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Connectedness in Digital Spaces with the Khalimsky Topology

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Abstract: In this paper we prove certain results related to connectedness in topological digital spaces.

Keywords: Digital spaces, Alexandroff space, Khalimsky topology, arc connected.

1. Introduction and Preliminaries

Digital geometry is considered in Z^n whereas continuous geometry can be used in R^n. To represent continuous geometrical objects in the computer we are limited to some sort of approximations. There are points in the Euclidean plane that can be described exactly on a computer. By introducing notions as connectedness and continuity on discrete sets we can represent discrete objects with the same accuracy as Euclid had in his geometry.

Definition 1.1 ([2]). *A digital space is a pair (V,π) where V is a non-empty set and π is a binary symmetric relation on V such that for any two elements x and y of V there is a finite sequence $(x^0,\dots,x^n)$ of elements in V such that $x=x^0$ and $y=x^n$ and $(x^j,x^{j+1})\in\pi$ for $j=0,\dots,n-1$.*

Remark 1.2. *All topological spaces are not digital and all digital spaces are not topological.*

Definition 1.3. *A topological space is said to be a smallest neighborhood space, or Alexandroff space, if arbitrary intersection of open sets is open.*

Definition 1.4 (Khalimsky topology)**.** *Khalimsky line is the topology space (Z,t) where t is the Alexandroff topology as follows. For any $z\in Z$*

$$t(z)=\begin{cases}\{z-1,z,z+1\} & \text{if } z \text{ is even}\\ \{z\} & \text{if } z \text{ is odd}\end{cases}$$

t is a $T_{1/2}$ topology.

Definition 1.5. *A topological space X is said to be Khalimsky arc connected if it satisfies the following conditions.*

(1). X satisfies T_0-axiom

(2). for all $x,y\in X, I=[a,b]_Z$ and $\phi: I\to X$ such that $\phi(a)=x$, $\phi(b)=y$ and ϕ is a homeomorphism of I into $\phi(I)$.

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2. Main Results

Theorem 2.1. *Let X be a connected smallest neighborhood space. Then for any pair of points x and y of X there is a finite sequence $\{x^0, x^1, \ldots, x^n\}$ such that $x = x^0$ and $y = x^n$ and $\{x^j, x^{j+1}\}$ is connected for $i = 0, 1, \ldots, n-1$.*

Proof. Let x be a point in X, and Y denote the set of points which can be connected to x by such a finite sequence. Clearly $x \in Y$. Suppose that $y \in Y$. Then $N(y) \subset Y$ and $\{\bar{y}\} \subset Y$. Thus Y is open, closed and non-empty. Since X is connected, this implies $Y = X$. □

Theorem 2.2. *A T_0 topological digital space is Khalimsky arc connected.*

Proof. Let X be a topological digital space and let $a, b \in X$. We get a finite, connected sequence of points $(x_0, \ldots, x_n)$ such that $x_0 = a$ and $x_n = b$. Since the sequence is finite, we get a subsequence $Y = \{y_0, \ldots, y_m\}$, that is minimal with respect to connectedness, and such that $y_0 = a$ and $y_m = b$. Hence Y has no connected proper subset containing a and b. Suppose that $i > j$ and $\{y_i, y_j\}$ is connected. Then $i = j + 1$ for if not, the sequence $\{y_0, \ldots, y_j, y_i, \ldots, y_m\}$ would be a connected proper subset of the minimal sequence, which is a contradiction.

So there are two possibilities for $N_Y(y_0)$; either $N_Y(y_0) = \{y_0\}$ or $N_Y(y_0) = \{y_0, y_1\}$. Let $s = 1$ in the first case and $s = 0$ in the second case. Re-index the sequence $\{y_i\}$ so that y_s is its first element and y_{m+s} its last. Let $s = 1$ in the first case and $s = 0$ in the second case. We will show that the function $\varphi : I = [s, s + m]_Z \to Y$, is a homeomorphism. Y is T_0 since X is T_0. φ is surjective and by minimality injective also. To show that φ is a homeomorphism, it is sufficient to show that

$$N_Y(y_i) = \varphi(N_I(i)) \text{ for every } i \in I. \tag{1}$$

For $i = s$ this holds by the choice of s. We use finite induction. Suppose that $s < k \leq s+m$, and that (1) holds for $i = k-1$. We consider two cases.

Case 1: k is odd. Then $N_I(k) = \{k\}$. We must show that y_{k-1} and y_{k+1} are not in $N_Y\{y_k\}$. But since equation (1) holds for $i = k - 1$, and $k \in N_I(k-1)$, we know that $y_k \in N_Y(y_{k-1})$. So $y_{k-1} \notin N_Y(y_k)(Y_k, T_0)$ and also that $y_{k+1} \notin N_Y(y_k)$ since $N_Y(y_{k-1}) \cap N_Y(y_k)$ is a neighborhood of y_k which does not include y_{k+1}.

Case 2: k is even. We must show that $y_{k-1} \in N_Y(y_k)$ and $y_{k+1} \in N_Y(y_k)$ provided $k < s + m$.

Now $N_Y(y_{k-1}) = \{y_{k-1}\}$ by assumption, so clearly y_{k-1} belongs to $N_Y(y_k)$. If $k < s + m$ and $y_k \in N_Y(y_{k+1})$ then $N_Y(y_k) \cap N_Y(y_{k+1})$ would be a neighborhood of y_k not containing y_{k-1}. This is a contradiction and hence $y_{k+1} \in N_Y(y_k)$. □

References

[1] Ulrich Echhardt and Longen J.Latecki, *Topologies for the digital spaces Z^2 and Z^3*, Computer Vision and Image Understanding, 90(2003), 295-312.

[2] Gabor T.Herman, *Geometry of digital spaces*, Birkhauser, (1998).

[3] K.D.Joshy, *Introduction to General Topology*, Wiley Eastern Limited, (1983).

[4] T.Y.Kong, Ralph Kopperman and Paul P.Meyer, *A topological approach to digital topology*, Amer. Math. Monthly, 98(1991), 901-917.

[5] V.A.Kovalevsky, *Finite topology as applied to image analysis*, Computer vision, Graphics, Image process, 46(1989), 141-151.

[6] S.Levialdi, *On shrinking binary patterns*, Commu. ACM, 15(1972), 7-10.