

International Journal of Mathematics And its Applications

Effect of Chemical Reaction and Rotation on MHD Unsteady Flow Past an Inclined Oscillating Infinite Porous Plate Embedded in Porous Medium for Heat Generation/ Absorption with Mass Transfer and Variable Temperature

Nidhi Pandya¹ and Mohammad Suleman Quraishi^{1,*}

1 Department of Mathematics & Astronomy, University of Lucknow, Lucknow, Uttar Pradesh, India.

Abstract: In this paper we have studied Chemical reaction, rotation and oscillation effect on MHD unsteady flow over an inclined infinite porous plate embedded in porous media. The dimensionless governing equation of flow field is solved analytically by Laplace Transform technique for different values of governing flow parameters. The primary velocity profile along the plate and secondary velocity profile perpendicular direction to plate, concentration profile and temperature profile are shown through graphs for different values of flow parameters.

Keywords: MHD, porous medium, heat and mass transfer. © JS Publication.

1. Introduction

Rotation and Chemical reaction effect on MHD flows arise in many areas of engineering and applied physics. The study of such flow has application in MHD generators, chemical-engineering, nuclear reactors, geothermal energy, reservoir engineering and astrophysical studies. In nature, the assumption of the pure fluid is rather impossible. The presence of foreign mass in the fluid plays an important role in flow of fluid. Thermal diffusion effect is one of the mechanisms in the transport phenomena in which molecules are transported in a multi-component mixture driven by temperature gradient. The inverse phenomena of thermal diffusion, if multi component mixture were initially at the same temperature, are allowed to diffuse into each other, there arises a difference of temperature in the system. Sparrow and Cess [1] analyzed the effect of magnetic field on free convection heat transfer Alametal. [2] Investigated Dufour effect and Soret effect on MHD free convective heat and mass transfer flow past a vertical flat plate embedded in porous medium. Dursunkaya [3] studied Diffusion thermo and thermal diffusion effect in transient and steady natural convection from vertical surface, Postelnicu [4] analyzed the Influence of a magnetic field on heat and mass transfer by natural convection from vertical surface in porous media considering Soret and Dufour effects. Raptis [5] discussed radiation and free convection flow past a moving plate. Rajesh and Vijaya kumar verma [6] analyzed radiation and mass transfer effects on MHD free convection flow past an exponentially accelerated vertical plate with variable temperature. Shivaiah [7] analyzed chemical reaction effects on an unsteady MHD free convective flow past an infinite vertical porous plate with constant suction and heat source. Alabraba [8] investigated the inter action of

^{*} E-mail: sulemanformaths86@gmail.com

mixed convection with thermal radiation in laminar boundary flow taking into account the binary chemical reaction and P.K.Sahu [9] Effects of chemical reactions on free convection MHD flow past an exponentially accelerated infinite vertical plate through a porous medium with variable temperature and mass. We have analyzed in the year 2017 [10] Effect of Soret and Rotation on MHD unsteady flow past an inclined infinite porous plate embedded in porous medium with heat generation/ absorption and mass transfer.

The object of this work is to investigate the effect of Chemical reaction, Oscillation and Rotation on MHD unsteady flow over an inclined porous plate embedded in porous medium with heat generation/ absorption, mass transfer with variable temperature. The dimensionless governing equation of flow field is solved analytically by Laplace Transform technique for different values of governing flow parameters. The primary velocity profile in the direction of plate and secondary velocity profile perpendicular direction to plate, concentration profile and temperature profile are shown through graphs for different values of flow parameters.

2. Mathematical Analysis

An unsteady flow of a viscous incompressible electrically conducting fluid past an impulsively started infinite inclined porous plate with variable temperature and constant mass diffusion in the presence of radiation are studied. The plate is inclined at angle Φ from vertical, is embedded in porous medium. x-axis is taken along the plate and z-axis is taken normal to it. It is also considered that the radiation heat flux in x-direction is negligible in comparison to z-direction. Initially the plate and fluid are at the same temperature and concentration level is zero and plate is at rest.

Now at time t > 0, the plate is moving impulsive motion along x-direction against gravitational field with constant velocity, the plate temperature and concentration raised by one unit. A transverse magnetic field of uniform strength B is assumed normal to the direction of flow. The transversely applied magnetic field and magnetic Reynolds number are very small and hence induced magnetic field is negligible, Cowling [11]. Due to infinite length in x-direction, the flow variables are functions of z and t only. Under the usual Boussinesq approximation, governing equations for this unsteady problem are given by Continuity equation

$$\frac{\partial v}{\partial y} = 0 \tag{1}$$

Momentum equation

$$\frac{\partial u}{\partial t} - 2\Omega\nu = v\frac{\partial^2 u}{\partial z^2} + g\beta Cos\phi(T - T_\infty) - \frac{\sigma B^2 u}{\rho} - \frac{vu}{K} + g\beta^* Cos\phi(C - C_\infty)$$
(2)

$$\frac{\partial\nu}{\partial t} + 2\Omega u = v \frac{\partial^2\nu}{\partial z^2} - \frac{\sigma B^2\nu}{\rho} - \frac{v\nu}{K}$$
(3)

Energy equation:

$$\rho C_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial z^2} - Q(T - T_\infty) \tag{4}$$

Equation of continuity of mass transfer:

$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial z^2} - K_r (C - C_\infty) \tag{5}$$

Where u and v is the primary velocity and secondary velocity components along x-direction and z-direction respectively. g is the acceleration due to gravity, β is the volumetric coefficient of thermal expansion, β^* is the coefficient of volume expansion for mass transfer, ν is the kinematic viscosity, μ is viscosity, ρ is the fluid density, B is magnetic parameter, k is the permeability of porous medium, σ is the electrical conductivity of the fluid, T is the dimensional temperature, T_{∞} is temperature of fluid, C_{∞} is concentration of fluid, D is the chemical molecular diffusivity, k is the thermal conductivity of the fluid, C_p is specific heat at constant pressure, C is the dimensional concentration, T_m is mean fluid temperature. T_w and C_w are the temperature and concentration on plate.

Initial and boundary conditions are given as:

$$t \le 0; \quad u = 0, \ v = 0, \ T = 0, \ C = 0 \ \forall \ z$$

$$t > 0; \quad u = u_0 \cos \omega t, \ v = 0, \ T = T_{\infty} + (T_w - T_{\infty}) \frac{u_0^2}{\nu} t, \ C = C_w \ at \ z = 0$$

$$u \to 0, \ v \to 0, \ T \to T_{\infty}, \ C \to C_{\infty} \ as \ z \to \infty$$
(6)

In order to form a non dimensional partial differential equation, introducing following non dimensional quantities:

$$\overline{u} = \frac{u}{u_0}, \ \overline{v} = \frac{v}{u_0}, \ \overline{t} = \frac{t u_0^2}{\nu}, \ \overline{z} = \frac{z u_0}{\nu}, \ \theta = \frac{T - T_\infty}{T_w - T_\infty}, \ \overline{C} = \frac{C - C_\infty}{C_w - C_\infty}, \ G_m = \frac{\nu \ g \ \beta^* (C_w - C_\infty)}{u_0^3}, \\
G_r = \frac{\nu \ g \ \beta (T_w - T_\infty)}{u_0^3}, \ \overline{K} = \frac{u_0^2}{\nu^2} K, \ S_c = \frac{\nu}{D}, \ \mu = \rho \nu, \ P_r = \frac{\mu \ C_p}{k}, \ M = \frac{\sigma B^2 \nu}{\rho u_0^2}, \\
\overline{\Omega} = \frac{\Omega \nu}{u_0^2}, \ S = \frac{Q \nu}{\rho C_p u_0^2}, \ \overline{\omega} = \frac{\omega \nu}{u_0^2}, \ \overline{K}_r = \frac{\nu \ K_r}{u_0^2}$$
(7)

 G_r , G_m , K_r , S_c , P_r , M, Ω , ω and S are Thermal Grashof number, Solutal Grashof number, Chemical Reaction parameter, Schmidt number, Prandtl number, Magnetic parameter, rotation parameter, oscillation parameter and heat absorption/generation parameter. By the substitution of above quantities we get the non dimensional form of equation (2) to (5) as:

$$\frac{\partial \overline{u}}{\partial \overline{t}} - 2\overline{\Omega}\overline{v} = \frac{\partial^2 \overline{u}}{\partial \overline{z}^2} + G_r \theta Cos\phi + G_m \overline{C} Cos\phi - M\overline{u} - \frac{\overline{u}}{\overline{K}}$$
(8)

$$\frac{\partial \overline{v}}{\partial \overline{t}} + 2\overline{\Omega}\overline{u} = \frac{\partial^2 \overline{v}}{\partial \overline{z}^2} - M\overline{v} - \frac{\overline{v}}{\overline{K}}$$

$$\tag{9}$$

$$\frac{\partial \overline{C}}{\partial \overline{t}} = \frac{1}{S_c} \frac{\partial^2 \overline{C}}{\partial \overline{z}^2} - \overline{K}_r \overline{C}$$
(10)

$$\frac{\partial\theta}{\partial\bar{t}} = \frac{1}{P_r} \frac{\partial^2\theta}{\partial\bar{z}^2} - S\theta \tag{11}$$

The following initial and boundary condition in non dimensional form as:

$$\overline{t} \leq 0; \quad \overline{u} = 0, \quad \overline{v} = 0, \quad \theta = 0, \quad \overline{C} = 0 \quad \forall \quad \overline{z}$$

$$\overline{t} > 0; \quad \overline{u} = \cos \bar{\omega} \overline{t}, \quad \overline{v} = 0, \quad \theta = \overline{t}, \quad \overline{C} = 1 \quad at \quad \overline{z} = 0$$

$$\overline{u} \to 0, \quad \overline{v} \to 0, \quad \theta \to 0, \quad \overline{C} \to 0 \quad as \quad \overline{z} \to \infty$$

$$\left. \right\}$$

$$(12)$$

For the sake of convenience dropping the bars then we have the system as

$$\frac{\partial u}{\partial t} - 2\Omega v = \frac{\partial^2 u}{\partial z^2} + G_r \theta \cos \phi + G_m C \cos \phi - Mu - \frac{u}{K}$$
(13)

$$\frac{\partial v}{\partial t} + 2\Omega u = \frac{\partial^2 v}{\partial z^2} - Mv - \frac{v}{K}$$
(14)

$$\frac{\partial C}{\partial t} = \frac{1}{S_c} \frac{\partial^2 C}{\partial z^2} - K_r C \tag{15}$$

$$\frac{\partial\theta}{\partial t} = \frac{1}{P_r} \frac{\partial^2\theta}{\partial z^2} - S\theta \tag{16}$$

With initial and boundary conditions as:

$$t \le 0; \ u = 0, \ v = 0, \ \theta = 0, \ C = 0 \ \forall z t > 0; \ u = \cos \omega t, \ v = 0, \ \theta = t, \ C = 1 \ at \ z = 0 u \to 0, \ v \to 0, \ \theta \to 0, \ C \to 0 \ as \ z \to \infty$$
 (17)

3. Method of Solution

Equations (13) to (16) are partial differential equation with initial and boundary conditions are solved analytically by Laplace Transform Technique using initial and boundary conditions given by equation (17). For solving equations (13) and (14) together we have assumed that q = u + iv then combined form of these equations is:

$$\frac{\partial q}{\partial t} = \frac{\partial^2 q}{\partial z^2} + G_r \theta C \cos\phi + G_m C \cos\phi - Mq - 2i\Omega q - \frac{q}{K}$$
(18)

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with boundary conditions

$$t \leq 0; \ q = 0, \ \theta = 0, \ C = 0 \ \forall z$$

$$q = \cos \omega t, \ \theta = t, \ C = 1, \ at \ z = 0$$

$$q \rightarrow 0, \ \theta \rightarrow 0, \ C \rightarrow 0 \ as \ z \rightarrow \infty$$

$$\left. \right\}$$

$$(19)$$

Now taking Laplace Transform (L.T.) on both sides of equation (16) we have

$$\theta = \frac{1}{4\sqrt{S}}e^{-\sqrt{SP_r}z}\left((1 + Erf[\frac{2\sqrt{St} - z\sqrt{P_r}}{2\sqrt{t}}]\right)\left(2\sqrt{St} - z\sqrt{P_r}\right) + e^{2z\sqrt{SP_r}} * Erf[\frac{2\sqrt{St} + z\sqrt{P_r}}{2\sqrt{t}}]\left(2\sqrt{St} + z\sqrt{P_r}\right)\right)$$

On solving equation (15) we have,

$$C = \frac{1}{2}e^{-z\sqrt{K_r S_c}} \left(1 + A_{21} + e^{2z\sqrt{K_r S_c}} Erfc\left[\frac{2t\sqrt{K_r} + z\sqrt{S_c}}{2\sqrt{t}}\right]\right)$$

Now putting the values of theta and C in equation (18), we get,

$$\begin{split} q &= \frac{1}{4} e^{-it\omega} \left(Exp[-z\sqrt{N_1 - i\omega}] + Exp[z\sqrt{N_1 - i\omega}] + Exp[-z\sqrt{N_1 + i\omega} + 2i\omega t] + Exp[z\sqrt{N_1 + i\omega} + 2i\omega t] \right) \\ &- e^{-z\sqrt{N_1 - i\omega}} Erf\left[\frac{z - 2t\sqrt{N_1 - i\omega}}{2\sqrt{t}} \right] - e^{z\sqrt{N_1 - i\omega}} Erf\left[\frac{z + 2t\sqrt{N_1 - i\omega}}{2\sqrt{t}} \right] - e^{-z\sqrt{N_1 + i\omega} + 2it\omega} Erf\left[\frac{z - 2t\sqrt{N_1 + i\omega}}{2\sqrt{t}} \right] \right) \\ &+ \frac{1}{2\sqrt{\pi}(N_1 - SP_r)^2} z\cos[\varphi] G_r\sqrt{P_r} \left(-\frac{1}{2\sqrt{S}} A_2 * N_1\sqrt{\pi}(1 - A_3 + A_1 + A_3 * A_4) + \frac{1}{z\sqrt{P_r}} A_2 * \sqrt{\pi^*} \right) \\ &- (-1 - A_3 - A_1 + A_3 * A_4) - \frac{1}{z\sqrt{P_r}} A_2 * N_1\sqrt{\pi}(1 - A_3 - A_1 + A_3 * A_4) - \frac{1}{z\sqrt{P_r}} A_2 * \sqrt{\pi^*} \\ &- (-1 - A_3 - A_1 + A_3 * A_4) - \frac{1}{z\sqrt{P_r}} A_2 * N_1\sqrt{\pi}(1 - A_3 - A_1 + A_3 * A_4) - \frac{1}{z\sqrt{P_r}} A_5\sqrt{\pi}(-1 - A_6 - A_7 + A_6 * A_8) \\ &- \frac{1}{z} A_2 * \sqrt{\pi}(-1 - A_3 - A_1 + A_3 * A_4) \sqrt{P_r} + \frac{1}{2} A_2 * \sqrt{\pi} St(-1 - A_3 - A_1 + A_3 * A_4) \sqrt{P_r} \\ &+ \frac{1}{z} A_6 * \sqrt{\pi}(-1 - A_6 - A_7 + A_6 * A_8) \sqrt{P_r} + \frac{1}{2} A_2 * \sqrt{\pi} St(-1 - A_3 - A_1 + A_3 * A_4) \sqrt{P_r} \\ &+ \frac{1}{4(N_1 - SP_r)^2} z\cos[\varphi] G_r \left(\frac{2A_9 * (1 + A_{10} + A_{11})}{z} + \frac{2A_{13}}{z} * (-1 - A_{14} + A_{15} + A_{14} * A_{16}) + \frac{2 * A_9 P_r(-1 - A_{10} - A_{11} + A_{10} * A_{12})}{z} \\ &- \frac{2 * A_9 St P_r(-1 - A_{10} - A_{11} + A_{10} * A_{12})}{z} - \frac{A_9 SP_r(1 - A_{10} + A_{11} + A_{10} * A_{12})}{\sqrt{N_1}} - \frac{2A_{13} P_r(-1 - A_{10} - A_{11} + A_{10} * A_{12})}{z} \\ &- \frac{1}{2(N_1 - K_r, S_c)} e^{-\left(z\sqrt{K_r} + z\sqrt{\frac{N_r + k_r}{N_r + k_r}} + \frac{K_r \sqrt{S_r}}{C_r + k_r + k_r + k_r + k_r + k_r + k_r}}{2\sqrt{k}} \right) \sqrt{S_r} \cos[\varphi] \\ &* \left(A_{17} + A_{18} - A_{19} - A_{20} - A_{20} * A_{21} + A_{17} * Erf\left[\frac{2t\sqrt{\frac{N_1 - K_r}{-1 + S_r}} - \frac{z\sqrt{N_r - k_r + k_r + k_r + k_r + k_r}}{2\sqrt{k}} \right] \right) G_m + \cos[\varphi] G_m \left(-\frac{1}{2(N_1 - K_r, S_c)} e^{-\frac{k_1N_r}{-1 + N_r}} - \frac{2N_r \sqrt{\frac{N_r - K_r}{-1 + N_r}} - \frac{2N_r \sqrt{N_r - k_r + k$$

4. Result and Discussion

Chemical reaction and rotation effect on MHD unsteady flow over an inclined oscillating infinite porous plate embedded in porous media. The dimensionless governing equation of flow field is solved analytically by Laplace Transform technique by different values of governing flow parameters. The primary velocity profile in the direction of plate and secondary velocity profile perpendicular direction to plate, concentration profile and temperature profile are shown through graphs for different values of flow parameters. The consequences of the relevant parameters on the flow field are broke down and discussed with the help of graphs of velocity profiles, temperature profiles and concentration profiles. Figures 1, 2, 3, and 9 depicts that the primary velocity 'u' increases with increase in G_m , G_r , K, and t. Figures 5, 7, 11, 13, 15, 17, 19 and 21 show that velocity 'u' decreases with increase in Φ , Ω , K_r , P_r , M, ω , S and S_c . Figures 4, 8 and 10 for Secondary velocity 'v' described that 'v' decreases with increase in K, Ω and t. Figures 6, 12, 14, 16, 18, 20 and 22 shows that 'v' increases with increase in Φ , K_r , P_r , M, ω , S and S_c . Temperature profile θ in Figures 25 and 23 show decreases with increase in S and P_r respectively and increase with increase in time in Figure 24 according to the boundary condition. Concentration profile 'C' increases with time't' in Figure 27 and decreases with S_c and K_r in Figures 28 and 26 respectively.



Figure 1: Velocity Profile 'u' for different values of G_m



Figure 3: Velocity profile 'u' for different values of 'K'



Figure 2: Velocity Profile 'u' for different values of G_r



Figure 4: Velocity profile 'v' for different values of 'K'



Figure 5: Velocity profile 'u' for different values of ' Φ '



Figure 7: Velocity profile 'u' for different values of Ω



Figure 9: Velocity profile 'u' for different values of 't'



Figure 6: Velocity profile 'v' for different values of ' Φ '



Figure 8: Velocity profile 'v' for different values of Ω



Figure 10: Velocity profile 'v' for different values of 't'



Figure 11: Velocity profile 'u' for different values of ' K_r '



Figure 13: Velocity profile 'u' for different values of P_r



Figure 15: Velocity profile 'u' for different values of 'M'



Figure 12: Velocity profile 'u' for different values of ' K_r '



Figure 14: Velocity profile 'v' for different values of P_r



Figure 16: Velocity profile 'v' for different values of 'M'



Figure 17: Velocity profile 'u' for different values of ω



Figure 19: Velocity profile 'u' for different values of 'S'



Figure 21: Velocity profile 'u' for different values of S_c



Figure 18: Velocity profile 'v' for different values of ω



Figure 20: Velocity profile 'v' for different values of 'S'



Figure 22: Velocity profile 'v' for different values of S_c



Figure 23: Temperature profile θ for different values of P_r Figure 24: Temperature profile θ for different values of time't'



Figure 25: Temperature profile θ for different values of 'S'



Figure 26: Concentration profile 'C' for different values of K_r



1.2 1.0 0.8 $S_c = 0.66, 1.4, 2$ $K_r = 0.2, t = 1$ 0.6 C 0.4 0.2 0.0 0 1 Z

Figure 27: Concentration profile 'C' for different values of 't'

Figure 28: Concentration profile 'C' for different values of

 S_c



5. Conclusion

In this work we have concluded that Chemical reaction and rotation effects on MHD unsteady flow over an inclined oscillating infinite porous plate embedded in porous media concluded the following conclusions:

- (1). Increasing inclination angle, velocities decreases rapidly.
- (2). Increasing rotation parameter, primary velocity decreases and secondary velocity decreases rapidly.
- (3). Velocities decrease with increase in oscillation parameter first then uniformly decrease.
- (4). Primary velocity increases and secondary velocity decreases with increase in time.
- (5). Temperature increases with time.
- (6). Concentration decreases with increase in Schmidt number and Chemical reaction parameter.
- (7). Increment in Chemical reaction parameter results decrement in primary velocity and decreases secondary velocity.

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Appendix

$$\begin{split} &A_{1} = Erf\left[\sqrt{S}\sqrt{t} - \frac{z\sqrt{P_{r}}}{2\sqrt{t}}\right] \\ &A_{2} = e^{-\sqrt{SP_{r}z}}, \\ &A_{3} = e^{-2\sqrt{SP_{r}z}}, \\ &A_{4} = Erf\left[\sqrt{S}\sqrt{t} + \frac{z\sqrt{P_{r}}}{2\sqrt{t}}\right] \\ &A_{5} = Exp\left[\frac{tN_{1}}{P_{r}-1} - z\sqrt{\frac{N_{1}-S}{-1+P_{r}}}\sqrt{P_{r}} - \frac{tSP_{r}}{-1+P_{r}}\right], \\ &A_{6} = Exp\left[2z\sqrt{\frac{N_{1}-S}{P_{r}-1}}\sqrt{P_{r}}\right] \\ &A_{7} = Erf\left[\sqrt{t}\sqrt{\frac{N_{1}-S}{P_{r}-1}} - \frac{z\sqrt{P_{r}}}{2\sqrt{t}}\right], \\ &A_{8} = Erf\left[\sqrt{t}\sqrt{\frac{N_{1}-S}{P_{r}-1}} + \frac{z\sqrt{P_{r}}}{2\sqrt{t}}\right], \\ &A_{9} = Exp\left[-z\sqrt{N_{1}}\right], \\ &A_{10} = Exp\left[2z\sqrt{N_{1}}\right], \\ &A_{10} = Exp\left[2z\sqrt{N_{1}}\right], \\ &A_{11} = Erf\left[\frac{2t\sqrt{N_{1}-z}}{2\sqrt{t}}\right], \\ &A_{12} = Erf\left[\frac{2t\sqrt{N_{1}+z}}{2\sqrt{t}}\right], \\ &A_{13} = Exp\left[\frac{tN_{1}}{P_{r}-1} - z\sqrt{\frac{(N_{1}-S)P_{r}}{-1+P_{r}}} - \frac{tSP_{r}}{-1+P_{r}}\right], \\ &A_{14} = Exp\left[2z\sqrt{\frac{(N_{1}-S)P_{r}}{P_{r}-1}}\right], \\ &A_{15} = Erf\left[\frac{z-2tz\sqrt{\frac{(N_{1}-S)P_{r}}{-1+P_{r}}}}{2\sqrt{t}}\right], \\ &A_{16} = Erf\left[\frac{z+2tz\sqrt{\frac{(N_{1}-S)P_{r}}{-1+P_{r}}}}{2\sqrt{t}}\right], \\ &A_{17} = Exp\left[\frac{tN_{1}}{-1+S_{c}} + z\sqrt{K_{r}S_{c}}\right], \\ &A_{18} = Exp\left[\frac{tN_{1}}{-1+S_{c}} + z\sqrt{K_{r}S_{c}}\right], \\ &A_{19} = Exp\left[2z\sqrt{K_{r}} + z\sqrt{\frac{N_{1}-K_{r}}{-1+S_{c}}} + \frac{tK_{r}\sqrt{S_{c}}}{-1+S_{c}}\right], \\ &A_{20} = Exp\left[z\sqrt{\frac{(N_{1}-K_{r})S_{c}}{2\sqrt{t}}}\right], \\ &A_{21} = Erf\left[\frac{2t\sqrt{K_{r}} - z\sqrt{S_{c}}}{2\sqrt{t}}\right], \\ &A_{22} = Exp\left[2z\sqrt{\frac{(N_{1}-K_{r})S_{c}}{-1+S_{c}}}\right], \\ &A_{12} = M\left[\frac{2t\sqrt{K_{r}} - z\sqrt{S_{c}}}{2\sqrt{t}}\right], \\ &A_{22} = Exp\left[2z\sqrt{\frac{(N_{1}-K_{r})S_{c}}{-1+S_{c}}}\right], \\ &A_{22} = Exp\left[2z\sqrt{\frac{(N_{1}-K_{r})S_{c}}{-1+S_{c}}}\right], \\ &A_{23} = M\left[\frac{1}{2}\sqrt{\frac{V(N_{1}-K_{r})S_{c}}}{2\sqrt{t}}\right], \\ &A_{24} = Krf\left[\frac{2t\sqrt{K_{r}} - z\sqrt{S_{c}}}{2\sqrt{t}}\right], \\ &A_{25} = Exp\left[2z\sqrt{\frac{(N_{1}-K_{r})S_{c}}}{2\sqrt{t}}\right], \\ &A_{26} = Exp\left[2z\sqrt{\frac{(N_{1}-K_{r})S_{c}}}{2\sqrt{t}}\right], \\ &A_{21} = Erf\left[\frac{2t\sqrt{K_{r}} - z\sqrt{S_{c}}}{2\sqrt{t}}\right], \\ &A_{22} = Exp\left[2z\sqrt{\frac{(N_{1}-K_{r})S_{c}}}{2\sqrt{t}}\right], \\ &A_{22} = Exp\left[2z\sqrt{\frac{(N_{1}-K_{r})S_{c}}}{2\sqrt{t}}\right], \\ &A_{24} = Krf\left[\frac{2t\sqrt{K_{r}} + 2i\Omega}{2\sqrt{t}}\right], \\ &A_{25} = Krf\left[\frac{2t\sqrt{K_{r}} + 2i\Omega}{2\sqrt{t}}\right], \\ &A_{26} = Krf\left[\frac{2t\sqrt{K_{r}} + 2i\Omega}{2\sqrt{t}}\right], \\ &A_{27} = Krf\left[\frac{2t\sqrt{K_{r}} + 2i\Omega}{2\sqrt{t}}\right], \\ &A_{28} = Krf\left[\frac{2t\sqrt{K_{r}} + 2i\Omega}{2\sqrt{t}}\right],$$