ We define the total coloring f, such that $f : S \to C$, where $S = V(S'(P_n)) \cup E(S'(P_n))$ and $C = \{1, 2, 3, 4, 5\}$. Now we assign the total coloring to these vertices and edges as follows.

$$f(v_i) = f({v_i}') = \begin{cases} 1, & \text{for all } \ i \equiv 1\ (mod\ 3) \\ 2, & \text{for all } \ i \equiv 2\ (mod\ 3) \ \text{ for } \ 1 \leq i \leq n \\ 3, & \text{for all } \ i \equiv 0\ (mod\ 3) \end{cases}$$

$$f(v_iv_{i+1}) = \begin{cases} 3, & \text{for all } \ i \equiv 1\ (mod\ 3) \\ 1, & \text{for all } \ i \equiv 2\ (mod\ 3) \ \text{ for } \ 1 \leq i \leq n \\ 2, & \text{for all } \ i \equiv 0\ (mod\ 3) \end{cases}$$

$$f({v_i}'v_{i+1}) = 4, \quad \text{for } 1 \leq i \leq n-1$$

$$f(v_i{v_{i+1}}') = 5, \quad \text{for } 1 \leq i \leq n-1$$

It is clear that the above method of total coloring, the graph $S'(P_n)$ is properly total colored with $\Delta(S'(P_n)) + 1 = 5$ colors. Hence the total chromatic number of the splitting graph of $P_n, \chi''(S'(P_n)) = \Delta(S'(P_n)) + 1 = 5$. □

Example 2.2. *Consider the splitting graph of path P_6.*

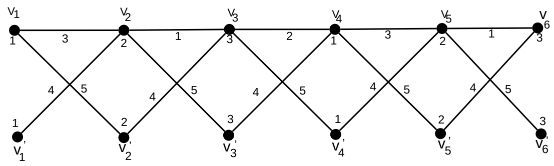

Figure 1. **Total Coloring of splitting graph of path graph** $S'(P_6)$

By using the total coloring pattern as given in Theorem 2.1, the colors $\{1, 2, 3, 4, 5\}$ are assigned to these vertices and edges as shown in figure 1. Thus the total chromatic number of splitting graph of P_6 is 5.

Theorem 2.3. *Let $S'(C_n)$ be the splitting graph of cycle graph of order n. Then*

$$\chi''(S'(C_n)) = \begin{cases} \Delta(S'(C_n)) + 1, & n \text{ is even} \\ \Delta(S'(C_n)) + 2, & n \text{ is odd} \end{cases}$$

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of cycle C_n and $\{e_1, e_2, \ldots, e_{n-1}\}$, where $\{e_i = v_iv_{i+1} : 1 \leq i \leq n-1\}$ and $\{e_n = v_nv_1\}$ be the edges of cycle C_n. By the definition of splitting graph, adding the new vertices $\{v_i' : 1 \leq i \leq n\}$ corresponding to the vertices $\{v_i : 1 \leq i \leq n\}$ of C_n, which are added to obtain $S'(C_n)$. In $S'(C_n)$, the vertex set and the edge set is given by $V(S'(C_n)) = \{v_i : 1 \leq i \leq n\} \cup \{v_i' : 1 \leq i \leq n\}$ and $E(S'(C_n)) = \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{v_iv_{i+1}' : 1 \leq i \leq n-1\} \cup \{v_i'v_{i+1} : 1 \leq i \leq n\} \cup \{v_nv_1\} \cup \{v_n'v_1\} \cup \{v_nv_1'\}$.
We construct the total coloring f, such that $f : S \to C$, where $S = V(S'(C_n)) \cup E(S'(C_n))$ and $C = \{1, 2, 3, 4, 5, 6\}$. Now we assign the total coloring to these vertices and edges as follows. we consider the following two cases

Case (i): When n is even

$$f(v_i) = \begin{cases} 1, & \text{if } i \text{ is odd, for } 1 \leq i \leq n \\ 2, & \text{if } i \text{ is even} \end{cases}$$

$$f(v_i') = 3, \qquad \text{for } 1 \leq i \leq n$$

$$f(v_iv_{i+1}) = \begin{cases} 3, & \text{if } i \text{ is odd, for } 1 \leq i \leq n-1 \\ 4, & \text{if } i \text{ is even} \end{cases}$$

$$f(v_nv_1) = 4$$

$$f(v_iv_{i+1}') = \begin{cases} 2, & \text{if } i \text{ is odd, for } 1 \leq i \leq n-1 \\ 1, & \text{if } i \text{ is even} \end{cases}$$

$$f(v_nv_1') = 1, \qquad f(v_n'v_1) = 5$$

$$f(v_i'v_{i+1}) = 5, \text{ for } 1 \leq i \leq n$$

Based on the above coloring pattern, the graph $S'(C_n)$ is total colored with $\Delta(S'(C_n)) + 1$ colors. Hence the total chromatic number of splitting graph of $C_n, \chi''(S'(C_n)) = \Delta(S'(C_n)) + 1$ for n is even.

Case (ii): When n is odd

$$f(v_i) = \begin{cases} 1, & \text{if } i \text{ is odd, for } 1 \leq i \leq n-1 \\ 2, & \text{if } i \text{ is even} \end{cases}$$

$$f(v_n) = 3$$

$$f(v_i{}') = 6, \quad \text{for } 1 \le i \le n$$

$$f(v_i v_{i+1}) = \begin{cases} 3, & \text{if } \ i \ \text{ is odd}, \ \text{ for } 1 \le i \le n-1 \\ 4, & \text{if } \ i \ \text{ is even} \end{cases}$$

$$f(v_n v_1) = 6$$

$$f(v_i v_{i+1}{}') = \begin{cases} 2, & \text{if } \ i \ \text{ is odd}, \quad \text{for } 1 \le i \le n-1 \\ 1, & \text{if } \ i \ \text{ is even} \end{cases}$$

$$f(v_n v_1{}') = 1, \qquad f(v_n{}' v_1) = 5$$

$$f(v_i{}' v_{i+1}) = 5, \quad \text{for } 1 \le i \le n-1$$

Based on the above coloring pattern, the graph $S'(C_n)$ is total colored with $\Delta(S'(C_n))+1$ colors. Hence the total chromatic number of splitting graph of $C_n, \chi''(S'(C_n)) = \Delta(S'(C_n)) + 2$ for n is odd. □

Example 2.4. *Consider the splitting graph of cycle C_8.*

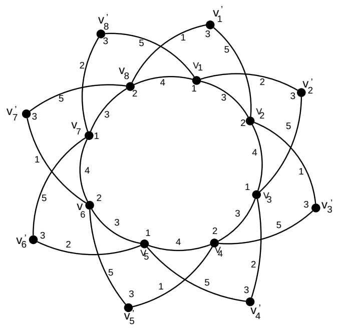

Figure 2. **Total Coloring of splitting graph of cycle graph** $S'(C_8)$

By using the total coloring pattern as given in Case (i) of Theorem 2.3, the colors {1, 2, 3, 4, 5} are assigned to these vertices and edges as shown in figure 2. Thus the total chromatic number of splitting graph of C_8 is 5.

Example 2.5. *Consider the splitting graph of cycle C_7.*

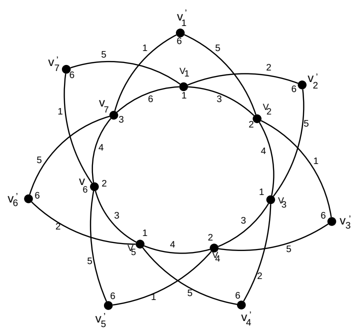

Figure 3. **Total Coloring of splitting graph of cycle graph** $S'(C_7)$

By using the total coloring pattern as given in Case (ii) of Theorem 2.3, the colors {1, 2, 3, 4, 5,6} are assigned to these vertices and edges as shown in figure 2. Thus the total chromatic number of splitting graph of C_7 is 6.

Theorem 2.6. *Let $S'(K_{1,n})$ be the splitting graph of star graph of order n. Then $\chi''(S'(K_{1,n})) = 2n+1,\ n \geq 2$.*

Proof. Let $V(K_{1,n}) = \{v\} \cup \{v_i : 1 \leq i \leq n\}$, where $\{v_i : 1 \leq i \leq n\}$ be the pendent vertices and $\{v\}$ be the root vertex of $K_{1,n}$. $E(K_{1,n}) = \{vv_i : 1 \leq i \leq n\}$. Now construct the splitting graph, adding the new vertices $\{v'\}$ and $\{v_i{}' : 1 \leq i \leq n\}$ corresponding to the vertices $\{v\}$ and $\{v_i : 1 \leq i \leq n\}$ of $K_{1,n}$, which are added to obtain $S'(K_{1,n})$. In $S'(K_{1,n})$, the vertex set and the edge sets are given by $V(S'(K_{1,n})) = \{v_i : 1 \leq i \leq n\} \cup \{v_i{}' : 1 \leq i \leq n\} \cup \{v\} \cup \{v'\}$ and $E(S'(K_{1,n})) = \{vv_i : 1 \leq i \leq n\} \cup \{vv_i{}' : 1 \leq i \leq n\} \cup \{v'v_i : 1 \leq i \leq n\}$. We define the total coloring f, such that $f : S \to C$, where $S = V(S'(K_{1,n})) \cup E(S'(K_{1,n}))$ and $C = \{1, 2, 3, \ldots 2n+1\}$. Now we assign the total coloring to these vertices and edges as follows

$$f(v) = 1, \qquad f(v') = 2$$

$$f(v_i) = f(vv_i{}') = \begin{cases} 2i+1, & \text{if } \ 2i+1 \not\equiv 0 \ (mod\ 2n+1) \ \text{ for } 1 \leq i \leq n \\ 2n+1, & \text{otherwise} \end{cases}$$

$$f(v_i{}') = f(vv_i) = \begin{cases} 2i, & \text{if } \ (2i) \not\equiv 0 \ (mod\ 2n) \ \text{ for } 1 \leq i \leq n \\ 2n, & \text{otherwise} \end{cases}$$

$$f(v'v_i) = \begin{cases} 2i-1, & \text{if } \ (2i-1) \not\equiv 0 \ (mod\ 2n-1) \ \text{ for } 1 \leq i \leq n \\ 2n-1, & \text{otherwise} \end{cases}$$

Based on the above coloring pattern, the graph $S'(K_{1,n})$ is properly total colored with $\Delta(S'(K_{1,n})) + 1$ colors. Hence the total chromatic number of splitting graph of $K_{1,n})$, $\chi''(S'(K_{1,n})) = 2n+1$. □

Example 2.7. *Consider the splitting graph of star graph S_6.*

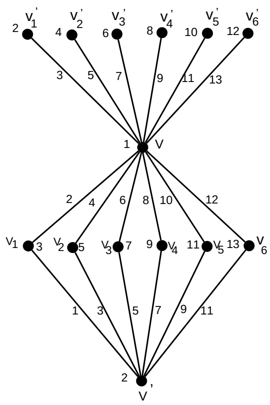

Figure 4. **Total Coloring of splitting graph of star graph $S'(S_6)$**

By apply the total coloring pattern given in the Theorem 2.6, the colors {1,2,. . . , 13} are assigned to these vertices and edges as shown in figure 4. Thus the total chromatic number of splitting graph of star $K_{1,6}$ is 13.

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