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Bounds for the Second q-Hankel Determinant of Certain Univalent Functions

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Abstract: We study the estimates for the Second q-Hankel determinant of analytic functions in a class which unifies a number of classes studied previously by Deepak Bansal, K. I. Noor, T. Yavuz, Sarika Verma, Shigeyoshi Owa and others. Our class includes q-convex, q-starlike functions and Ma-Minda starlike and convex functions.

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1. Introduction

The Hankel determinants $H_q(n)$ of Taylor's coefficients of function $f \in \mathcal{A}$ where $\mathcal{A}$ denotes the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \ \ and \ \ |z| < 1\}$. is defined by

$$\mathbf{H_q(n)} := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}$$

where $(n = 1, 2, ... \ and \ q = 1, 2, ...,)$. $H_2(1)$ is the classical Fekete-Szegö functional. Fekete-Szegö in [2] found the maximum value of $H_2(1)$. Pommerenke in [13] proved that the Hankel determinant of univalent functions satisfy

$$|H_q(n)| < K n^{-(\frac{1}{2}+\beta)q+\frac{3}{2}} \quad (n = 1, 2, \ldots, q = 2, 3, \ldots),$$

where $\beta > \frac{1}{4000}$ and K depends only on q. Hayman [5] showed that

$$|H_2(n)| = |a_n a_{n+2} - a_{n+1}^2| < A n^{\frac{1}{2}}, \ n = 2, 3, \ldots,$$

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where A is an absolute constant for a really mean univalent functions. Hankel determinants are useful in showing that a function of bounded characteristic in $\mathcal{U}$, i.e, a function which is ratio of tow bounded analytic functions with its Laurent series around the origin having integral coefficients, is rational. In recent years, several authors investigated bounds for the Hankel determinant belonging tow various subclasses of univalent and multivalent functions in a class which unifies a number of classes studied earlier by Deepak Bansal, K. I. Noor, T. Yavuz, Sarika Verma, Shigeyoshi Owa and others. Our class include q-starlike, q-convex functions and Ma-Minda starlike and convex functions. We use the concept of principle of subordination and q-calculus to define our classes. Recently in the second half of the twentieth century q-calculus aroused interest due to lot of applications in the various mathematical fields such as combinatorics, number theory, quantum theory and the theory of relativity. The q-analogue of a function is defined as follows:

Definition 1.1 ([6]). *The q-analogue of f is given by*

$$\partial_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{z(1-q)}, & z \neq 0, \\ f'(0), & z = 0. \end{cases} \tag{2}$$

where $(0 < q < 1)$.

Equivalently (2), may be written as

$$\partial_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \quad z \neq 0$$

where

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1 \\ n, & q = 1 \end{cases}$$

Note that as $q \to 1$, $[n]_q \to n$. The class of functions with positive real part plays a significant role in complex function theory. Using principle of subordination we define the functions with positive real part.

Definition 1.2 ([14]). *Let f and g be analytic in $\mathcal{U}$, then f is said to be subordinate to the function g, written $f(z) \prec g(z)$, if there exists an analytic function $\omega : \mathcal{U} \to \mathcal{U}$ satisfying $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$, $z \in \mathcal{U}$.*

Definition 1.3 ([1]). *Let $\mathcal{P}$ denote the class of analytic functions $p : \mathcal{U} \to \mathbb{C}$, $p(0) = 1$, and $\Re\{p(z)\} > 0$, then $p(z) \prec \frac{1+z}{1-z}$.*

The class $\mathcal{P}$ can be completely characterized in terms of subordination. We need the following lemmas to derive our results.

Lemma 1.4 ([1]). *If the function $p \in \mathcal{P}$ is given by the series*

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots, \tag{3}$$

then the following sharp estimate holds:

$$|c_n| \leq 2, \ (n = 1, 2, \ldots).$$

Lemma 1.5 ([4]). *If the function $p \in \mathcal{P}$ is given by the series (3), then*

$$2c_2 = c_1^2 + x(4 - c_1^2), \tag{4}$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z, \tag{5}$$

for some x, z with $|x| \leq 1$ and $|z| \leq 1$.

The q-derivative of starlike and convex functions can be expressed as follows

$$S_q^* = \left\{ f \in \mathcal{A} : \frac{z\partial_q f(z)}{f(z)} \prec \frac{1+z}{1-qz} \right\} \ \ \textit{and} \ \ K_q = \left\{ f \in \mathcal{A} : \frac{\partial_q (z\partial_q f(z))}{\partial_q f(z)} \prec \frac{1+z}{1-qz} \right\}.$$

2. Main Results

Now we define q-drivative of starlike and convex functions with respect to φ which generalize Ma and Minda starlike and convex functions [10].

Definition 2.1. *Let $\varphi : U \to \mathbb{C}$ be analytic, and let the Maclaurin series of φ is given by*

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots \quad (B_1, B_2 \in \mathbb{R}, B_1 > 0). \tag{6}$$

The class $S_q^*(\varphi)$ of q-starlike functions with respect to φ consists of functions $f \in \mathcal{A}$ satisfying the subordination

$$\frac{z\partial_q f(z)}{f(z)} \prec \varphi(z).$$

For the function φ given by $\varphi_\alpha(z) := \frac{1+(1-2\alpha)z}{(1-z)}$, $0 < \alpha \leq 1$, the class $S_q^*(\alpha) := S_q^*(\varphi_\alpha)$ is the well-known class of q-starlike functions of order α. For $\varphi_P(z) := 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$, the class

$$S_q^*(\varphi_P) := \left\{ f \in \mathcal{A} : Re\left(\frac{z\partial_q f(z)}{f(z)}\right) > \left|\frac{z\partial_q f(z)}{f(z)} - 1\right| \right\}$$

is the *parabolic q-starlike* functions, and for $\varphi_H(z) := 1 + \frac{2}{1-k^2}\sinh^2\left[\left(\frac{2}{\pi}\arccos k\right)\arctan h\sqrt{z}\right]$ $, 0 < k < 1$, the class

$$S_q^*(\varphi_H) = \left\{ f \in \mathcal{A} : Re\left(\frac{z\partial_q f(z)}{f(z)}\right) > \left|\frac{z\partial_q f(z)}{f(z)} - 1\right| \right\}$$

is the *hyperbolic q-starlike* functions, and for

$$\varphi_E(z) := \frac{1}{k^2-1}\sin\left(\frac{\pi}{2R(t)}\int_0^{\frac{u(z)}{\sqrt{t}}} \frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx\right) + \frac{1}{k^2-1}, \quad k > 1,$$

where $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{t}z}$ $, t \in (0,1)$ $z \in \mathcal{U}$ and z is chosen such that $k = \cosh\left(\frac{\pi R'(t)}{4R(t)}\right)$, $R(t)$ is the Legendre's complete elliptic integral of the first kind and $R'(t)$ is complementary integral of $R(t)$, the class

$$S_q^*(\varphi_E) = \left\{ f \in \mathcal{A} : Re\left(\frac{z\partial_q f(z)}{f(z)}\right) > \left|\frac{z\partial_q f(z)}{f(z)} - 1\right| \right\}$$

is the *elliptic q-starlike* functions [16], which as $q \to 1$ the class reduces to the class introduced by Rønning [15].

Theorem 2.2. *Let the function $f \in S_q^*(\varphi)$ be given by (1)*

(1). If B_1, B_2 and B_3 satisfy the conditions $MB_1^2 + N|B_2| + NB_1 \leq 0$, $\left|([3]_q - 1)^2 B_1 B_3 + KB_1^4 + MB_1^2 B_2 - LB_2^2\right| - LB_1^2 \leq 0$, then the second Hankel determinant satisfies

$$|a_2 a_4 - a_3^2| \leq \frac{B_1^2}{([3]_q - 1)^2}.$$

(2). If B_1, B_2 and B_3 satisfy the conditions $MB_1^2 + N|B_2| + NB_1 \geq 0$, $\left|([3]_q - 1)^2 B_1 B_3 + KB_1^4 + MB_1^2 B_2 - LB_2^2\right| - NB_1|B_2| - \frac{M}{2}B_1^3 - \frac{([3]_q-1)^2}{2}B_1^2 \geq 0$, or the conditions $MB_1^2 + N|B_2| + NB_1 \geq 0$, $\left|([3]_q - 1)^2 B_1 B_3 + KB_1^4 + MB_1^2 B_2 - LB_2^2\right| - LB_1^2 \geq 0$, then the second Hankel determinant satisfies

$$|a_2 a_4 - a_3^2| \leq \frac{1}{([4]_q - 1)([3]_q - 1)^2([2]_q - 1)}\left|([3]_q - 1)^2 B_1 B_3 + KB_1^4 - LB_2^2 + MB_1^2 B_2\right|.$$

(3). If B_1, B_2 and B_3 satisfy the $MB_1^2 + N|B_2| + NB_1 > 0$, $\left|([3]_q-1)^2 B_1 B_3 + KB_1^4 + MB_1^2 B_2 - LB_2^2\right| - NB_1|B_2| - \frac{M}{2}B_1^3 - \frac{([3]_q-1)^2}{2}B_1^2 \leq 0$, then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{4([4]_q-1)([3]_q-1)}\left[\frac{\frac{4L}{([3]_q-1)^2}\left|([3]_q-1)^2B_1B_3 + KB_1^4 - LB_2^2 + MB_1^2B_2\right| - 4NB_1|B_2| - 2MB_1^3 - ([3]_q-1)^2B_1^2 - \frac{4N^2B_2^2}{([3]_q-1)^2} - \frac{4NMB_1^2|B_2|}{([3]_q-1)^2}}{|([3]_q-1)^2B_1B_3 + KB_1^4 - LB_2^2 + MB_1^2B_2| - 2NB_1|B_2| - MB_1^3 - NB_1^2}\right],$$

where

$$K = [3]_q - [4]_q, L = ([4]_q-1)([2]_q-1), M = \frac{([3]_q-1)([2]_q+[3]_q-2) - 2([4]_q-1)([2]_q-1)}{[2]_q-1)} \text{ and}$$
$$N = ([3]_q-1)^2 - 2([4]_q-1)([2]_q-1). \tag{7}$$

Proof. Since $f \in S_q^*(\varphi)$, there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ in $\mathcal{U}$ such that

$$\frac{z\partial_q f(z)}{f(z)} = \varphi(w(z)). \tag{8}$$

Define the function p_1 by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1z + c_2z^2 + \ldots,$$

or, equivalently

$$w(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left(c_1z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)z^3 + \ldots\right) \tag{9}$$

Then p_1 is analytic in $\mathcal{U}$ with $p_1(0) = 0$ and has a positive real part in $\mathcal{U}$. By using(9) together with (6), it is evident that

$$\varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1 + \frac{1}{2}B_1c_1z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right)z^2 + \ldots \tag{10}$$

By (8) we have

$$\frac{z\partial_q f(z)}{f(z)} = 1 + \frac{1}{2}B_1c_1z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right)z^2 + \ldots, \tag{11}$$

so that

$$\left(\frac{z + [2]_qa_2z^2 + [3]_qa_3z^3 + \ldots}{z + a_2z^2 + a_3z^3 + \ldots}\right) = 1 + \frac{1}{2}B_1c_1z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right)z^2 + \ldots$$

which implies the equality

$$z + [2]_qa_2z^2 + [3]_qa_3z^3 + \cdots = z + \left(\frac{1}{2}B_1c_1 + a_2\right)z^2 + \left(\frac{1}{2}B_1c_1a_2 + a_3 + \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right)z^3 + \ldots,$$

Equating the coefficients on both sides we have

$$a_2 = \frac{B_1c_1}{2([2]_q-1)},$$
$$a_3 = \frac{1}{4([3]_q-1)([2]_q-1)}\left((B_1^2 - ([2]_q-1)B_1 + ([2]_q-1)B_2)c_1^2 + 2([2]_q-1)B_1c_2\right),$$
$$a_4 = \frac{1}{8([4]_q-1)([3]_q-1)([2]_q-1)}\left[\left(-2([3]_q-1)([2]_q-1)B_2 + ([3]_q-1)([2]_q-1)B_1 + B_1^3 - ([2]_q+[3]_q-2)B_1^2\right)c_1^3\right.$$
$$+ \left(([2]_q+[3]_q-2)B_1B_2 + ([3]_q-1)([2]_q-1)B_3\right)c_1^3 + 4([3]_q-1)([2]_q-1)B_1c_3$$

$$+2\left(([2]_q+[3]_q-2)B_1^2-2([3]_q-1)([2]_q-1)B_1+2([3]_q-1)([2]_q-1)B_2\right)c_1c_2\right].$$

Therefore

$$\begin{aligned}a_2a_4-a_3^2=&\frac{B_1}{16([4]_q-1)([3]_q-1)([2]_q-1)}\left[\frac{c_1^4}{[3]_q-1}\left(KB_1^3+NB_1-NB_2+([3]_q-1)^2B_3-L(B_2^2/B_1)-MB_1^2+MB_1B_2\right)\right.\\&\left.+\frac{2c_1^2c_2}{([3]_q-1)}\left(MB_1^2+2NB_2-2NB_1\right)+4([3]_q-1)B_1c_1c_3-\frac{4L}{([3]_q-1)}B_1c_2^2\right].\end{aligned}$$

where K, L, M and N are given by (7). Let

$$\begin{aligned}&d_1=4([3]_q-1)B_1,\ d_2=\frac{2}{([3]_q-1)}\left(MB_1^2+2NB_2-2NB_1\right),\ d_3=-\frac{4L}{([3]_q-1)}B_1,\\&d_4=\frac{1}{([3]_q-1)}\left(NB_1^3+KB_1-2KB_2+QB_3-M\frac{B_2^2}{B_1}-LB_1^2+LB_1B_2\right)\end{aligned}\tag{12}$$

and $T=\frac{B_1}{16([4]_q-1)([3]_q-1)([2]_q-1)}$. Then

$$\left|a_2a_4-a_3^2\right|=T\left|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4\right|.\tag{13}$$

Since the function $p(e^{i\theta}z)(\theta\in\mathbb{R})$ is in the class $\mathcal{P}$ for any $p\in\mathcal{P}$, there is no loss of generality in assuming $c_1>0$. Write $c_1=c, c\in[0,2]$. Substituting the values of c_2 and c_3 respectively from (4) and (5) in (13), we obtain

$$|a_2a_4-a_3^2|=\frac{T}{4}\left|c^4\left(d_1+2d_2+d_3+4d_4\right)+2xc^2(4-c^2)\left(d_1+d_2+d_3\right)+(4-c^2)x^2\left(-d_1c^2+d_3(4-c^2)\right)+2d_1c(4-c^2)(1-|x|^2z)\right|.$$

Replacing $|x|$ by μ and substituting the values of d_1, d_2, d_3 and d_4 from (12) yield

$$\begin{aligned}|a_2a_4-a_3^2|\le&\frac{T}{4}\left[\frac{c^4}{[3]_q-1}\left|4KB_1^3+4([3]_q-1)^2B_3-4L(B_2^2/B_1)+4LB_1B_2\right|+\frac{4\mu c^2(4-c^2)}{([3]_q-1)}\left(MB_1^2+2N|B_2|\right)\right.\\&\left.+\frac{4\mu^2(4-c^2)}{([3]_q-1)}\left(NB_1c^2+4LB_1+8([3]_q-1)^2B_1c(4-c^2)(1-\mu^2)\right)\right].\\=&T\left[\frac{c^4}{4([3]_q-1)}\left|4KB_1^3+4([3]_q-1)^2B_3-4L(B_2^2/B_1)+4LB_1B_2\right|+2([3]_q-1)B_1c(4-c^2)\right.\\&\left.+\frac{\mu c^2(4-c^2)}{([3]_q-1)}\left(MB_1^2+2N|B_2|\right)+\frac{\mu^2(4-c^2)}{([3]_q-1)}B_1\left(Nc^2-2([3]_q-1)^2c+4L\right)\right].\\\equiv&F(c,\mu).\end{aligned}\tag{14}$$

Note that for $(c,\mu)\in[0,2]\times[0,1]$, differentiating $F(c,\mu)$ in(14) partially with respect to μ yields

$$\frac{\partial F}{\partial\mu}=T\left[\frac{c^2(4-c^2}{([3]_q-1)})\left(MB_1^2+2N|B_2|\right)+\frac{2\mu(4-c^2)B_1}{([3]_q-1)}\left(Nc^2-2([3]_q-1)^2c+4L\right)\right].\tag{15}$$

Then, for $0<\mu<1$, $\quad 0<q<1$ and for any fixed c with $0<c<2$, it is clear from (15) that $\frac{\partial F}{\partial\mu}>0$, that is, $F(c,\mu)$ is an increasing function of μ. Hence, for fixed $c\in[0,2]$, the maximum of $F(c,\mu)$ occurs at $\mu=1$, and

$$\max F(c,\mu)=F(c,1)\equiv G(c).$$

Also note that

$$G(c)=T\left[\frac{c^4}{4([3]_q-1)}\left(\left|4KB_1^3+4([3]_q-1)^2B_3-4L(B_2^2/B_1)+4LB_1B_2\right|-8N|B_2|-4MB_1^2-4NB_1\right)\right.$$

$$+\frac{4c^2}{([3]_q-1)}\left(2NB_2+MB_1^2+RB_1\right)+\frac{16L}{([3]_q-1)}B_1\Big].$$

where $R=\left(([3]_q-1)^2-2([4]_q-1)([2]_q-1)\right)$. Let

$$\begin{aligned} X &= \frac{1}{4([3]_q-1)}\left(\left|4KB_1^3+4([3]_q-1)^2B_3-4L(B_2^2/B_1)+4LB_1B_2\right|-8N|B_2|-4MB_1^2-4NB_1\right) \\ Y &= \frac{4}{([3]_q-1)}\left(2NB_2+MB_1^2+RB_1\right), \qquad (16) \\ Z &= \frac{16L}{([3]_q-1)}B_1. \end{aligned}$$

Since

$$\max(Xt^2+Yt+Z)=\begin{cases} Z, & Y\le 0,\ X\le \frac{-Y}{4}; \\ 16X+4Y+Z, & Y\ge 0,\ X\ge \frac{-Y}{8} \text{ or } Y\le 0, X\ge \frac{-Y}{4}; \\ \frac{4XZ-Y^2}{4X}, & Y>0,\ X\le \frac{-y}{8}; \end{cases} \qquad (17)$$

where $0\le t\le 4$. Then we have

$$|a_2a_4-a_3^2|\le \frac{B_1}{16([4]_q-1)([3]_q-1)([2]_q-1)}\begin{cases} Z, & Y\le 0,\ X\le \frac{-Y}{4}; \\ 16X+4Y+Z, & Y\ge 0,\ X\ge \frac{-Y}{8} \text{ or } Y\le 0, X\ge \frac{-Y}{4}; \\ \frac{4XZ-Y^2}{4X}, & Y>0,\ X\le \frac{-y}{8}; \end{cases}$$

where X, Y and Z are given by (16). □

Remark 2.3.

- *As $q\to 1$ Theorem 2.2 reduces to Theorem 1 in [8].*
- *As $q\to 1$ and $B_1=B_2=B_3=2$, Theorem 2.2 reduces to Theorem 3.1 in [7].*

Definition 2.4. *Let $\varphi: U\to\mathbb{C}$ be analytic, and let $\varphi(z)$ be given as in (6). The class $K_q(\varphi)$ of q-convex functions with respect to φ consists of functions f satisfying the subordination*

$$\frac{\partial_q\left(z\partial_q f(z)\right)}{\partial_q f(z)}\prec\varphi(z).$$

Theorem 2.5. *Let the function $f\in K_q(\varphi)$ be given by (1).*

(1). If B_1, B_2 and B_3 satisfy the conditions $MB_1^2+2N|B_2|-2NB_1\le 0$, $\left|[2]_q[3]_qB_1B_3+MB_1^2B_2-KB_1^4-[4]_qB_2^2\right|-[4]_qB_1^2\le 0$, then the second Hankel determinant satisfies

$$|a_2a_4-a_3^2|\le\frac{B_1^2}{[3]_q([2]_q)^2([4]_q-1)([2]_q-1)}.$$

(2). If B_1, B_2 and B_3 satisfy the conditions $MB_1^2+2N|B_2|-2NB_1\ge 0$, $2\left|[2]_q[3]_qB_1B_3+MB_1^2B_2-KB_1^4-[4]_qB_2^2\right|-MB_1^3-2NB_1|B_2|-[3]_q[2]_qB_1^2\ge 0$, or the conditions $MB_1^2+2N|B_2|-2NB_1\le 0$, $\left|[2]_q[3]_qB_1B_3+MB_1^2B_2-KB_1^4-[4]_qB_2^2\right|-[4]_qB_1^2\ge 0$, then the second Hankel determinant satisfies

$$|a_2a_4-a_3^2|\le\frac{B_1}{([4]_q!)([2]_q)([4]_q-1)([2]_q-1)}\left|[2]_q[3]_qB_1B_3+MB_1^2B_2-KB_1^4-[4]_qB_2^2\right|.$$

(3). If B_1, B_2 and B_3 satisfy the conditions $MB_1^2 + 2N|B_2| - 2NB_1 > 0$, $2\left|[2]_q[3]_qB_1B_3 + MB_1^2B_2 - KB_1^4 - [4]_qB_2^2\right| - MB_1^3 - 2NB_1|B_2| - [3]_q[2]_qB_1^2 \leq 0$, *then the second Hankel determinant satisfies*

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{4[4]_q![2]_q([4]_q-1)([2]_q-1)}\left[\frac{\begin{array}{r}4[4]_q\left|[3]_q!B_1B_3 + MB_1^2B_2 + KB_1^4 - [4]_qB_2^2\right| - 2[3]_q!MB_1^3 - 4[3]_q!NB_1|B_2| \\ -([3]_q!)^2B_1^2 - M^2B_1^4 - 4MNB_1^2|B_2| - 4N^2B_2^2\end{array}}{\left|[3]_q!B_1B_3+MB_1^2B_2+KB_1^4-[4]_qB_2^2\right|-MB_1^3-2N|B_2|-NB_1^2}\right],$$

where

$$K = \frac{[3]_q! - [4]_q[2]_q}{[2]_q}, \quad M = \frac{[3]_q!(1+[2]_q) - 2[4]_q[2]_q}{[2]_q}, \quad N = [3]_q! - [4]_q. \tag{18}$$

Proof. Since $f \in K_q(\varphi)$, there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ in $\mathcal{U}$ such that

$$\frac{\partial_q\left(z\partial_q f(z)\right)}{\partial_q f(z)} = \varphi(w(z)) = 1 + \frac{1}{2}B_1c_1z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right)z^2 + \ldots \tag{19}$$

so that

$$\left(\frac{([2]_q)^2a_2z + ([3]_q)^2a_3z^2 + ([4]_q)^2a_4z^3 + \ldots}{1 + [2]_qa_2z + [3]_qa_3z^2 + [4]_qa_4z^3 + \ldots}\right) = 1 + \frac{1}{2}B_1c_1z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right)z^2 + \ldots$$

which implies the equality

$$\begin{aligned}([2]_q)^2a_2z + ([3]_q)^2a_3z^2 + ([4]_q)^2a_4z^3 + \ldots = {} & \left(1 + \frac{1}{2}B_1c_1\right)z + \left(\frac{1}{2}a_2B_1c_1 + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right)\right)z^2 \\ & + \left(\frac{1}{2}[3]_qa_3B_1c_1 + [2]_qa_2\left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right)\right. \\ & \left. + \left(B_1\left(\frac{c_3}{2} - \frac{c_1c_2}{2} + \frac{c_1^3}{8}\right) + B_2c_1\left(\frac{c_2}{2} - \frac{c_1^2}{4}\right) + \frac{B_3c_1^3}{8}\right)\right)z^3 + \ldots\end{aligned}$$

Equating the coefficients on both sides we have

$$\begin{aligned}a_2 &= \frac{B_1c_1}{2[2]_q([2]_q-1)}, \\ a_3 &= \frac{1}{4[3]_q([3]_q-1)}\left(\left(B_1^2 - B_1 + B_2\right)c_1^2 + 2B_1c_2\right), \\ a_4 &= \frac{1}{8[2]_q[4]_q([4]_q-1)}\left[\left(-2[2]_qB_2 + [2]_qB_1 + B_1^3 - (1+[2]_q)B_1^2 + (1+[2]_q)B_1B_2 + [2]_qB_3\right)c_1^3\right. \\ &\quad \left. +2\left((1+[2]_q)B_1^2 - 2[2]_qB_2[2]_qB_2\right)c_1c_2 + 4[2]_qB_1c_3\right].\end{aligned}$$

Therefore

$$\begin{aligned}a_2a_4 - a_3^2 = {} & \frac{B_1}{16([2]_q)^2[4]_q([4]_q-1)([2]_q-1)}\left[\frac{c_1^4}{[3]_q}\left(-2NB_2 + NB_1 + LB_1^3 - MB_1^2 + MB_1B_2 + [2]_q[3]_qB_3 - [4]_q\frac{B_2^2}{B_1}\right)\right. \\ & \left. + \frac{2c_1^2c_2}{[3]_q}\left(MB_1^2 - 2NB_1 + 2NB_2\right) + 4[2]_qB_1c_1c_3 - 4\frac{[4]_q}{[3]_q}B_1c_2^2\right].\end{aligned}$$

where $L = \frac{[3]_q! - [4]_q[2]_q}{[2]_q}$ and K, M, N are given by (18). By writing

$$\begin{aligned}& d_1 = 4[2]_qB_1, \; d_2 = 2\left(MB_1^2 - 2NB_1 + 2NB_2\right), \; d_3 = 4\frac{[4]_q}{[3]_q}B_1, \\ & d_4 = \left(-2NB_2 + NB_1 + LB_1^3 - MB_1^2 + MB_1B_2 + [2]_q[3]_qB_3 - [4]_q\frac{B_2^2}{B_1}\right), \; T = \frac{B_1}{16([2]_q)^2[4]_q([4]_q-1)([2]_q-1)}\end{aligned} \tag{20}$$

we have

$$\left|a_2a_4-a_3^2\right| = T\left|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4\right| \tag{21}$$

Since the function $p(e^{i\theta}z)(\theta\in\mathbb{R})$ is in the class $\mathcal{P}$ for any $p\in\mathcal{P}$, there is no loss of generality in assuming $c_1>0$. Write $c_1=c, c\in[0,2]$. Substituting the values of c_2 and c_3 respectively from (4) and (5) in (21), we obtain

$$|a_2a_4-a_3^2| = \frac{T}{4}\left|c^4\left(d_1+2d_2+d_3+4d_4\right)+2xc^2(4-c^2)\left(d_1+d_2+d_3\right)+(4-c^2)x^2\left(-d_1c^2+d_3(4-c^2)\right)+2d_1c(4-c^2)(1-|x|^2z)\right|.$$

Replacing $|x|$ by μ and substituting the values of d_1,d_2,d_3 and d_4 from (20) yield

$$\begin{aligned}|a_2a_4-a_3^2| &\le \frac{T}{4}\left[\frac{c_1^4}{[3]_q}\left|4LB_1^3+4MB_1B_2+4([3]_q!)B_3-4[4]_q(B_2^2/B_1)\right|+\frac{2\mu c^2(4-c^2)}{[3]_q}\left(2MB_1^2+4N|B_2|\right)\right.\\ &\left.+\frac{\mu^2(4-c^2)}{[3]_q}\left(4NB_1c^2+16[4]_qB_1\right)+8([3]_q!)B_1c(4-c^2)(1-\mu^2)\right]. \qquad (22)\\ &= T\left[\frac{c_1^4}{[3]_q}\left|4LB_1^3+4MB_1B_2+4([3]_q!)B_3-4[4]_q(B_2^2/B_1)\right|+2[2]_qB_1c(4-c^2)+\frac{2\mu c^2(4-c^2)}{[3]_q}\left(MB_1^2+2N|B_2|\right)\right. \qquad (23)\\ &\left.+\frac{\mu^2(4-c^2)B_1}{[3]_q}\left(Nc^2-2([3]_q!)c+4[4]_q\right)\right].\\ &\equiv F(c,\mu), \qquad (24)\end{aligned}$$

where L, M and N are given by (18). Again, differentiating $F(c,\mu)$ in(24) partially with respect to μ yields

$$\frac{\partial F}{\partial \mu} = T\left[\frac{c^2}{[3]q}(4-c^2)\left(MB_1^2+2N|B_2|\right)+\frac{2\mu(4-c^2)B_1}{[3]_q}\left(Nc^2-2([3]_q!)c+4[4]_q\right)\right]. \tag{25}$$

Then, for $0<\mu<1$, $0<q<1$ and for any fixed c with $0<c<2$, it is clear from (25) that $\frac{\partial F}{\partial \mu}>0$, that is, $F(c,\mu)$ is an increasing function of μ. Hence, for fixed $c\in[0,2]$, the maximum of $F(c,\mu)$ occurs at $\mu=1$, and

$$\max F(c,\mu) = F(c,1)\equiv G(c).$$

Also note that

$$G(c) = T\left[\tfrac{c^4}{[3]_q}\left(\left|4LB_1^3+4MB_1B_2+4([3]_q!)B_3-4[4]_q(B_2^2/B_1)\right|-MB_1^2-2N|B_2|-NB_1\right)+\tfrac{4c^2}{[3]_q}\left(MB_1^2+2N|B_2|-RB_1\right)+16\tfrac{[4]_q}{[3]_q}B_1\right],$$

where $R=\frac{2[4]_q-[3]_q!}{[3]_q}$, L, M and N are given by (18). Let

$$\begin{aligned}X &= \frac{1}{[3]_q}\left(\left|4LB_1^3+4MB_1B_2+4([3]_q!)B_3-4[4]_q(B_2^2/B_1)\right|-MB_1^2-2N|B_2|-NB_1\right),\\ Y &= \frac{4}{[3]_q}\left(MB_1^2+2N|B_2|-RB_1\right), \qquad (26)\\ Z &= 16\frac{[4]_q}{[3]_q}B_1.\end{aligned}$$

By using (17) we get

$$|a_2a_4-a_3^2| \le \frac{B_1}{16([2]_q)^2[4]_q([4]_q-1)([2]_q-1)}\begin{cases} Z, & Y\le 0,\ X\le \frac{-Y}{4};\\ 16X+4Y+Z, & Y\ge 0,\ X\ge\frac{-Y}{8} \text{ or } Y\le 0, X\ge \frac{-Y}{4};\\ \frac{4XZ-Y^2}{4X}, & Y>0,\ X\le\frac{-y}{8};\end{cases}$$

where X, Y and Z are given by (26). □

Remark 2.6.

- *As $q\to 1$ Theorem 2.5 reduces to Theorem 2 in [8].*
- *As $q\to 1$ for the choice $\varphi(z)=((1+z)/(1-z))$, Theorem 2.5 reduces to Theorem 3.2 in [7].*

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