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On Commutativity of Assosymmetric Rings With Some Identities

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Abstract: In this paper we present some results on assosymmetric rings satisfying some identities. A non-associative ring is defined as a nilpotent if there exists $k \geq 0$ such that any product having k elements is zero. A ring is solvable if the chain of sub rings $S \supseteq S^2 \supseteq (S^2)^2 \supseteq \dots$ reaches zero in a finite number of steps. While solvable associative rings are obviously nilpotent, solvable alternative rings need not be nilpotent [1].

Keywords: Assosymmetric rings, Non-associative ring, Solvable associative rings, commutative ring.

1. Introduction

An assosymmetric ring is one which satisfying condition $(x, y, z) = (P(x), P(y), P(z))$ for every permutation P of x, y, z. These rings are introduced by Kleinfeld [2]. He derived that an assosymmetric ring having characteristics different from 2 and 3 was either associative or it had a nonzero ideal whose square was zero.

Throughout this paper, A denotes an assosymmetric ring having characteristic different from 2 and 3. The main purpose of this note is to show that A is solvable if and only if A is nilpotent and let A' denote the ring generated by the right and left multiplication operators R_x and L_x, x belongs to A. In section 1, we discuss solvable assosymmetric rings. We see that a 2-divisible solvable ring with each associator in the nucleus is nilpotent and hence a solvable 2-and 3-divisible assosymmetric ring is nilpotent. Consequently a 2-and 3-divisible assosymmetric nil ring with descending chain condition on right ideals is nilpotent. Using these results, we establish the Wedderburn Principal theorem for assosymmetric algebras. Rings satisfying the identity $x(yz) = y(xz)$ are called strongly Novikov and rings satisfying $(w, x, yz) = y(w, x, z)$ are called weakly Novikov. Weakly Novikov rings are a subclass of associative rings where as strongly Novikov rings are not. In section 2 we find that the square of every element of an assosymmetric ring R is in the nucleus. Using this we prove that the non-zero idempotent e in a prime assosymmetric ring R, is the identity element of R. Many sufficient conditions are known under which a given ring becomes commutative. Notable among them are some given by Jacobson, Kaplansky and Herstein. In all these results, they take the ring to be associative. In 1968 Johnson, Outcalt and Yaqub [3] proved that if a non-associative ring R with unity, satisfying the identity $(ab)^2 = a^2b^2\ \forall\ a, b \in R$, then R is commutative. In section 3 we prove that an assosymmetric ring satisfying $((ba)^2, b) = 0$, $(b, ba - a^2b^2) = 0$.

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2. Solvable Assosymmetric Rings

There is a well known theorem for alternative algebras called Wedderburn Principal Theorem [10]. We now establish an analogous of this theorem for assosymmetric algebras. Obviously solvable associative rings are nilpotent, but solvable alternative rings are not nilpotent [1]. If A is a 2-and 3-divisible assosymmetric ring, Pokrass and Rodabaugh [7] proved that A is solvable if and only if A is nilpotent. From here on A will denote a 2-and 3-divisible assosymmetric ring. In the first approach [10], we consider A^1, the ring generated by the right and left multiplication operators R_x, and L_x, where $x \in A$. Then we say that A is right nilpotent (of index n) if for some fixed n, $R_{X_1} \dots R_{X_n} = 0$ for all x_i. Similarly we define A to be left nilpotent. It is easy to show that all nilpotent rings are right nilpotent and that all right nilpotent rings are solvable. The following indentities which obtain in A^1 are used.

$$RyLx = LxRy - RyRx + Ryx \tag{1}$$

$$LyLx = Lxy - RxRy + Rxy \tag{2}$$

$$RxRy = Rxy - Ryx + RyRx \tag{3}$$

$$0 = RyRzRwRx - RyzRwRx - RyRzRwx + RyzRwx \tag{4}$$

Identities (1) and (2) are both equivalent to the law $(x, y, z) = (z, x, y)$. Identity (3) is equivalent to $(x, y, z) = (x, z, y)$. Identity (4) is a restatement of the equation $((x, y, z), K, u) = 0$.

Lemma 2.1. *Every product $S_{X_1}S_{X_2} \dots S_{X_K}$ in A^1 may be written as a sum of terms of the form $LL \dots LRR \dots R$, where the number of R's appearing in each term is atleast as great as the number of R's in $S_{X_1}S_{X_2} \dots S_{X_K}$.*

Proof. Let $T = S_{X_1}S_{X_2} \dots S_{X_K}$. We define $D(S_{X_i})$ to be 0 if $S = R$, and if $S = L$ define $D(Sx_i)$ to be the number of S's preceding it. Finally let $D(T) = \sum D(Sx_i)$, and we will call this the degree of T. We induct on $D(T)$. If $D(T) = 0$ there is nothing to prove. Assume the lemma for each product of degree less than $m = D(T) > 0$. We assume that T begins with an R,

$$T = R_{X_1} \dots R_{X_{i-1}} L_{X_i} S_{X_{i+1}} \dots S_{X_K} \text{ say}$$

Using (1) this becomes.

$$(R_{X_1} \dots L_{X_i} R_{X_{i-1}} S_{X_{i+1}} \dots S_{X_K}) - (R_{X_1} \dots R_{X_{i-1}} R_{X_i} S_{X_{i+1}} \dots) + (R_{X_1} \dots R_{X_{i-1}} L_{X_i} S_{X_{i+1}} \dots S_{X_K}),$$

a sum of three terms, each of degree less than m, and each having atleast as many R's as T has. By induction we are done. □

From Lemma 2.1 and a symmetric argument we get.

Lemma 2.2. *If A is right nilpotent of index n then any product involving atleast n R's is zero. If A is left nilpotent of index m then any product involving m L's is zero.*

Lemma 2.3. *If A is both left and right nilpotent then A is nilpotent.*

Proof. We assume that index of right nilpotence is n and the index of left nilpotence is m. To show that A is nilpotent it is sufficient to show that A^1 is nilpotent [10]. However any product of n + m elements in A^1 is a sum of terms each involving atleast n R's or m L's. By Lemma 2.2 each term is zero. □

Lemma 2.4. *If A is right nilpotent, then A is nilpotent.*

Proof. By Lemma 2.3, we need only show that A is also left nilpotent. Let us say the index of right nil potence of A is $n-1$, so that $((x_1x_2)x_3)\dots x_n = 0$ for any n elements in A. Consider the Equation (2),

$$L_{X_3}L_{X_1} = L_{X_1X_2} - R_{X_1}R_{X_2} + R_{X_1X_2}$$

Left multiplication by L_{X_3} shows that

$$L_{X_3}L_{X_2}L_{X_1} = L_{X_3}L_{X_1}L_{X_2} - L_{X_3}R_{X_1}R_{X_2} + L_{X_2}R_{X_1X_2}$$

Using (2) again, now on the term $L_{X_3}L_{X_1X_2}$ gives

$$L_{X_3}L_{X_2}L_{X_1} = L_{(X_1X_2)X_3} - R_{X_1X_2}R_{X_3} + R_{(X_1X_2)X_3} - L_{X_3}R_{X_1}R_{X_2} + L_{X_3}R_{X_1X_2}$$

Repeating, we multiply this last equation by L_{X_4} and applying (2) to the term $L_{X_4}L_{(X_1X_2)X_3}$. Continuing, this process, we get

$$L_{X_n}\dots L_{X_2}L_{X_1} = L_{((X_1X_2)\dots)X_n} + \sum T_i$$

Where each T_i is a term containing atleast one R. Since A is right nilpotent,

$$L_{X_n}\dots L_{X_2}L_{X_1} = \sum T_i.$$

This shows that any product of n^2. L's is a sum of terms each containing atleast n R's. By right nilpotency and Lemma 2.2 each term is zero, and so A is left nilpotent. □

We remark that Lemma 2.4 actually holds for rings satisfying only the law $(x, y, z) = (z, x, y)$, since the only identities used are (1) and (2). Consequently, right nilpotent alternative rings are nilpotent.

Theorem 2.5. *Let A be a 2- and 3-divisible solvable assosymmetric ring. Then A is nilpotent.*

Proof. By Lemma 2.4, we need only show that A is right nilpotent.Consider the ideal $B = A^2$. Since B is solvable of lesser length than A, we assume that B is nilpotent. Then we know that B^1 is also nilpotent [10]. Next let us denote the subring in A^1 generated by $\{R_{X_i}/X_i \in A\}$ by $\hat{A}$. Identity (4) shows

$$R_Y R_Z R_W R_X = R_{YZ} R_W R_X mod \hat{A}B^1$$

Equation (3) shows

$$RyzRwRx = R(yz)wRx - Rw(yz)Rx + RwRyzRx$$

Using (3) again on each of the three terms on the right hand side of the last equation will give us $R_{YZ}R_WR_X \in \hat{A}B^1 + B$. This implies that, $R_YR_ZR_WR_X \in AB^1 + B^1$ or $(\hat{A})^4 \subseteq \hat{A}B^1 + B^1$. So that $(\hat{A})^5 \subseteq \hat{A}B^1$. Finally, an induction argument shows that $(\hat{A})^{4i+1} \subseteq \hat{A}(B^1)$ for each i. This means, since B^1 is nilpotent, that A is nilpotent. Hence, A is right nilpotent. □

Usually the concept of a nilring is reserved for power-associative rings. However we shall define A to be nil if each subring generated by a single element is nilpotent.

Corollary 2.6. *Let A be a 2- and 3-divisible assosymmetric nil ring with descending chain condition on right ideals. Then A is nilpotent.*

Proof. Let V be the ideal generated by all associators (a, b, c) it is shown in [4] that $v^2 = 0$. Since A/V is an associative nilring with descending chain condition on right ideals, it is well known that A/V is solvable. The solvability of V and A/V now generates that A is solvable. By Theorem 2.5, A is nilpotent. □

Theorem 2.7. *Let R be a 2-divisible ring with each associator in the nucleus. Then if R is solvable, then R is nilpotent.*

Proof. We know that in any ring R, $V = (R, R, R) + (R, R, R)R$ is an ideal. First we show that $V \subseteq N$. Using the Teichmuller identity and the fact that $(R, R, R) \subseteq N$, we get,

$$\begin{aligned}
(a, b, c)(x, y, z) &= (a, b, c(x, y, z)) \\
&= -(a, b, (c, x, y)z) \\
&= -(a, b(c, x, y), z) \\
&= (a, (b, c, x)y, z) \\
&= -(a, (b, c, x)y, z) \\
&= -((a, b, c)x, y, z) \\
&= -(a, b, c)(x, y, z)
\end{aligned}$$

By the divisibility assumption all the above expressions became 0. This shows $(R, R, R)R \subseteq N$. So $V \subseteq N$. Now if R is solvable, then R/V is a solvable associative ring and therefore nilpotent. Hence, $R^t \subseteq V \subseteq N$ for some t. Also V is associative, so $V^n = 0$. Now we apply the Lemma 2.4 taking $S = R$ and $m = t$. Then there is an I for which $R^I \subseteq (R^t)^n \subseteq V^n = 0$. This shows that R is nilpotent. □

We define R to be nil if each subring generated by a single element is nilpotent.

Corollary 2.8. *Let R be a 2- and 3-divisible assosymmetric subring with descending chain condition on right ideals. Then R is nilpotent.*

Proof. Let V be the ideal generated by all associators. Since R/V is an associative subring with descending chain condition on right ideals, R/V is solvable. The solvability of V and R/V shows that R is solvable. By the above theorem R is nilpotent. □

Using these results, let us now prove Wedderburn Prinicpal theorem for assosymmetirc algebras, which is analogous of this theorem for assosymmetric algebras [10].

Theorem 2.9 (Wedderburn Principal Theorem for Assosymmetric Algebra)**.** *Let V be a finite dimensional assosymmetric algebra over a 2- and 3-divisible field F with radical V. If U/V is separable, then $U = T \oplus V$, where T is a sub-algebra of U and T is isomorphic to U/V.*

Proof. It is sufficient to prove the existence of T isomorphic to U/V Since the theorem is trivial unless $V \neq 0$, and since V is solvable, we have proper inclusions in the series

$$V = V^{(1)} \supseteq V^{(2)} \supseteq \cdots \supseteq V^{(r)} = 0.$$

Also $V^2(=V^{(2)})$ is an ideal of T. For a in U and x, y in V imply

$$\begin{aligned} a(xy) &= (ax)y - (a,x,y) \\ &= (ax,y) - (y,a,x) \\ &= (ax,y) - (ya)x + y(ax) \end{aligned}$$

is in V^2, since V is an ideal. Hence V^2 is a left ideal of U. Reciprocally, V^2 is a right ideal of U. □

The same inductive argument based on the dimension of U which is used for associative algebras suffices to reduce the proof of the theorem to the case $V^2 = 0$. The remaining steps are those of the proof of alternative algebras [10]. From the results above and in [4], one can see that the assosymmetric identities are powerful identities. In the presence of these theorems, one would expect the Wedderburn Principal theorem, to be proved in short order. In [9] the idempotent lifting theorem is proved under fairly general conditions. The problem with assosymmetric not associative rings is that they are not power-associative.

Example 2.10. *Let A be an algebra over F spanned by e, n with $e^2 = e + n$, $ne = n$, $en = n^2 = 0$. Then $N = \{\beta_n/\beta \in F\}$ is the radical of A and A/N is isomorphic to F. The ring A is assosymmetric. However, if $e + \beta_n = (e + \beta_n)^2 = e + n + \beta_n$ then $\beta + 1 = \beta$. Thus, there are no-zero idempotent in A. The Wedderburn Principal theorem and the idempotent lifting theorem both fail for A. the fact that A is only two dimensional would indicate that no meaningful results in these directions could be expected.*

3. Assosymmetric Rings with Weak Novikov Identity

Right alternative rings satisfying the weak Novikov identities are studied in [5] and it is shown that the square of every element of the ring is in the nucleus. Paul [6] proved that if R is a prime non-associative ring satisfying $(a,b,c) = (a,c,b)$ and with commutators in the left nucleus, then a non-zero idempotent e is the identity element of R if and only if e belongs to the nucleus. Now we prove that in a non-associative 2- and 3-divisible prime assosymmetric ring R satisfying the weak Novikov identity, the square of every element of R is in the nucleus and the non-zero idempotent e in R is the identity element of R. Following identities hold in an arbitrary ring

$$(da,b,c) - (d,ab,c) + (d,a,bc) = d(a,b,c) + (d,a,b)c \tag{5}$$

$$f(d,a,b,c) = (da,b,c) - a(d,b,c) - (a,b,c)d,$$

$$(a,b,c) + (b,c,a) + (c,a,b) = (ab,c) + (bc,a) + (ca,b) \tag{6}$$

$$\text{and}\ \ (ab,c) - a(b,c) - (a,c)b = (a,b,c) - (a,c,b) + (c,a,b) \tag{7}$$

Putting $c = a$ in (7) gives

$$(ab,a) + a(a,b) = (a,b,a) \tag{8}$$

In any assosymmetric ring (7) becomes

$$(ab,c) - a(b,c) - (a,c)b = (a,b,c) \tag{9}$$

It is proved that in [4] a 2- and 3-divisible assosymmetric ring R the following identities hold for all d,a,b,c,t in R : $f(d,a,b,c) = 0$, that is,

$$(da,b,c) = a(d,b,c) + (a,b,c)d \tag{10}$$

$$((d,a),b,c)=0 \tag{11}$$

$$((d,a,b),c,e)=0 \tag{12}$$

That is, every commutator and associator is in the nucleus N. Suppose that $n \in N$. Then with $d=n$ in (5) we get $(na,b,c)=n(a,b,c)$. Combining this with (11) yields

$$(na,b,c)=n(a,b,c)=(an,b,c) \tag{13}$$

From (11), we have

$$(R,N) \subseteq N \tag{14}$$

Lemma 3.1. *If R is a non-associative 2- and 3-divisible prime assosymmetric ring, then all commutators are in the center.*

Proof. By forming associators on each side of (8) and using that (11) Gives

$$((a,b,a),r,s)=(a(a,b),r,s)=((a,b)a,r,s)$$

Using (5) and (11), we have $((a,b)a,r,s)=(a,b)(a,r,s)$. We conclude that $((a,b,a),r,s)=(a,b)(a,r,s)$. By linearising this (replacing a by $a+t$), we obtain

$$((a,b,t)+(t,b,a),r,s)=(a,b)(t,r,s)+(t,b)(a,r,s).$$

If we substitute a commutator v for t, we see that $(v,b)(a,r,s)=0$. This can be restated as $((R,R),R)(R,R,R)=0$. But now the ideal generated by commutators $((R,R),R)$ (which can be characterized as all sums of double commutators plus right multiples of double commutators, because of (11) annihilates the associator ideal. Since R is prime and not associative, we conclude that

$$((R,R),R)=0 \tag{15}$$

Thus the commutators are in the center. By forming the commutators of each side of (6) with w and using (15) it follows that $3((a,b,c),d)=0$. Thus $((a,b,c),d)=0$. □

Theorem 3.2. *If R is a non-associative 2- and 3-divisible assosymmetric ring satisfying weak Novikov identity*

$$(d,a,bc)=b(d,a,c) \tag{16}$$

Then a^2 is in the nucleus N.

Proof. By taking $d=a$ in (10), we get

$$(a^2,b,c)=a(a,b,c)+(a,b,c)a \tag{17}$$

In an assosymmetric ring we have $(a^2,b,c)=(b,c,a^2)$. On the other hand (16) implies that

$$(b,c,a^2)=a(b,c,a)=a(a,b,c).$$

Thus from (17) we must have $(a,b,c)a=0$. Since $((a,b,c),a)=0$ and $(a,b,c)a=0$, we have $a(a,b,c)=0$, so that using (17), we get $(a^2,b,c)=0$. Therefore a^2 is in the nucleus N. □

Theorem 3.3. *If R is a non-associative 2- and 3-divisible prime assosymmetric ring satisfying the weak Novikov identity, then non-zero idempotent e in R is the identity element of R.*

Proof. From Theorem 3.2, $e \in N$. By Lemma 2.1 in Rich [8], we have a decomposition $R = \oplus R_{ij}$, $i, j = 0, 1$, relative to e with $R_{ij}R_{kI} \subseteq d_{jk}R_{il}$ (d denotes the Kronecker delta). Now $R_{10} = (e, R_{10}) = -(R_{10}, e)$ and $R_{01} = (R_{01}, e)$. Since $e \in N$ and $(R, N) \subseteq N$, R_{10} and R_{01} are contained in N. Now N is a subring of R. It follows that $R_{01}R_{01} + R_{10}R_{10} \subseteq N$. This, together with the property $R_{ij}R_{kl} \subseteq d_{jk}R_{il}$, allows us to conclude that $B = R_{10}R_{01} + R_{10} + R_{01} + R_{01}R_{10}$ is an ideal of R contained in N. Let I be the associator ideal of R. We shall show that $XI = (0)$. Let $x \in X$ where X is an ideal contained in N. Then using (5) we get

$$(xa, b, c) - (x, ab, c) + (x, a, bc) = x(a, b, c) + (x, a, b)c.$$

Since B is an ideal contained in N and $b \in B$, we have,

$$\begin{aligned}(xa, b, c) &= (x, ab, c)\\ &= (x, a, bc)\\ &= (x, a, b)c\\ &= 0.\end{aligned}$$

Thus, from the above equation, we get $x(a, b, c) = 0$. Also, since $x \in N$, $x((a, b, c)d) = (x(a, b, c))d = 0$. Thus we have proved that $xI = (0)\ \forall\ x$ in X. Hence $XI = (0)$. But R is prime non-associative. This implies that $X = (0)$. So we have $R = R_{11} \oplus R_{00}$. Thus, R_{11} and R_{00} are ideals of R such that $R_{11}R_{00} = (0)$. From the primeness of R again $R_{11} = (0)$, $R_{00} = (0)$. But $0 \neq e \in R_{11}$. So that $R_{11} \neq (0)$. We must have $R_{00} = (0)$. Thus implies that e is the identity element of R. ☐

4. Assosymmetric Rings with $((ba)^2, b) = 0$

M. Janjic [2] and Ashraf and Quadri [11] proved some commutativity theorems for associative rings with conditions like a, $[a, (ab)^n, b] = 0$ and $[a, ab - b^m a^n] = 0$. In this direction we prove the commutativity of non-associative assosymmetric ring with unity 1 satisfying any one of the following identities

(i). $((ba)^2, b) = 0$

(ii). $[b, ba - a^2b^2]$

(iii). $(a, ba^2 - a^4b) = 0$

Theorem 4.1. *Unity such Let R be a 2-divisible assosymmetric ring with that $((ba)^2, b) = 0$ for all a,b in R. Then R is commutative and associative.*

Proof. Let a, b be in R. Then $[(ba)^2, b] = 0$. That is,

$$(ba)^2b - b(ba)^2 = 0 \tag{18}$$

Replacing b by $b + 1$ in (18) and using (18), we get,

$$a^2b + [(ba)a]b + a(ba)b - b(ba)a - b(a(ba)) - ba^2 = 0 \tag{19}$$

Replacing b by $b+1$ in (19), we get

$$2a^2b - 2ba^2 = 0 \tag{20}$$

Thus $a^2b - ba^2 = 0$. Replacing b by $b+1$ in (20) and using (20), we get $2ab - 2ba = 0$. Thus $ab - ba = 0$. That is, $ab = ba$. Hence R is commutative. In every assosymmetric ring we have the identity

$$(ab, c) = a(b, c) + (a, c)b + (a, b, c).$$

Since R is commutative, $(a, b, c) = 0$. Therefore R is associative. □

Similarly we can prove the following theorem.

Theorem 4.2. *Let R be a 2-divisible, assosymmetric ring with unity 1 such that $[(ba)^2, a] = 0$ for all a,b in R. Then R is commutative and associative.*

Theorem 4.3. *Let R be a 2 divisible assosymmetric ring with unity 1 such that $[b, ba - a^2b^2]$ for all a, b in R. Then R is commutative.*

Proof. Let a, b be in R. Then $[b, ba - a^2b^2] = 0$. That is,

$$b(ba - a^2b^2) - (ba - a^2b^2)b = 0 \tag{21}$$

Replacing b by $b+1$ in (21) and using (21) we get

$$2ba - ba^2 - ba - ab + a^2b = 0$$
$$ba - ab - ba^2 + a^2b = 0$$
$$ba - ba^2 - ab + a^2b = 0 \tag{22}$$

Replacing a by $a+1$ in (22) and using (22) we get

$$ba + b - ba^2 - 2ba - b + a^2b + 2ab + b - ab - b = 0 \Rightarrow 2ab - 2ba = 0$$

Thus, $ab - ba = 0$. That is $ab = ba$. Hence R is commutative □

Similarly we can prove the following theorem

Theorem 4.4. *Let R be a 2 divisible assosymmetric ring with unity 1 such that $[a, ba - a^2b^2] = 0$ for all a,b in R. Then R is commutative.*

Theorem 4.5. *Let R be a 2 and 3 divisible assosymmetric ring with unity 1 such that $(b, ba^2 - a^4b) = 0$ for all a, b in R. Then R is commutative.*

Proof. Let a, b be in R. Then $(b, ba^2 - a^4b) = 0$. That is,

$$b(ba^2 - a^4b) - (ba^2 - a^4b)b = 0 \tag{23}$$

Replacing b by $b+1$ in (23) and using (23), we get

$$b(ba^2 - a^4b) - (ba^2 - a^4b)b = 0$$

$$ba^2 - ba^4 - a^2b + a^4b = 0 \tag{24}$$

Replacing a by $a+1$ in (24) and using (24), we get

$$ab - ba + 3a^2b - 3ba^2 + 2a^3b - 2ba^3 = 0 \tag{25}$$

Replacing a by $a+1$ in (25) and using (25), we get

$$12ab - 12ba + 9a^2b - 9ba^2 = 0$$

$$4ab - 4ba + 3a^2b - 3ba^2 = 0 \tag{26}$$

Replacing a by $a+1$ in (26) and using (26), we get

$$4ab + 4b - 4ba - 4b + 3a^2b + 6ab + 3b - 3ba^2 - 6ba - 3b = 0 \Rightarrow -6ba + 6ab = 0$$

Thus $-ba + ab = 0$. That is $ab = ba$. Hence R is commutative □

Similarly we can prove the following theorem.

Theorem 4.6. *Let R be a 2 and 3 divisible assosymmetric ring with unity 1 such that $(a, ba^2 - a^4b) = 0$ for all a, b in R. Then R is commutative.*

Example 4.7. *Let* $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} / a,b,c \text{ are integers} \right\}$. *It can be verified that for any a, b, in R,* $[(ba)^2], b] = 0$ *and* $[b, ba - a^2b^2] = 0$. *However R is not commutative. Therefore unity 1 essential in the hypothesis of the theorems.*

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