Int. J. Math. And Appl., 6(1–D)(2018), 707–709
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of *Mathematics And its Applications*

Countable KC Functions and Inversely Countable KC Functions

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Abstract: In this paper, we introduce a new class of functions to be called countable KC and inversely countable KC functions. We study some relations between countable KC functions and inversely countable KC functions. We prove that if f is a continuous function from a space X to a Frechet, countable KC space Y, then $f(H)$ is closed in Y whenever H is a limit point compact set of X. It is also proved that if f is a countable KC function from a Frechet space X to a compact space Y, having closed point inverses, then f is continuous.

MSC: 54C10, 54D50.

Keywords: Countable KC function, Inversely countable KC function, Continuous function, Frechet space.

1. Introduction

By a space, we shall mean a topological space. In [1], Fuller studies a number of conditions on a function between two spaces or it's inverse, preserving closedness, openness, or compactness. He also studies relationship between the class of functions having closed graph and the class of functions which map compact sets into closed sets; the functions in later class have been called as KC functions by A. Wilansky in [6]. In [3], some relations between KC functions and inversely KC functions have been studied. In this paper, we introduce a new class of functions to be called class of countable KC and inversely countable KC functions. Notation, definitions and preliminaries are given in the Section 2. The main results of the paper appear in Section 3. In Section 3, we prove that if f is a continuous function from a space X to a Frechet, countable KC space Y, then $f(H)$ is closed in Y whenever H is a limit point compact set of X. Suppose $f : X \to Y$ is inversely countable KC and Y is T_1 at $f(X)$. If Y is a Frechet space, then image of every limit point compact set of X is closed in Y. It is proved that if f is a countable KC function from a Frechet space X to a space Y, having closed point inverses, then inverse image of every limit point compact set of Y is closed in X. This proves that if f is a countable KC function from a Frechet space X to a compact space Y, having closed point inverses, then f is continuous.

2. Notation and Definition

For completeness, we have included some of the standard notation and definitions.

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Let X be a space. For $H \subset Y \subset X$, $cl_Y(H)$ denotes the closure of H in Y. A subset K of X is called limit point compact [4] if every infinite subset of K has a limit point in K. If A is a subset of X, we say that X is T_1 at A ([2]) if each point of A is closed in X. X is said to be Frechet space (or closure sequential in the terminology of Wilansky [6]) if for each subset A of X, $x \in cl(A)$ implies there exists a sequence $\{x_n\}$ in A converging to x. X is said to be KC ([5]) if every compact set of X is closed in X. We call a space X to be a Countable KC space if every countably and compact set of X is closed in X. Let X and Y be two spaces. Let f be a function from X to Y. f is said to be compact (compact preserving) if inverse image (image) of each compact set is compact. f is called KC ([6]) if image of every compact set of X is closed in Y. f is called inversely KC if inverse image of every compact set of Y is closed in X. We say that f is countable KC if image of every countable and compact set of X is closed in Y. f is called inversely countable KC if inverse image of every countable and compact set of Y is closed in X.

3. Countable KC Functions and Inversely Countable KC Functions

Theorem 3.1. *Suppose $f : X \to Y$ is continuous and Y is a countable KC space. If Y is a Frechet space, then image of every limit point compact set of X is closed in Y.*

Proof. Let H be a limit point compact subset of X and let $y \in cl_Y f(H)$ such that $y \notin f(H)$. There exists a sequence $\{x_n\}$ of points in H such that $f(x_n) \to y$ as Y is a Frechet space. Since H is limit point compact set, the sequence $\{x_n\}$ has a cluster point x, say, in H. Since Y is countable KC, $V = Y - \{f(x)\}$ is an open set containing y. Therefore there exists an integer n_o such that $f(x_n) \in V$ for all $n \geq n_o$ as $f(x_n) \to y$. Let $K = \{f(x_n) : n \geq n_o\} \cup \{y\}$. Then K being a countable and compact set of Y is closed in Y as Y is countable KC. Since $x \in (cl_X(f^{-1}(K)) - f^{-1}(K)), f^{-1}(K)$ is not closed in X. This is a contradiction as f is continuous. Thus $f(H)$ is closed in Y. □

Corollary 3.2. *Suppose $f : X \to Y$ is continuous and Y is a countable KC space. If Y is a Frechet space, then f is KC.*

The following theorem is a generalization of Theorem 4.2 of [3].

Theorem 3.3. *Suppose $f : X \to Y$ is inversely countable KC and Y is T_1 at $f(X)$. If Y is a Frechet space, then image of every limit point compact set of X is closed in Y.*

Proof. Let H be a limit point compact subset of X and let $y \in cl_Y f(H)$ such that $y \notin f(H)$. There exists a sequence $\{x_n\}$ of points in H such that $f(x_n) \to y$. Since H is limit point compact set, the sequence $\{x_n\}$ has a cluster point x, say, in H. Since Y is T_1 at $f(X), V = Y - \{f(x)\}$ is an open set containing y. Therefore there exists an integer n_o such that $f(x_n) \in V$ for all $n \geq n_o$ as $f(x_n) \to y$. Let $K = \{f(x_n) : n \geq n_o\} \cup \{y\}$. Then K is countable and compact. Since $x \in (cl_X(f^{-1}(K)) - f^{-1}(K)), f^{-1}(K)$ is not closed in X. This is a contradiction as f is inversely countable KC. Thus $f(H)$ is closed in Y. □

The following two corollaries are generalizations of Corollaries 4.3 and 4.4 of [3] respectively.

Corollary 3.4. *Suppose $f : X \to Y$ is inversely countable KC and Y is T_1 at $f(X)$. If Y is a Frechet space, then f is KC.*

Corollary 3.5. *Suppose $f : X \to Y$ is inversely countable KC and Y is T_1 at $f(X)$. If X is compact and Y is a Frechet space, then f is closed.*

The following theorem is a generalization of Theorem 4.7 of [3].

Theorem 3.6. *Suppose $f : X \to Y$ is countable KC and has closed point inverses. If X is Frechet, then inverse image of every limit point compact set of Y is closed in X.*

Proof. Let K be any limit point compact subset of Y, and let $x \in cl_X(f^{-1}(K))$ such that $x \notin f^{-1}(K)$. Since X is a Frechet space, there exists a sequence $\{x_n\}$ of points in $f^{-1}(K)$ such that $x_n \to x$. Then $\{f(x_n)\}$ is a sequence in K and K is limit point compact implies $\{f(x_n)\}$ has a cluster point, say, y in K. Since $y \neq f(x)$, $U = X - f^{-1}(\{y\})$ is an open set due to closed point inverses and contains x. Therefore, there exists a positive integer n_o such that $x_n \in U$ for all $n \geq n_o$ as $x_n \to x$. Then $H = \{x_n : n \geq n_o\} \cup \{x\}$ is countable and compact, but $f(H)$ is not closed as $y \in (cl_Y f(H)) - f(H))$. This gives a contradiction to the given condition. Thus $f^{-1}(K)$ is closed in X. □

The following two corollaries are generalizations of Corollaries 4.8 and 4.9 of [3] respectively.

Corollary 3.7. *Suppose $f : X \to Y$ is countable KC and has closed point inverses. If X is a Frechet space, then f is inversely KC.*

Corollary 3.8. *Suppose $f : X \to Y$ is countable KC and has closed point inverses. If Y is compact and X is a Frechet space, then f is continuous.*

Theorem 3.9. *Let $f : X \to Y$ be inversely countable KC, compact preserving and closed. Then f is countable KC.*

Proof. If K is any countable and compact subset of X, then $f(K)$ is countable and compact, $f^{-1}(f(K))$ is closed and so $f(f^{-1}(f(K)))$ is closed. But for any subset A of X, $A \subset f^{-1}(f(A))$ and for any subset B of Y, $f(f^{-1}(B)) \subset B$. Therefore, $f(K) = f(f^{-1}(f(K)))$ is closed. Hence f is countable KC. □

Corollary 3.10. *Let $f : X \to Y$ be inversely countable KC, continuous and closed. Then f is countable KC.*

Theorem 3.11. *Let $f : X \to Y$ be a countable KC, compact and continuous injection. Then f is inversely countable KC.*

Proof. If K is any countable and compact subset of Y, then $f^{-1}(K)$ is countable and compact, $f(f^{-1}(K))$ is closed and so $f^{-1}(f(f^{-1}(K)))$ is closed. But for any subset A of X, $A \subset f^{-1}(f(A))$ and for any subset B of Y, $f(f^{-1}(B)) \subset B$. Therefore, $f^{-1}(K) = f^{-1}(f(f^{-1}(K)))$ is closed. Hence f is inversely countable KC. □

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