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Degrees and Degree Sequences of PAN Critical Graphs

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Abstract: A pseudo-complete coloring of a graph G is an assignment of colors to the vertices of G such that for any two distinct colors, there exist adjacent vertices having those colors. The maximum number of colors used in a pseudo-complete coloring of G is called the pseudo-achromatic number of G and is denoted by $\psi_s(G)$. A graph G is called edge critical if $\psi_s(G-e) < \psi_s(G)$ for any edge e of G. A graph G is called vertex critical if $\psi_s(G-v) < \psi_s(G)$ for every vertex v of G. These graphs are generally called as pseudo-achromatic number critical graphs (shortly as PAN Critical graphs). In this paper, we investigate the properties of these critical graphs.

Keywords: Pseudo-complete coloring, Pseudo-achromatic number, k-edge critical graph, k-vertex critical graph.

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1. Introduction

By a graph we mean a finite undirected graph without loops, multiple edges and isolated vertices. An assignment of colors to the vertices of a graph $G = (V, E)$ is called a proper coloring, if any two adjacent vertices receive distinct colors and is called a pseudo–complete coloring if for any two distinct colors, there exist adjacent vertices having those colors. A pseudo-complete proper coloring of G is called a complete coloring of G. The minimum number of colors used in a proper coloring of G is called the chromatic number of G and is denoted by $x(G)$. The maximum number of colors used in a complete coloring of G is called the achromatic number of G and is denoted by $(\psi G))$ [6]. The maximum number of colors used in a pseudo-complete coloring of G is called the achromatic number of G and is denoted by $\psi_s(G)$ [4]. Several bounds for these coloring parameters were obtained in [4–7]. A graph which admits a pseudo-complete coloring by k colors is called a k-pseudo complete colorable graph. The concept of critical graphs with respect to chromatic number, was introduced by Dirac [2, 3] in a bid to settle the four color conjecture. In [1], Sureshkumar introduced the concepts of criticality in graphs with respect to pseudo-achromatic number and obtained characterizations of edge critical graphs, critical cycles and critical paths. In this paper, we further investigate some properties of these critical graphs such as degrees, degree sequences, diameter and traversibility.

2. Critical Graphs

Definition 2.1. *A graph G is called k-edge critical if $\psi_s(G) = k$ and $\psi_s(G-e) < k$ for any edge e of G. A graph G is called k-vertex critical if $\psi_s(G) = k$ and $\psi_s(G-v) < k$ for any vertex v of G.*

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Definition 2.2. *Let G be a graph and $v \in V(G)$ be a vertex of degree d. Let n be a positive integer less than d. Then an n-splitting of v is the replacement of v by a set of n new pairwise independent vertices $\{u_i\}_{i=1}^{n}$ with deg $u_i > 1$, for all $i, 1 < i < n, \sum_{i=1}^{n} deg(u_i) = d$ and $N(\{u_i\}_{i=1}^{n}) = N(v)$, where for any subset S of $V(G), N(S)$ means the set of all neighbors of vertices in S.*

The following simple observations, which are quite useful later, follow directly from the definitions of critical graphs.

Proposition 2.3. *A graph G is k-edge critical if and only if G is k-pseudo-complete colorable and $|E(G)| = \binom{n}{2}$.*

Proposition 2.4. *Any k-edge critical graph is k-vertex critical.*

Proposition 2.5. *If G is a k-edge critical graph and II is the graph obtained from G by n-splitting a vertex of G. Then H is k-edge critical.*

Proposition 2.6. *Let G be a k-edge critical graph and H be the graph obtained from G by identifying a pair of vertices, having same color with respect to a k-pseudo-complete coloring of G. Then H is k-edge critical.*

Proposition 2.7. *If G is k-edge critical, then $G + K_n, is (n + k)$-vertex critical.*

Proposition 2.8. *Let k be an odd integer. Then, the cycle of order $\binom{k}{2}$) is k–edge critical.*

3. Degrees and Degree Sequences

Theorem 3.1. *If G is k-edge critical, then $\Delta(G) \leq k - 1$. Also, given any two positive integers in and n such that in $< n - 1$, there exists an n-edge critical graph G with $\Delta(G) = m$.*

Proof. Suppose there is a vertex v in $V(G)$ with degree at least k. Then in any k-pseudo-complete coloring of G, at least two vertices in $N[v]$ have the same color, so that G is not k-edge critical. Hence, $\Delta(G) < k - 1$. Also, given any two positive integers in and n such that $m < n - 1, m \leq n - 1$, the graph, $G = K_{1,m} UrK_2$, where$r = \binom{n}{2} - m$ is n-edge critical and $\Delta(G) = m$. □

Theorem 3.2. *Let m and k be two positive integers such that $m \leq k - 1$ and let G be any k-edge critical graph with $\Delta(G) = m$. Then, $k\{\frac{k-1}{m}\} \leq |V(G)| \leq k^2 - k - m + 1$. Also, $|V(G)| \leq k^2 - k - m + 1$ if and only if $G = K_{1,m} UrK_2$, where $r = \binom{n}{2} - m$. Where $\{n\}$ denote the smallest integer greater than or equal to n.*

Proof. Consider any k-pseudo-complete coloring of G. Then for any color c there are at least $\{\frac{k-1}{m}\}$ vertices in G with color c so that $k\{\frac{k-1}{m}\} \leq |V(G)|$. Now, let G_1 be a graph with maximum number of vertices among all the k-edge critical graphs G with $\Delta(G) = m$. Then G_1 has a component isomorphic to $K_{1,m}$ and each of the other components of G_1 is a K_2. Hence $G_1 = G = K_{1,m} UrK_2$, where $r = \binom{n}{2} - m$. Thus, $|V(G)| \leq |V(G_1)| = k^2 - k - m + 1$ and the last statement follows. □

Remark 3.3. *The following example shows that the lower bound in Theorem 3.2 is sharp. Let in and k be any two positive integers with $m \leq n - 1$. Let G be a graph obtained from K_k by p-splitting each of its vertices where $p = \{\frac{k-1}{m}\}$, such that for each vertex one of the p new vertices has degree m and all others have degrees at most m. It follows from Proposition 2.5 that G is k-edge critical. Also $\Delta(G) = m$ and $|V(G)| = k\{\frac{k-1)}{m}\}$.*

Theorem 3.4. *If G is k-edge critical and $G \neq K_K$ then $\delta(G) \neq [\frac{k-1}{2}]$. Also, given any two positive integers m and n such that $m \leq [\frac{n-1}{2}]$, there exists an n-edge critical graph G with $\delta(G) = m$.*

Proof. Suppose that $(G) > [\frac{k-1}{2}]$. Then for any pair of vertices u, v of $G, degu + degv > k-1$. Consider any k-pseudo-complete coloring of G. If u and v have same color, then u and v are non-adjacent and have no common neighbors. Hence by Proposition 2.6, the graph H obtained from G by identifying u and v is k-edge critical and $\Delta(H) > k-1$, which contradicts Theorem 3.1. Hence any two vertices of G have distinct colors, so that $G = K_k$, a contradiction. Thus $\delta(G) \leq \{\frac{k-1}{2}\}$. Given any two positive integers m and n such that $m \leq [\frac{n-1}{2}]$, let G be the graph obtained from K_n, by 2-splitting one of its vertices such that one of the new vertices has degree m and the other has degree $n-1-m$. Clearly G is n-edge critical and $\delta(G) = m$. □

Theorem 3.5. *Let m and k be two positive integers such that $m \leq \{\frac{k-1}{2}\}$ and let G be a k-edge critical graph with $\delta(G) = m$. Then, $k+1 \leq |V(G)| \leq k[\frac{k-1}{m}]$.*

Proof. Since G is k-edge critical and $m \leq \{\frac{k-1}{2}\}, |V(G)| \geq k$ and G is not complete, so that $|V(G)| \geq k+1$. Now, consider a k-pseudo-complete coloring of G. Then for a given color c, the sum of the degrees of all the vertices having color c is $k-1$. Since $\delta(G) = m$, there can be at most $[\frac{k-1}{m}]$ vertices with color c. Hence, $|V(G)| \leq k[\frac{k-1}{m}]$. □

Lemma 3.6. *If G is k-vertex critical, then $k \leq |V(G)| \leq k(k-1)$.*

Proof. Let G be a k-vertex critical graph and $v \in V(G)$. Consider a k-pseudo-complete coloring of G and c_i be the color of v. Since G is-k-vertex critical, there is a color $c_1 \neq c_2$ such that any'edge having the color $\{c1, c2\}$ is incident at v. Thus, for any vertex $v \in V(G)$, there exists a pair of distinct colors such that all the edges having that pair of colors are incident at v. Hence, $|V(G)| \leq 2\binom{k}{2} = k(k-1)$. Also, since k colors are to be represented in $G, |V(G)| \geq k$. □

Theorem 3.7. *If G is k-vertex critical, $\Delta(G) < (k–1)(k–2)$. Also given any two positive integers m and n such that $m < (n–1)(n–2)$, there exists an n-vertex critical graph G with $\Delta(G) = rn$.*

Proof. Among all k-vertex critical graphs, choose a graph, G for which $\Delta(G)$ is maximum. Let v be a vertex of G with degree $= \Delta(G)$. If there is a vertex u in G having the same color as that of v with respect to a k-pseudo-complete coloring of G, then the graph G' obtained from G by identifying u and v is a k-vertex critical graph with $\Delta(G') > \Delta(G)$, a contradiction. Hence in any k-pseudo-complete coloring of G, v is the only vertex having a particular color. Also, by the maximality of G, v is adjacent to all the vertices of $G–v$, so that $\Delta(G) = |V(G–v)|$ and $G–v$ is a (k_1)-vertex critical graph with the maximum number of vertices. Hence by Lemma 3.1, $\Delta(G) = (k–1)(k–2)$.

Now, let m and n be any two positive integers such that $m < (n-1)(n-2)$. Let G_o be the graph consisting of $n\binom{n-1}{2}$ disjoint copies of K_2. Consider an $(n–1)$-pseudo-complete coloring of G_o and let $\{1, 2, 3, ..., n–1\}$ be the colors used. For $1 < i < (n1)(n–3)$, let G_i be the graph obtained from G_{i-1}, by identifying a vertex of color $\{i/(n–3\}$, which has the highest degree among the vertices of color $\{i/(n–3\}$, with another vertex of color $\{i/n–3\}$. By Proposition 2.4, G_i is $(n–1)$-edge critical and hence by Proposition 2.5, $G_i + K_1$ is n-vertex critical. Clearly, $\Delta(G_i + K_1) = (n-1)(n-2) - i$, where $0 < i < (n1)(n–3)$. If $1 < m < n–2$, the existence of an n-vertex critical graph with $\Delta(G) = m$ follows from Theorem 3.1 and Proposition 2.2. □

Theorem 3.8. *Let $k > 2$ be an integer. A sequence $\pi = (d_i)_{i=1}^{n}$ of positive integers is potentially k-edge critical iff there exist k subsequences $\{\pi_j\}_{j=1}^{k}$ with each d_i belongs to exactly one π_j and $\sum\limits_{d_i \in \pi j} d_i = k-1$ for all $j, 1 < j < k$.*

Proof. Suppose $\pi = (d_i)_{i=1}^{n}$ is potentially k-edge critical. Let G be a k-edge critical graph with degree sequence $(d_i)_{i=1}^{n}$ Consider a k-pseudo-complete coloring of G and let $\{1, 2, \ldots, k\}$ be the colors used. For each $j, 1 < j < k$, let π_j be the subsequence of π formed by the degrees of the vertices of color j in G. Clearly, $\{\pi_j\}_{j=1}^{k}$ satisfies the required conditions.

Conversely, suppose $\pi = (d_i)_{i=1}^{n}$ is a sequence of positive integers having k subsequences, $(\pi_j)_{j=1}^{k}$ satisfying the given conditions. Let $n(j)$ denote the length of π_j and let $\pi_j = (d_{j,m})_{m=1}^{n(j)}$. Consider the complete graph, K_k and let its vertex set be $V(K_k) = \{v_1, v_2, \ldots, v_k\}$. Let G be the graph obtained from K_k, by doing an $n(j)$-splitting operation on v_i, for each $j, 1 < j < k$, such that the new $n(j)$ vertices, which replace v_i, have degrees $d_{j,1}, d_{j,2}, \ldots, d_{j.n(j)}$. By Proposition 2.3, G is k-edge critical and the degree sequence of G is π. $\square$

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