Int. J. Math. And Appl., 6(1–E)(2018), 847–852
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

# On Certain Janowski Symmetrical Functions with Negative Coefficients

**K. Renuka Devi[1,*], Fuad. S. Al Sarari[1] and S. Latha[1]**

1 Department of Studies in Mathematics, University of Mysore, Manasagangotri, Mysore, India.

**Abstract:** In this paper we study a class $\mathcal{S}^{(j,k)}(A,B)$ of analytic functions with negative coefficients defined by using notions of Janowski symmetrical functions. We derive necessary and sufficient conditions, extreme point results and integral means inequalities for functions belonging to this class.

**MSC:** 30C45.

**Keywords:** Janowski functions, Subordination, Extreme points, $(j,k)$-symmetric points.



## 1. Introduction

Let $\mathcal{A}$ denote the class of functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

that are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \ \ and \ \ |z| < 1\}$, and suppose $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all functions that are univalent in $\mathcal{U}$. Also, let $\Omega$ be the family of functions $w$, analytic in $\mathcal{U}$ and satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathcal{U}$. If $f$ and $g$ are analytic in $\mathcal{U}$, we say that a function $f$ is subordinate to a function $g$ in $\mathcal{U}$, if there exists a function $w \in \Omega$ such that $f(z) = g(w(z))$, and we denote this by $f \prec g$. If $g$ is univalent in $\mathcal{U}$ then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$. The Hadamard product or convolution of two functions $f$ and $g$ in $\mathcal{A}$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathcal{U}.$$

Using the principle of the subordination we define the class $\mathcal{P}$ of functions with positive real part.

**Definition 1.1** ([9]). *Let $\mathcal{P}$ denote the class of analytic functions of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ defined on $\mathcal{U}$ and satisfying $p(0) = 1$, $\operatorname{Re} p(z) > 0$, $z \in \mathcal{U}$.*

* *E-mail: renukk84@gmail.com*

Any function $p$ in $\mathcal{P}$ has the representation $p(z) = \dfrac{1+w(z)}{1-w(z)}$, where $w \in \Omega$ and

$$\Omega = \{w \in \mathcal{A} : w(0) = 0, |w(z)| < 1\}. \tag{2}$$

The class of functions with positive real part $\mathcal{P}$ plays a crucial role in geometric function theory. Its significance can be seen from the fact that simple subclasses like class of starlike $\mathcal{S}^*$, class of convex functions $\mathcal{C}$, class of starlike functions with respect to symmetric points have been defined by using the concept of class of functions with positive real part.

**Definition 1.2** ([2]). *Let $\mathcal{P}[A, B]$, with $-1 \leq B < A \leq 1$, denote the class of analytic function $p$ defined on $\mathcal{U}$ with the representation $p(z) = \dfrac{1+Aw(z)}{1+Bw(z)}$, $z \in \mathcal{U}$, where $w \in \Omega$.*

**Remark 1.3.** *$p \in \mathcal{P}[A, B]$ if and only if $p(z) \prec \dfrac{1+Az}{1+Bz}$.*

In order to define a new class of Janowski symmetrical functions with negative coefficients defined in the open unit disk $\mathcal{U}$, we first recall the notion of $k$-fold symmetric functions defined in $k$-fold symmetric domain, where $k$ is any positive integer. A domain $\mathcal{D}$ is said to be $k$-fold symmetric if a rotation of $\mathcal{D}$ about the origin through an angle $\frac{2\pi}{k}$ carries $\mathcal{D}$ onto itself. A function $f$ is said to be $k$-fold symmetric in $\mathcal{D}$ if for every $z$ in $\mathcal{D}$ we have

$$f\left(e^{\frac{2\pi i}{k}} z\right) = e^{\frac{2\pi i}{k}} f(z), \; z \in \mathcal{D}.$$

The family of all $k$-fold symmetric functions is denoted by $\mathcal{S}^k$, and for $k = 2$ we get class of odd univalent functions. In 1995, Liczberski and Polubinski [3] constructed the theory of $(j, k)$-symmetrical functions for $j = 0, 1, 2, \ldots, k-1)$ and $(k = 2, 3, \ldots$. If $\mathcal{D}$ is $k$-fold symmetric domain and $j$ any integer, then a function $f : \mathcal{D} \to \mathbb{C}$ is called $(j, k)$-symmetrical if for each $z \in \mathcal{D}$, $f(\varepsilon z) = \varepsilon^j f(z)$. We note that the $(j, k)$-symmetrical functions is a generalization of the notions of even, odd, and $k$-symmetrical functions.

The theory of $(j, k)$-symmetrical functions has many interesting applications; for instance, in the investigation of the set of fixed points of mappings, for the estimation of the absolute value of some integrals, and for obtaining some results of the type of Cartan's uniqueness theorem for holomorphic mappings, see [3]. Denote the family of all $(j, k)$-symmetrical functions by $\mathcal{S}^{(j,k)}$. We observe that , $\mathcal{S}^{(0,2)}$, $\mathcal{S}^{(1,2)}$ and $\mathcal{S}^{(1,k)}$ are the classes of even, odd and $k$-symmetric functions respectively. We have the following decomposition theorem:

**Theorem 1.4** ([3]). *For every mapping $f : \mathcal{U} \mapsto \mathbb{C}$, and a $k$-fold symmetric set $\mathcal{U}$, there exists exactly one sequence of $(j, k)$-symmetrical functions $f_{j,k}$ such that*

$$f(z) = \sum_{j=0}^{k-1} f_{j,k}(z),$$

*where*

$$f_{j,k}(z) = \frac{1}{k} \sum_{v=0}^{k-1} \varepsilon^{-vj} f\left(\varepsilon^v z\right), \; z \in \mathcal{U}. \tag{3}$$

**Remark 1.5.** *Equivalently, (3) may be written as*

$$f_{j,k}(z) = \sum_{n=1}^{\infty} \delta_{n,j} a_n z^n, \quad a_1 = 1, \tag{4}$$

*where*

$$\delta_{n,j} = \frac{1}{k} \sum_{v=0}^{k-1} \varepsilon^{(n-j)v} = \begin{cases} 1, & n = lk + j; \\ 0, & n \neq lk + j; \end{cases}, \quad (l \in \mathbb{N}, \; k = 1, 2, \ldots, \; j = 0, 1, 2, \ldots, k-1). \tag{5}$$

Al Sarari and Latha [4] introduced and studied the classes $\mathcal{S}^{(j,k)}(A,B)$ and $\mathcal{K}^{(j,k)}(A,B)$ which are starlike and convex with respect to $(j,k)$-symmetric points. For more details about the classes with $(j,k)$-symmetrical functions see [6–8].

**Definition 1.6.** *A function $f \in \mathcal{A}$ is said to belongs to the class $\mathcal{S}^{(j,k)}(A,B)$, with $-1 \leq B < A \leq 1$ if*

$$\frac{zf'(z)}{f_{j,k}(z)} \prec \frac{1+Az}{1+Bz},$$

*where $f_{j,k}$ are defined by (3).*

New denote by $\mathcal{T}$ for the class of functions $f$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0), \tag{6}$$

that are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \ \ \textit{and} \ \ |z| < 1\}$. We denote by $\mathcal{ST}^{(j,k)}(A,B)$, the class obtained by taking the intersection of $\mathcal{S}^{(j,k)}(A,B)$ with $\mathcal{T}$,

$$\mathcal{ST}^{(j,k)}(A,B) = \mathcal{S}^{(j,k)}(A,B) \cap \mathcal{T}.$$

In this paper, we derive a necessary and sufficient condition, results regarding extreme points, and integral means inequalities for this new class.

## 2. Main Results

We need the following lemma.

**Lemma 2.1** ([11])**.** *If $f$ and $g$ are analytic functions in $\mathcal{U}$ with $f \prec g$, then*

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\beta} \leq \int_0^{2\pi} \left|g(re^{i\theta})\right|^{\beta},$$

*where $\beta > 0, z = re^{i\theta}$ and $0 < r < 1$.*

**Theorem 2.2.** *Let $f(z) = z - \sum_{n=2}^{\infty} a_n z^n, (a_n \geq 0)$, be analytic function in $\mathcal{U}$, then $\mathcal{ST}^{(j,k)}(A,B)$ if and only if*

$$\sum_{n=2}^{\infty} \{(n - \delta_{n,j}) + |Bn - A\delta_{n,j}|\} \, a_n \leq (A - B), \tag{7}$$

*for $-1 \leq B < A \leq 1$ and $B \geq \frac{A}{2}$. The result is sharp.*

*Proof.* Assume the inequality (7) holds and let $|z| = 1$, it needs to show that the values satisfies the condition

$$\left|\frac{zf'(z) - f_{j,k}(z)}{Af_{j,k}(z) - Bzf'(z)}\right| \leq 1, \tag{8}$$

We have

$$\left|\frac{zf'(z) - f_{j,k}(z)}{Af_{j,k}(z) - Bzf'(z)}\right| = \left|\frac{\sum\limits_{n=2}^{\infty}(n-\delta_{n,j})a_n z^{n-1}}{(A-B) - \sum\limits_{n=2}^{\infty}(Bn - A\delta_{n,j})a_n z^{n-1}}\right|$$

$$\leq \frac{\sum_{n=2}^{\infty}(n-\delta_{n,j})a_n|z|^{n-1}}{(A-B)-\sum_{n=2}^{\infty}|Bn-A\delta_{n,j}a_n|z|^{n-1}}$$
$$\leq \frac{\sum_{n=2}^{\infty}(n-\delta_{n,j})a_n}{(A-B)-\sum_{n=2}^{\infty}|Bn-A\delta_{n,j}|a_n}$$

This last expression is bounded above by 1, which implies that $f \in \mathcal{ST}^{(j,k)}(A,B)$.

Conversely, assume that $f \in \mathcal{ST}^{(j,k)}(A,B)$, then

$$\Re\left\{\frac{zf'(z)-f_{j,k}(z)}{Af_{j,k}(z)-Bzf'(z)}\right\} \leq \left|\frac{zf'(z)-f_{j,k}(z)}{Af_{j,k}(z)-Bzf'(z)}\right| \leq 1,$$

so or

$$\Re\left\{\frac{zf'(z)-f_{j,k}(z)}{Af_{j,k}(z)-Bzf'(z)}\right\} \leq 1, \tag{9}$$
$$\Re\left\{\frac{\sum_{n=2}^{\infty}(n-\delta_{n,j})a_n z^{n-1}}{(B-A)-\sum_{n=2}^{\infty}(Bn-A\delta_{n,j})a_n z^{n-1}}\right\} \leq 1,$$

Choose values of $z$ on the real axis so that $\frac{zf'(z)-f_{j,k}(z)}{Af_{j,k}(z)-Bzf'(z)}$, is real. Upon clearing the denominator in (9) and letting $z \to 1^-$ through real values, we obtain

$$\sum_{n=2}^{\infty}(n-\delta_{n,j})a_n \leq (A-B)-\sum_{n=2}^{\infty}|Bn-A\delta_{n,j}|a_n,$$

which gives (7). The coefficient inequality (7) is sharp for the analytic function

$$g(z) = z - \frac{A-B}{(n-\delta_{n,j})+|Bn-A\delta_{n,j}|}z^n,$$

where $-1 \leq B < A, B \geq \frac{A}{2}$ and $\delta_{n,j}$ is defined by (5). □

**Corollary 2.3.** *Let $f \in \mathcal{ST}^{(j,k)}(A,B)$, then*

$$a_n \leq \frac{A-B}{(n-\delta_{n,j})+|Bn-A\delta_{n,j}|}, \qquad n \geq 2. \tag{10}$$

Next we derive certain results about extreme points of the class $\mathcal{ST}^{(j,k)}(A,B)$

**Theorem 2.4.** *Let $f_1(z) = z$, and*

$$f_n(z) = z - \frac{A-B}{(n-\delta_{n,j})+|Bn-A\delta_{n,j}|}z^n \tag{11}$$

*then, $f(z) \in \mathcal{ST}^{(j,k)}(A,B)$ if and only if it be expressed in the following form*

$$f(z) = \sum_{n=1}^{\infty}\lambda_n f_n(z), \qquad \lambda_n \geq 0, \ \sum_{n=1}^{\infty}\lambda_n = 1. \tag{12}$$

*Proof.* Suppose that

$$f(z) = \sum_{n=1}^{\infty}\lambda_n f_n(z)$$

$$= \lambda_1 z - \sum_{n=2}^{\infty} \lambda_n \frac{A-B}{(n-\delta_{n,j}) + |Bn - A\delta_{n,j}|} z^n.$$
$$= z - \sum_{n=2}^{\infty} a_n z^n,$$

where

$$a_n = \frac{A-B}{(n-\delta_{n,j}) + |Bn - A\delta_{n,j}|}, (n \geq 2). \tag{13}$$

It follows from (13) that

$$\sum_{n=2}^{\infty} \left[\lambda_n \frac{A-B}{(n-\delta_{n,j}) + |Bn - A\delta_{n,j}|}\right] \cdot \left[\frac{(n-\delta_{n,j}) + |Bn - A\delta_{n,j}|}{(A-B)}\right] = 1 - \lambda_1 \leq 1$$

Thus by Theorem 2.2, we get $f \in \mathcal{ST}^{(j,k)}(A,B)$.

Conversely suppose that $f \in \mathcal{ST}^{(j,k)}(A,B)$, by Corollary 2.3, we have

$$a_n \leq \frac{A-B}{(n-\delta_{n,j}) + |Bn - A\delta_{n,j}|}, \qquad n \geq 2,$$

setting

$$\lambda_n = \frac{A-B}{(n-\delta_{n,j}) + |Bn - A\delta_{n,j}|} a_n, \qquad \lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n.$$

We have

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z),$$

which completes the proof. □

**Theorem 2.5.** *Let the functions* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0$ *and* $g(z) = z - \sum_{n=2}^{\infty} b_n z^n, b_n \geq 0$ *be in the class* $\mathcal{ST}^{(j,k)}(A,B)$. *Then for* $0 \leq \zeta \leq 1$,

$$h(z) = (1-\zeta)f(z) + \zeta g(z) = z - \sum_{n=2}^{\infty} c_n z^n, (c_n \geq 0),$$

*is in the class* $\mathcal{ST}^{(j,k)}(A,B)$.

*Proof.* Suppose that $f(z), g(z) \in \mathcal{ST}^{(j,k)}(A,B)$. From Theorem 2.2 we have

$$\sum_{n=2}^{\infty} (n-\delta_{n,j}) + |Bn - A\delta_{n,j}| a_n \leq (A-B),$$

and

$$\sum_{n=2}^{\infty} (n-\delta_{n,j}) + |Bn - A\delta_{n,j}| b_n \leq (A-B).$$

We can see that

$$\begin{aligned}
\sum_{n=2}^{\infty} (n-\delta_{n,j}) + |Bn - A\delta_{n,j}| c_n &= \sum_{n=2}^{\infty} (n-\delta_{n,j}) + |Bn - A\delta_{n,j}| [(1-\zeta)a_n + \zeta b_n] \\
&= (1-\zeta)\left\{\sum_{n=2}^{\infty} (n-\delta_{n,j}) + |Bn - A\delta_{n,j}| a_n\right\} + \zeta \left\{\sum_{n=2}^{\infty} (n-\delta_{n,j}) + |Bn - A\delta_{n,j}| b_n\right\} \\
&\leq (1-\zeta)(A-B) + \zeta(A-B) \\
&= (A-B)
\end{aligned}$$

which completes the proof. □

**Theorem 2.6.** *Let $\beta > 0$. If $f \in \mathcal{ST}^{(j,k)}(A,B)$, then for $z = re^{i\theta}, 0 < r < 1$, we have*

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^\beta d\theta \le \int_0^{2\pi} \left|g(re^{i\theta})\right|^\beta d\theta,$$

*where*

$$g(z) = z - \frac{A-B}{(2-\delta_{2,j}) + |2B - A\delta_{2,j}|} z^2. \tag{14}$$

*Proof.* Let $f(z)$ is defined by (6), and $g(z)$ is given by (14). We need to prove that

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} a_n z^{n-1}\right|^\beta d\theta \le \int_0^{2\pi} \left|1 - \frac{A-B}{(2-\delta_{2,j}) + |2B - A\delta_{2,j}|} z\right|^\beta d\theta.$$

By Lemma 2.1, it suffices to show that

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{A-B}{(2-\delta_{2,j}) + |2B - A\delta_{2,j}|} z,$$

setting

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{A-B}{(2-\delta_{2,j}) + |2B - A\delta_{2,j}|} w(z). \tag{15}$$

From (15) and Theorem 2.2, we obtain

$$\begin{aligned} |w(z)| &= \left|\sum_{n=2}^{\infty} \frac{(2-\delta_{2,j}) + |2B - A\delta_{2,j}|}{A-B} a_n z^{n-1}\right| \\ &\le |z| \sum_{n=2}^{\infty} \frac{(n-\delta_{n,j}) + |nB - A\delta_{n,j}|}{A-B} a_n \le 1. \end{aligned}$$

This completes the proof of the theorem. □

## References


[1] H.Silverman, *Univalent functions with negative coefficients*, Proceedings of the American Mathematical Society, 51(1)(1975), 109-116.

[2] W.Janowski, *Some extremal problems for certain families of analytic functions*, I. Ann. Polon. Math., 28(1973), 297-326.

[3] P.Liczberski and J.Polubinski, *On $(j,k)$-symmtrical functions*, Mathematica Bohemica, 120(1995), 13-25.

[4] F. Alsarari and S. Latha, *A few results on functions that are Janowski starlike related to $(j,k)$-symmetric opints*, Octo. Math. Maga., 21(2)(2013), 556-563.

[5] F. Alsarari and S. Latha, *A note on coefficient inequalities for $(j,i)$-symmetrical functions with conic regions*, Bull. Int. Math. Vit. Ins., 6(2016), 77-87.

[6] F. Alsarari and S. Latha, *A note on functions defined with related to $(j,k)$-symmetric points*, International Journal of Mathematical Archive, 6(8)(2015), 1-6.

[7] F. Alsarari and S. Latha, *A note on coefficient inequalities for symmetrical functions with conic regions*, An. Univ. Oradea, fasc. Mat., 23(1)(2016), 67-75.

[8] F. Alsarari and S. Latha, *On symmetrical functions with bounded boundary rotation*, J. Math. Comput Sci., 4(3)(2014), 494-502.

[9] P.L.Duren, *Univalent Functions*, Springer-Verlag, (1983).

[10] A.W.Goodman, *Univalent functions and nonanalytic curves*, Proc. Amer. Math. Soc., 8(1957), 598601.

[11] J.E.Littlewood, *On inequalities in the theory of functions*, Proc. London Math. Soc., 23(1925), 481519.