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E_k-Super Vertex Magic Labeling of Graphs

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Abstract: Let G be a graph with p vertices and q edges. An E_k-super vertex magic labeling (E_k-SVML) is a bijection $f : V(G) \cup E(G) \to \{1, 2, \ldots, p + q\}$ with the property that $f(E(G)) = \{1, 2, \ldots, q\}$ and for each $v \in V(G)$, $f(v) + w_k(v) = M$ for some positive integer M. For an integer $k \geq 1$ and for $v \in V(G)$, let $w_k(v) = \sum\limits_{e \in E_k(v)} f(e)$, where $E_k(v)$ is the set of all edges which are at distance at most k from v. The graph G is said to be E_k*-regular* with regularity r if and only if $|E_k(e)| = r$ for some integer $r \geq 1$ and for all $e \in E(G)$. A graph that admits an E_k-SVML is called E_k-super vertex magic (E_k-SVM). This paper contain several properties of E_k-SVML in graphs. A necessary and sufficient condition for the existence of E_k-SVML in graphs has been obtained. Also, the magic constant for E_k-regular graphs has been obtained. Further, we establish E_2-SVML of some classes of graphs such as cycles, complement of cycles, prism graphs and a family of circulant graphs.

MSC: 05C78.

Keywords: E_k-super vertex magic labeling, E_k-regular graphs, circulant graphs.

1. Introduction

Throughout this paper, we consider only finite simple and undirected graphs. The set of vertices and edges of a graph $G(p, q)$ will be denoted by $V(G)$ and $E(G)$ respectively, $p = |V(G)|$ and $q = |E(G)|$. A general reference for graph-theoretic terminology, we follow [2]. A graph *labeling* is an assignment of integers (usually positive or non-negative integers), which assigned to vertices /or edges /or both into a set of numbers. A comprehensive survey of graph labelings is given in Gallian [1]. In 1963, Sedlàček [7] introduced the concept of magic labeling in graphs. A graph G is *magic* if the edges of G can be labeled by the numbers $\{1, 2, \ldots, q\}$ so that the sum of labels of all the edges incident with any vertex is the same ([5]).

In 2002, MacDougall et al. [3] introduced the notion of vertex magic total labeling (VMTL) in graphs. A *VMTL* of G is a bijection $f : V(G) \cup E(G) \to \{1, 2, \ldots, p + q\}$ such that for each vertex $v \in V(G)$, $f(v) + \sum\limits_{u \in N(v)} f(uv) = M$ for some positive integer M, called as the *magic constant of VMTL* of G. They studied some basic properties of vertex magic graphs and showed some families of graphs having a VMTL. In 2004, MacDougall et al. [4] defined the super vertex-magic total labeling (SVMTL) in graphs. They call a VMTL is *super* if $f(V(G)) = \{1, 2, \ldots, p\}$. In this labeling, the smallest labels are assigned to the vertices. Swaminathan and Jeyanthi [8] introduced another labeling called super vertex magic labeling (SVML). They call a VMTL is super if $f(E(G)) = \{1, 2, \ldots, q\}$. Here, the smallest labels are assigned to the edges. To avoid confusion, Marimuthu and Balakrishnan [5] called a VMTL is *E-super* if $f(E(G)) = \{1, 2, \ldots, q\}$. A graph G is called *E-super vertex magic (E-SVM)* if it admits an E-super vertex magic labeling (E-SVML).

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This paper generalize the definition of E-SVML and define a new labeling called E_k-super vertex magic labeling (E_K-SVML). For an integer $k \geq 1$ and for $v \in V(G)$, let $w_k(v) = \sum_{e \in E_k(v)} f(e)$, where $E_k(v)$ is the set of all edges which are at distance at most k from v. An E_k-SVML of G is a bijection $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ such that $f(E(G)) = \{1, 2, \ldots, q\}$ and for each $v \in V(G)$, $f(v) + w_k(v) = M$ for some positive integer M. This constant is called as the *magic constant of E_k-SVML* of G. A graph that admits an E_k-SVML is called *E_k-super vertex magic (E_k-SVM)*. Let k be an integer such that $1 \leq k \leq \text{diam}(G)$. For $e \in E(G)$, we define $E_k(e)$ as the set of all vertices which are at distance at most k from e. Note that if uv is an edge, then the vertices u and v are at distance 1 from the edge uv. The graph G is said to be *E_k-regular* with regularity r if and only if $|E_k(e)| = r$ for some integer $r \geq 1$ and for all $e \in E(G)$. Note that all nontrivial graphs are E_1-regular. Consider the following graph $G(V, E)$, with $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$.

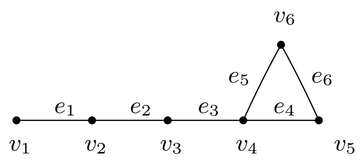


Figure 1: G

The following table gives the values of $E_k(v)$ and $E_k(e)$ when $k = 2$.

$E_2(v)$	$E_2(e)$
$E_2(v_1) = \{e_1, e_2\}$	$E_2(e_1) = \{v_1, v_2, v_3\}$
$E_2(v_2) = \{e_1, e_2, e_3\}$	$E_2(e_2) = \{v_1, v_2, v_3, v_4\}$
$E_2(v_3) = \{e_1, e_2, e_3, e_4, e_5\}$	$E_2(e_3) = \{v_2, v_3, v_4, v_5, v_6\}$
$E_2(v_4) = \{e_2, e_3, e_4, e_5, e_6\}$	$E_2(e_4) = \{v_3, v_4, v_5, v_6\}$
$E_2(v_5) = \{e_3, e_4, e_5, e_6\}$	$E_2(e_5) = \{v_3, v_4, v_5, v_6\}$
$E_2(v_6) = \{e_3, e_4, e_5, e_6\}$	$E_2(e_6) = \{v_4, v_5, v_6\}$

Table 1. $E_2(v)$ **and** $E_2(e)$ **in** G

This paper contain several properties of E_k-SVML in graphs. A necessary and sufficient condition for the existence of E_k-SVML in graphs has been obtained. Also, the magic constant for E_k-regular graphs has been obtained. Further, we establish E_2-SVML of some classes of graphs such as cycles, complement of cycles, prism graphs and a family of circulant graphs.

2. Main Section

This section will explore the basic properties of E_k-SVML. Let G be a graph of order $p(\geq 2)$. Suppose $E_k(u) = E_k(v)$ for a pair of vertices u and v $(u \neq v)$ of G. Then $f(u) + w_k(u) \neq f(v) + w_k(v)$ for any E_k-SVML f of G (since f is one to one). In this case, G does not admit E_k-SVML and hence the next result follows.

Lemma 2.1. *Let G be a graph of order $p(\geq 2)$. If $E_k(u) = E_k(v)$ for some $u, v \in V(G)$ $(u \neq v)$, then G is not E_k-SVM.*

If a graph G admits E_k-SVML, then $1 \leq k \leq \text{diam}(G)$ (If $k > \text{diam}(G)$, then $E_k(u) = E_k(v)$ for any two different vertices $u, v \in V(G)$).

Corollary 2.2. *The star graph S_n does not admit E_k-SVML for $k \geq 2$.*

Proof. Suppose there exists an E_k-SVML f on S_n. Since $\text{diam}(S_n) = 2$, S_n does not admit E_k-SVML for $k > 3$. When $k = 2$, we get $E_k(u) = E_k(v)$ for any two different vertices $u, v \in V(S_n)$. Then by Lemma 2.1, S_n does not admit E_k-SVML for $k \geq 2$. □

Theorem 2.3. *Let G be a graph and g is a bijection from $E(G)$ onto $\{1, 2, \ldots, q\}$. Then g can be extended to an E_k-SVML of G if and only if $\{w_k(u)/u \in V(G)\}$ consists of p sequential integers.*

Proof. Assume that $\{w_k(u)/u \in V(G)\}$ consists of p sequential integers. Let $t = \min\{w_k(u)/u \in V(G)\}$. Define $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ as $f(xy) = g(xy)$ for $xy \in E(G)$ and $f(x) = t+p+q-w_k(x)$. Then $f(E(G)) = \{1, 2, \ldots, q\}$ and $f(V(G)) = \{p+q, p+q-1, \ldots, q+1\}$ (since $\{w_k(x) - t : x \in V(G)\}$ is a set of sequential integers). Hence f is an E_k-SVML with magic constant $M = t + p + q$.

Conversely, suppose that g can be extended to an E_k-SVML f of G with a magic constant M. Since $f(u) + w_k(u) = M$ for every $u \in V(G)$, $\{w_k(u)/u \in V(G)\} = \{M - q - p, M - q - p + 1, \ldots, M - q - 1\}$ is a set of p sequential integers. □

Lemma 2.4. *If a graph $G(p, q)$ is E_k-SVM and E_k-regular with regularity r, then the magic constant is given by $M = q + \frac{p+1}{2} + \frac{r}{p}\frac{q(q+1)}{2}$.*

Proof. Let f be an E_k-SVML of G with the magic constant M. Then $f(E(G)) = \{1, 2, \ldots, q\}$, $f(V(G)) = \{q+1, q+2, \ldots, q+p\}$ and $M = f(v) + w_k(v)$ for all $v \in V(G)$. By summing over all $v \in V(G)$, we get

$$\begin{aligned} pM &= \sum_{v \in V(G)} f(v) + \sum_{v \in V(G)} w_k(v) \\ &= \sum_{v \in V(G)} f(v) + \sum_{v \in V(G)} \sum_{e \in E_k(v)} f(e) \\ &= (q+1) + (q+2) + \ldots + (q+p) + r \sum_{e \in E(G)} f(e) \end{aligned}$$

(since each edge is counted exactly r times in the sum $\sum_{v \in V(G)} \sum_{e \in E_k(v)} f(e)$). Thus $pM = pq + \frac{p(p+1)}{2} + r\frac{q(q+1)}{2}$ and hence $M = q + \frac{p+1}{2} + \frac{r}{p}\frac{q(q+1)}{2}$. □

Lemma 2.4 gives the magic constant only for E_k-regular graphs which admit E_k-SVML for $k \geq 1$. In 2003, Swaminathan and Jeyanthi [8] obtained the following result which gives the magic constant for all non trivial graphs which admit E-SVML.

Lemma 2.5 ([8]). *If a nontrivial graph G is super vertex magic then the magic number M is given by $M = q + \frac{p+1}{2} + \frac{q(q+1)}{p}$.*

When $k = 1$, we have $r = |E_1(e)| = 2$ for all $e \in E(G)$. The above result is a corollary of Lemma 2.4, when $k = 1$.

Lemma 2.6. *For $k \geq 2$, there is no tree which is E_k-regular and E_k-SVM.*

Proof. Let T be a tree and $\text{diam}(T) = d (\geq 3)$. Let $P : u_0 u_1 \ldots u_{d-1} u_d$ be a path of length d. Then $u_0 u_1$ and $u_{d-1} u_d$ must be pendent edges. When $k = d$, we have $E_k(u_0) = E_k(u_d)$ and hence T is not E_k-SVM. Also when $k \leq d - 1$, we have $E_k(u_1 u_2) > E_k(u_0 u_1)$ and hence T is not E_k-regular. Thus $\text{diam}(T) \leq 2$ and hence T is a star graph. Thus by Corollary 2.2, T is not E_k-SVM for $k \geq 2$. □

Theorem 2.7. *Let G be a connected E_k-regular graph with regularity r. If G is E_k-SVM, then $M \geq \frac{5p-3}{2}$ when $k = 1$ and $M \geq \frac{(p+1)(r+3)}{2} - 1$ when $k \geq 2$.*

Proof. For $k = 1$, we have $r = 2$. Since G is connected, $q \geq p - 1$. Thus by Lemma 2.4, $M \geq (p-1) + \frac{p+1}{2} + \frac{(p-1)p}{p}$ $= \frac{5p-3}{2}$ (This part is proved in [5]). Let $k \geq 2$. Suppose $q = p - 1$. Then G is a tree and by Lemma 2.6, there is no tree which is E_k-regular and E_k-SVM. If $q \geq p$, then by Lemma 2.4, $M \geq p + \frac{p+1}{2} + \frac{r}{p}\frac{p(p+1)}{2} = \frac{(p+1)(r+3)}{2} - 1$. □

Remark 2.8. The lower bounds obtained in Theorem 2.7 are sharp.

(i) The path P_5 is E-SVM and $M = \frac{5p-3}{2}$=11.

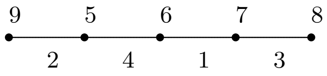


Figure 2: E-SVML of P_5

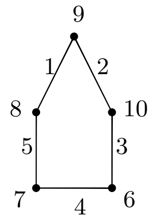


Figure 3: E_2-SVML of C_5

(ii) The cycle C_5 is E_2-regular with regularity $r = 4$ and C_5 is E_2-SVM with $M = \frac{(p+1)(r+3)}{2} - 1 = 20$.

Remark 2.9. *P_5 dose not admit E_2-SVML.*

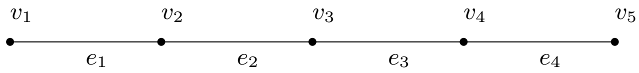


Figure 4: The graph P_5

Suppose P_5 admits an E_2-SVML, say $'f'$. Then $f(E(G)) = \{f(e_1), f(e_2), f(e_3), f(e_4)\} = \{1, 2, 3, 4\}$ and hence $w_2(v_3) = 10$. Thus by Theorem 2.3, $\{w_2(v)/v \in V(G)\} = \{10, 9, 8, 7, 6\}$. Since $w_2(v_3) = 10$, we have $w_2(v_1) = f(e_1) + f(e_2) \in \{6, 7, 8, 9\}$. Thus either $f(e_1)$ or $f(e_2)$ must be 4. In this case $w_2(v_5) \leq 5$, a contradiction.

Marimuthu and Kumar [6] proved the following result.

Theorem 2.10 ([6])**.** *Let G be a regular graph having an E-super vertex magic labeling in which the label 1 is assigned to some edge e. Then the graph $G - \{e\}$ has an E-super vertex magic labeling.*

Remark 2.11. *The above result fails in the case of E_2-SVML. For example, consider the cycle C_5. By Remark 2.8, the cycle C_5 is E_2-SVM and by Remark 2.9, $C_5 - e$ $(\cong P_5)$ is not E_2-SVM.*

3. E_2-SVML of Cycles and Prism Graphs

In this section, we identified some classes of graphs such as cycles, complement of cycles, prism graphs and a family of circulant graphs which admit E_2-SVML. Since $E_2(u)$ is same for all $u \in V(C_3)$, by Lemma 2.1, C_3 does not admit E_2-SVML.

Lemma 3.1 ([9])**.** *For any integers a and b, we have $gcd(a, b) = gcd(b, a) = gcd(\pm a, \pm b) = gcd(a, b - a) = gcd(a, b + a)$.*

Theorem 3.2. *Let $n(\geq 5)$ be an integer. Then the cycle C_n is E_2-SVM if and only if n is odd.*

Proof. Suppose there exists an E_2-SVML f of C_n. Since $|E_2(e)| = r = 4$ for all $e \in E(C_n)$, by taking $k = 2$, $p = q = n$ and $r = 4$ in Lemma 2.4, we get $M = \frac{7n+5}{2}$. Since M is an integer, n must be odd.
Conversely, assume that n is odd and $n \geq 5$. Let $V(C_n) = \{a_i/1 \leq i \leq n\}$ and $E(C_n) = \{a_i a_{i \oplus_n 1}/1 \leq i \leq n\}$, where the operation $\oplus_n$ stands for addition modulo n.
Case A: Suppose $n = 4\ell + 1$ for some integer $\ell \geq 1$. Define a function $f : V(C_n) \cup E(C_n) \to \{1, 2, \ldots, 2n\}$ as follows: $f(a_i) = n - 3 + i$ when $4 \leq i \leq n$ and $f(a_i) = 2n - 3 + i$ when $1 \leq i \leq 3$; $f(a_i a_{i \oplus_n 1}) = [(i - 1)\ell + 1] (\text{mod } n)$, where $[(i - 1)\ell + 1] (\text{mod } n)$ is the positive residue when $(i - 1)\ell + 1$ divides n.
Next we prove that $\gcd(\ell, 4\ell + 1) = 1$. By taking $b = 4\ell + 1$ and $a = \ell$ in Lemma 3.1, we get $\gcd(\ell, 4\ell + 1) = \gcd(\ell, 3\ell + 1) = \gcd(\ell, 2\ell + 1) = \gcd(\ell, \ell + 1) = \gcd(\ell, 1) = 1$. Thus ℓ is a generator for the finite cyclic group $(Z_n, \oplus_n)$ and hence $f(E(C_n)) = \{1, 2, \ldots, n\}$.

Claim 1: $w_2(a_i) = 10\ell + 8 - i$ for $4 \le i \le n$ and $w_2(a_i) = (\ell+1)6 - (i-1)$ for $1 \le i \le 3$.

Case i: Suppose $i = 4x$ for some $1 \le x \le \ell$. Now

$$\begin{aligned} w_2(a_i) &= f(a_{i-2}a_{i-1}) + f(a_{i-1}a_i) + f(a_ia_{i+1}) + f(a_{i+1}a_{i+2}) \\ &= [(i-3)\frac{n-1}{4} \oplus_n 1] + [(i-2)\frac{n-1}{4} \oplus_n 1] + [(i-1)\frac{n-1}{4} \oplus_n 1] + [(i)\frac{n-1}{4} \oplus_n 1] \\ &= [nx - x - \frac{3n}{4} + \frac{3}{4} \oplus_n 1] + [nx - x - \frac{n}{2} + \frac{1}{2} \oplus_n 1] + [nx - x - \frac{n}{4} + \frac{1}{4} \oplus_n 1] + [nx - x \oplus_n 1] \\ &= [-x - \frac{3n}{4} + \frac{3}{4} \oplus_n 1] + [-x - \frac{n}{2} + \frac{1}{2} \oplus_n 1] + [-x - \frac{n}{4} + \frac{1}{4} \oplus_n 1] + [-x \oplus_n 1]. \end{aligned}$$

Since $1 \le x \le \ell$, the above four terms (brackets) are not positive. Thus

$$w_2(a_i) = [n - x - \frac{3n}{4} + \frac{3}{4} + 1] + [n - x - \frac{n}{2} + \frac{1}{2} + 1] + [n - x - \frac{n}{4} + \frac{1}{4} + 1] + [n - x + 1].$$

Since $n = 4\ell + 1$, we get $w_2(a_i) = 10\ell + 8 - i$.

Case ii: Suppose $i = 4x + 1$ for some $1 \le x \le \ell$. In this case,

$$w_2(a_i) = [-x - \frac{n}{2} + \frac{1}{2} \oplus_n 1] + [-x - \frac{n}{4} + \frac{1}{4} \oplus_n 1] + [-x \oplus_n 1] + [-x + \frac{n}{4} - \frac{1}{4} \oplus_n 1].$$

Here the first three terms are not positive (since $1 \le x \le \ell$). Thus

$$w_2(a_i) = [n - x - \frac{n}{2} + \frac{1}{2} + 1] + [n - x - \frac{n}{4} + \frac{1}{4} + 1] + [n - x + 1] + [-x + \frac{n}{4} - \frac{1}{4} + 1] = 10\ell + 8 - i.$$

Similarly, we can show that $w_2(a_i) = 10\ell + 8 - i$ when $i = 4x + 2$ and $i = 4x + 3$ for $1 \le x \le \ell - 1$. Consider the vertex a_1.

$$\begin{aligned} w_2(a_1) &= f(a_1a_2) + f(a_2a_3) + f(a_na_1) + f(a_{n-1}a_n) \\ &= 1 + [\frac{n-1}{4} \oplus_n 1] + [(n-1)\frac{(n-1)}{4} \oplus_n 1] + [(n-2)\frac{(n-1)}{4} \oplus_n 1] \\ &= 1 + [\frac{n-1}{4} \oplus_n 1] + [(4\ell)\frac{(n-1)}{4} \oplus_n 1] + [(4\ell-1)\frac{(n-1)}{4} \oplus_n 1] \\ &= 1 + [\frac{n}{4} - \frac{1}{4} \oplus_n 1] + [-\ell \oplus_n 1] + [-\ell - \frac{n}{4} + \frac{1}{4} \oplus_n 1] \\ &= 1 + [\frac{n}{4} - \frac{1}{4} + 1] + [n - \ell + 1] + [n - \ell - \frac{n}{4} + \frac{1}{4} + 1] \ \ \text{(since the last two terms are not positive)} \\ &= 6\ell + 6. \end{aligned}$$

Similarly, we can prove $w_2(a_2) = 6\ell + 5$ and $w_2(a_3) = 6\ell + 4$.

Note that $\ell = \frac{n-1}{4}$. Thus by Claim 1, $f(a_i) + w_2(a_i) = n - 3 + i + 10l + 8 - i = \frac{7n+5}{2} = M$ for $4 \le i \le n$. Again by Claim 1, $f(a_i) + w_2(a_i) = n - 3 + i + 6l + 7 - i = \frac{7n+5}{2} = M$ for $i = 1, 2, 3$.

Case B: Suppose $n = 4\ell+3$ for some integer $\ell \ge 1$. Define $f : V(C_n) \cup E(C_n) \to \{1, 2, \ldots, 2n\}$ as follows: $f(a_i) = 2n - i$ when $1 \le i \le n-1$ and $f(a_n) = 2n$; $f(a_ia_{i\oplus_n 1}) = [(i-1)(\ell+1)+1] (\text{mod } n)$, where $[(i-1)(\ell+1)+1] (\text{mod } n)$ is the positive residue when $(i-1)(\ell+1)+1$ divides n. By Lemma 3.1, $\gcd(\ell+1, 4\ell+3) = \gcd(\ell+1, 3\ell+2) = \gcd(\ell+1, 2\ell+1) = \gcd(\ell+1, \ell) = \gcd(\ell, \ell+1) = \gcd(\ell, 1) = 1$. Hence $\ell+1$ is a generator for the finite cyclic group $(Z_n, \oplus_n)$ and hence $f(E(C_n)) = \{1, 2, \ldots, n\}$. As proved in Case A, we can prove that the above labeling is an E_2-SVML with magic constant $M = \frac{7n+5}{2}$. □

Theorem 3.3. *Let $G = \overline{C_n}$ be the complement of the cycle C_n, where $n(\ge 5)$ is an integer. Then G is E_2-SVM with the magic constant* $\frac{n^4 - 6n^3 + 15n^2 - 18n}{8}$

Proof. Define $f : V(\overline{C_n}) \cup E(\overline{C_n}) \to \{1, 2, \ldots, \frac{n^2-n}{2}\}$ as follows:
First we label the n edges $\{a_1a_3, a_2a_4, \ldots, a_na_2\}$ by $f(a_{i\oplus n-1}a_{i\oplus 1}) = i$ for $1 \le i \le n$. And the remaining $\frac{n^2-3n}{2} - n$ edges are randomly labeled with the labels $\{n+1, n+2, \ldots, \frac{n^2-3n}{2}\}$. The vertices are labeled as $f(a_i) = \frac{n^2-3n}{2} + i$. Then for each a_i with $1 \le i \le n$, we have $f(a_i) + w_2(a_i) = [\frac{n^2-3n}{2} + i] + [1 + 2 + \ldots + \frac{n^2-3n}{2} - i] = \frac{n^4-6n^3+15n^2-18n}{8}$. □

Theorem 3.4. *Let $n(\ge 3)$ be an integer. Then the prism D_n is E_2-SVM if and only if n is even.*

Proof. Suppose there exists an E_2-SVML f of D_n with the magic constant M. Since $|E_2(e)| = r = 6$ for all $e \in E(D_n)$, by taking $k = 2$, $p = 2n$, $q = 3n$ and $r = 6$ in Lemma 2.4, we get $M = \frac{35n+10}{2}$. Since M is an integer, n must be even.
Conversely, suppose n is even. Let $V(D_n) = \{a_i, b_i/1 \le i \le n\}$ and $E(D_n) = \{(a_ib_i)/1 \le i \le n\} \cup \{(a_ia_{i\oplus_n 1}), (b_ib_{i\oplus_n 1})/1 \le i \le n\}$. Define $f : V(D_n) \cup E(D_n) \to \{1, 2, \ldots, 5n\}$ as follows:
$f(a_i) = 4n + \frac{n}{2} - \frac{i-1}{2}$ if i is odd; The range is given by $\{4n+1, 4n+2, \ldots, 4n+\frac{n}{2}\}$,
$f(a_i) = 5n - (\frac{i}{2} - 2)$ if $i \ge 4\ \forall\ i$ is even; $\{4n + \frac{n}{2} + 2, 4n + \frac{n}{2} + 3, \ldots, 5n\}$,
$f(a_2) = 4n + \frac{n}{2} + 1$; $\{4n + \frac{n}{2} + 1\}$,
$f(b_i) = 3n + \frac{i+1}{2}$ if i is odd; $\{3n+1, 3n+2, \ldots, 3n+\frac{n}{2}\}$,
$f(b_i) = 3n + \frac{n}{2} + \frac{i}{2} - 1$ if $i \ge 4\ \forall\ i$ is even; $\{3n + \frac{n}{2} + 1, 3n + \frac{n}{2} + 2, \ldots, 4n-1\}$,
$f(b_2) = 4n$; $\{4n\}$,
$f(a_ib_i) = \frac{i+1}{2}$ if i is odd; $\{1, 2, \ldots, \frac{n}{2}\}$,
$f(a_ib_i) = \frac{n}{2} + \frac{i}{2}$ if i is even; $\{\frac{n}{2}+1, \frac{n}{2}+2, \ldots, n\}$,
$f(a_ia_{i\oplus_n 1}) = n + \frac{n}{2} - \frac{i-1}{2}$ if i is odd; $\{n+1, n+2, \ldots, n+\frac{n}{2}\}$,
$f(b_ib_{i\oplus_n 1}) = 2n - (\frac{i}{2} - 1)$ if i is even; $\{n + \frac{n}{2} + 1, n + \frac{n}{2} + 2, \ldots, 2n\}$,
$f(a_ia_{i\oplus_n 1}) = 2n + \frac{i}{2}$ if i is even; $\{2n+1, 2n+2, \ldots, 2n+\frac{n}{2}\}$,
$f(b_ib_{i\oplus_n 1}) = 3n - \frac{i-1}{2}$ if i is odd; $\{2n + \frac{n}{2} + 1, 2n + \frac{n}{2} + 2, \ldots, 3n\}$.
It is easily seen that f is an E_2-SVML with the magic constant $M = \frac{35n+10}{2}$. □

Let Γ be a finite group with e as the identity. A generating set of Γ is a subset A such that every element of Γ can be expressed as a product of finitely many elements of A. Assume that $e \notin A$ and $a \in A$ implies $a^{-1} \in A$ (A is called as symmetric generating set). A Cayley graph is a graph $G = (V, E)$, where $V(G) = \Gamma$ and $E(G) = \{(x, a)/x \in V(G), a \in A\}$ and it is denoted by $Cay(\Gamma, A)$. Since A is a generating set for Γ, G is a connected regular graph of degree $|A|$. When $\Gamma = Z_n$, the corresponding Cayley graph is called as a circulant graph, denoted by $Cir(n, A)$.
In Lemma 2.4, we find the magic constant of E_k-regular graphs which admit E_k-SVML. When $A = \{1, 2, n-1, n-2\}$, the circulant graph $Cir(n, A)$ is not E_2-regular. In the next Theorem, we find the magic constant of this family of circulant graphs.

Theorem 3.5. *Let $n(\ge 7)$ be an integer. Then $G = Cir(n, \{1, 2, n-1, n-2\})$ is E_2-SVM with the magic constant $16n+7$.*

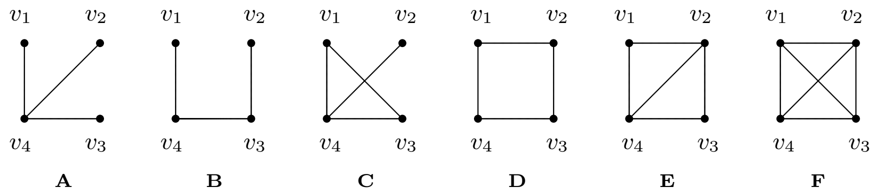
Proof. Let $V(G) = \{a_1, a_2, \ldots, a_n\}$ and $E(G) = \{a_ia_{i\oplus 1}, a_ia_{i\oplus 2}/1 \le i \le n\}$. Define $f : V(G) \cup E(G) \to \{1, 2, \ldots, 3n\}$ as follows:
$f(a_i) = 2n + i - 4$ for $5 \le i \le n$; $f(a_i) = 3n + i - 4$ for $1 \le i \le 4$; $f(a_ia_{i\oplus 1}) = i$ for $1 \le i \le n$ and $f(a_ia_{i\oplus 2}) = 2n + 1 - i$ for $1 \le i \le n$. Let $v \in V(G)$. Suppose $v = a_i$, $5 \le i \le n-2$. Then $f(a_i) + w_2(a_i) = f(a_i) + f(a_{i-3}a_{i-2}) + f(a_{i-2}a_{i-1}) + f(a_{i-1}a_i) + f(a_ia_{i\oplus 1}) + f(a_{i\oplus 1}a_{i\oplus 2}) + f(a_{i\oplus 2}a_{i\oplus 3}) + f(a_{i-4}a_{i-2}) + f(a_{i-3}a_{i-1}) + f(a_{i-2}a_i) + f(a_{i-1}a_{i\oplus 1}) + f(a_ia_{i\oplus 2}) + f(a_{i\oplus 1}a_{i\oplus 3}) + f(a_{i\oplus 2}a_{i\oplus 4}) = [2n + i - 4] + [i-3] + [i-2] + [i-1] + i + [i+1] + [i+2] + [2n + 1 - (i-4)] + [2n + 1 - (i-3)] + [2n + 1 - (i-2)] + [2n + 1 - (i-1)] + [2n + 1 - i] + [2n + 1 - (i+1)] + [2n + 1 - (i+2)] = 16n + 7$. Similarly, we can prove that $f(a_i) + w_2(a_i) = 16n + 7$ for $i = 1, 2, 3, 4, n-1, n$. □

4. Some Results on E-SVML

In this section, we obtained some results on E-SVML.

Lemma 4.1. *Any connected graph on four vertices is not E-SVM.*

Proof. Suppose there exists an E-SVML with magic constant M. All the non-isomorphic connected graphs on four vertices are given below.



Then by Lemma 2.5, $M = q + \frac{p+1}{2} + \frac{q(q+1)}{p}$. Thus for the graphs A, B, C and D, the magic constant is not an integer and hence they are not E-SVM. Suppose the graph E admits an E-SVML, say f. Then $M = 15$, $f(E(E)) = \{1,2,3,4,5\}$ and $f(V(E)) = \{6,7,8,9\}$.

Case 1: Suppose $f(v_1v_2) = 1$. Since $f(v_1)+w(v_1) = 15$, we must have $f(v_1) = 9$ and $f(v_1v_4) = 5$. Since $f(v_4)+w(v_4) = 15$, we must have $f(v_2v_4), f(v_3v_4) \in \{2,3,4\}$ and $f(v_4) \in \{6,7,8\}$ such that $f(v_4)+f(v_2v_4)+f(v_3v_4) = 10$, which is not possible. Similarly, the cases $f(v_2v_3) = 1$, $f(v_3v_4) = 1$ and $f(v_4v_1) = 1$ are not possible.

Case 2: Suppose $f(v_2v_4) = 1$. Since $5 \in f(E)$, with out loss of generality, assume that $f(v_1v_2) = 5$. Suppose $f(v_2v_3) = 2$, then $w(v_2) = w(v_4) = 8$, a contradiction by Lemma 2.1. Suppose $f(v_1v_4) = 2$, then $w(v_1) = w(v_3) = 7$, a contradiction by Lemma 2.1. Thus $f(v_3v_4)$ must be equal to 2. Since $f(v_2) + w(v_2) = 15$, we must have $f(v_2) = 6$ and $f(V_2v_3) = 3$. Thus $f(v_3)$ must be equal to 10, which is not possible. Hence the graph E is not E-SVM.

Next, we consider the graph F. Suppose the graph F admits E-SVML, say f. Then $M = 19$, $f(E(F)) = \{1,2,3,4,5,6\}$ and $f(V(F)) = \{7,8,9,10\}$ and hence $w(v) \leq 12$ for all $v \in V(F)$.

Claim : $f(v_4v_2, v_4v_3, v_4v_1) = \{2,3,4\}$ or $\{1,3,5\}$. Suppose $f(v_4v_3) = 6$ or $f(v_4v_1) = 6$, then $w(v_3) \geq 13$ or $w(v_1) \geq 13$, which is not possible. Suppose $f(v_4v_2) = 6$, then $w(v_2) \geq 13$, which is not possible. Thus any edge adjacent with v_4 must not receive the label 6. Since $f(v_4)+w(v_4) = 19$ and $f(v_4) = 10$, from the above fact, we must have $f(v_4v_2, v_4v_3, v_4v_1) = \{2,3,4\}$ or $\{1,3,5\}$. Suppose $f(v_1v_3) = 6$. Since $f(v_3) + w(v_3) = 19$, by above claim, we must have $f(v_3v_4) = 3$ and $f(v_2v_3) = 1$. Since $f(v_2)+w(v_2) = 19$, we must have $f(v_1v_2)+f(v_2v_4) = 10$ and $f(v_1v_2), f(v_2v_4) \in \{2,4,5\}$, which is not possible. Suppose $f(v_2v_3) = 6$. Since $f(v_3)+w(v_3) = 19$, by above claim, we must have $f(v_3v_4) = 3$ and $f(v_1v_3) = 1$. Since $f(v_1)+w(v_1) = 19$, we must have $f(v_1v_2) + f(v_1v_4) = 11$ and $f(v_1v_2), f(v_1v_4) \in \{2,4,5\}$, which is not possible. Suppose $f(v_1v_2) = 6$. Since $f(v_1) + w(v_1) = 19$ and $f(v_1) = 7$, we must have $f(v_1v_3, v_1v_4) = \{1,5\}$ or $\{2,4\}$, which is not possible by the above claim. Thus we proved that we cannot label any edge by the label 6, which is a contradiction to $f(E(F)) = \{1,2,3,4,5,6\}$. □

Theorem 4.2. *Let G be a (p,q) graph. If $q = p+1$, then G is not E-SVM.*

Proof. Suppose $q = p+1$. Then by Lemma 2.5, $M = p+1+\frac{p+1}{2}+\frac{(p+1)(p+2)}{p} = \frac{5p+9}{2}+\frac{2}{p}$ which is an integer only when $p = 4$. Thus by Lemma 4.1, G is not E-SVM. □

Corollary 4.3. *For $n \geq 4$, the cycle with one chord is not E-SVM.*

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