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Fixed Point Theorems in Quasi Semi linear 2-Normed Space

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Abstract: C. Park [9] introduced the term of a quasi 2-normed space. Also he proved some properties of quasi 2-norm and M. Kir and M. Acikgoz [7] elaborated the procedure for completing the quasi 2-normed space. In this paper, we introduced quasi semi linear 2-normed spaces and φ'-contraction mapping in these spaces are defined. It is investigated that under suitable conditions, φ'-contraction mapping have fixed points in quasi semi linear 2-normed spaces.

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1. Introduction

Theory of 2-Banach spaces was investigated by S. Gahler [5] and K. Iseki [6] who had proved some fixed point theorems in 2-Banach spaces. Y.J. Cho, N. Huang and X. Long proved some fixed point theorems for nonlinear mappings in 2-Banach spaces. M.S. Khan and M.D. Khan [8] worked for Involutions with fixed points in 2-Banach spaces. In this paper we have proved some fixed point theorems in quasi semi linear 2-normed space by working with φ'-contraction.

1.1. Preliminaries

Definition 1.1. *Let X be a linear space of dimension greater than 1 and let $\|.,.\|$ be a real-valued function on $X \times X$ satisfying the following conditions:*

(1). $\|x, y\| = 0$ if and only if x and y are linearly dependent.

(2). $\|x, y\| = \|y, x\|$ for all $x, y \in X$.

(3). $\|x, ay\| = |a|\|x, y\|$, a being real, for all $x, y \in X$.

(4). $\|x, y + z\| \leq \|x, y\| + \|x, z\|$, for all $x, y, z \in X$.

Then $\|.,.\|$ is called a 2-norm and the pair $(X, \|.,.\|)$ is called a linear 2-normed space.

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Definition 1.2. *A nonempty set X, together with a nonnegative function $\|.,.\| : X^3 \to R$ is called a quasi semi linear 2-normed space such that*

(1). to each pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $\|x - y, z\| \neq 0$.

(2). $\|x - y, z\| = 0$ if atleast two of x,y,z are equal.

Definition 1.3. *φ'-contraction in Quasi Semi linear 2-Normed spaces Consider the set φ', the set of all real valued functions $\varphi' : R_+^3 \to R_+$ satisfying the following properties:*

(a). $\varphi'(1,1,1) = h < 1$, where $h \in R_+$.

(b). Let $u, v \in R_+$ be such that if either $u \leq \varphi'(u, v, v)$ or $u \leq \varphi'(v, u, v)$ or $u \leq \varphi(v, v, u)$, then $u \leq kv$, for some $k \in [h, 1)$.

Definition 1.4. *A self mapping T on a quasi semi linear 2-normed space $(X, \|.,.\|)$ is called a φ'-contraction, if*

$$\|Tx - Ty, a\| \leq \varphi'[\|x - y, a\|, \|x - Tx, a\|, \|y - Ty, a\|] \;\; \forall \;\; x, y, a \in X. \tag{1}$$

Throughout this paper, $(X, \|.,.\|)$ is the quasi semi linear 2-normed space and using φ'-contraction mapping, we proved the following theorems.

2. Main Results

Theorem 2.1. *Let $(X, \|.,.\|)$ be a quasi semi linear 2-normed space and T is a φ'-contraction. If there exists a point $x_0 \in X$ such that for all $a \in X$*

$$\|x_0 - Tx_0, a\| = \inf\{\|x - Tx, a\| : x \in X\} \tag{2}$$

then T has a unique fixed point.

Proof. Suppose $x_0 \neq Tx_0$, put $x = x_0, y = Tx_0$ in (1). Therefore, $\|Tx_0 - T^2x_0, a\| \leq \varphi'[\|x_0 - Tx_0, a\|, \|x_0 - Tx_0, a\|, \|Tx_0 - T^2x_0, a\|]$. $\|Tx_0 - T^2x_0, a\| \leq k\|x_0 - Tx_0, a\|$ for some $k \in [h, 1)$ [From φ'-contraction] Since $k < 1$, we have $\|Tx_0 - T^2x_0, a\| \leq \|x_0 - Tx_0, a\|$, which is a contradiction to (2). Hence, $Tx_0 = x_0$. Therefore, x_0 is the fixed point of T.
Uniqueness: Let y_0 be another fixed point of T. That is $y_0 = Ty_0$. Now,

$$\begin{aligned}
\|x_0 - y_0, a\| = \|Tx_0 - Ty_0, a\| &\leq \varphi'(\|x_0 - y_0, a\|, \|x_0 - Tx_0, a\|, \|y_0 - Ty_0, a\|) \\
&\leq \varphi'(\|x_0 - y_0, a\|, \|x_0 - x_0, a\|, \|y_0 - y_0, a\|) \\
&\leq \varphi'(\|x_0 - y_0, a\|, 0, 0)
\end{aligned}$$

Therefore by φ'-contraction, we obtain $\|x_0 - y_0, a\| \leq 0$(or) $\|x_0 - y_0, a\| = 0 \Rightarrow x_0 = y_0$. That is, the fixed point is unique. □

Remark 2.2. *The Example 2.3 shows that the conditions (1) and (2) are essential in Theorem 2.1.*

Example 2.3. *Let $X = \{1, 2, 3, 4\}$ be a finite set with a 2-normed linear space defined as follows:*
$\|x - y, z\| = 0$, if at least any two of x, y, z are equal. Take

$$\|1 - 2, 3\| = 3$$

$$\|1 - 2, 4\| = 4$$

$$\|2-3,4\| = 5$$
$$\|1-3,4\| = 6.$$

We define $T : X \to X$ by $T(1) = 2$; $T(2) = 3$; $T(3) = 4$; $T(4) = 1$. Clearly, $\inf \|x - Tx, T^2x\|$ exists. The property $\|Tx - Ty, a\| \le \varphi'(\|x-y,a\|, \|x-Tx,a\|, \|y-Ty,a\|)$ for all $x, y, a \in X$ does not exists. Since T is not a φ'-contraction. So, in particular, let us take $x = 1$; $y = 2$; $a = 4$, we have $\|T(1) - T(2), 4\| \le \varphi'(\|1-2,4\|, \|1-T(1),4\|, \|2-T(2),4\|)$. That is, $\|2-3,4\| \le \varphi'(\|1-2,4\|, \|1-2,4\|, \|2-3,4\|)$. Using φ'-contraction, we get $\|2-3,4\| \le k\|1-2,4\|$ or $5 \le k.4$ which is not possible since $k < 1$. From this, T has no fixed point.

Corollary 2.4. *Let $(X, \|.,.\|)$ be a quasi semi linear 2-normed space and T a self map of $(X, \|.,.\|)$ satisfying the following conditions:*

(c). there exists an integer n such that $\|T^nx - T^ny, a\| \le \varphi'(\|x-y,a\|, \|x-T^nx,a\|, \|y-T^ny,a\|)$ for all $x, y, a \in X$.

(d). there exists a point $x_0 \in X$ such that $\|x_0 - T^nx_0, a\| = inf\{\|x - T^nx, a\| : x \in X\}$, then T has a unique fixed point.

Proof. Suppose $S = T^n$, then by the above theorem, S has a unique fixed point. Hence, T^n has a unique fixed point Let x_0 be the unique fixed point of T^n. So, $T^n(Tx_0) = T(T^nx_0) = Tx_0$. Therefore, Tx_0 is a fixed point of T^n. If $Tx_0 \ne x_0$, then it is a contradiction to the existence of unique fixed point of T^n. Thus $Tx_0 = x_0$. □

Theorem 2.5. *Let S and T be self mappings of a quasi semi linear 2-normed space $(X, \|.,.\|)$ satisfying the condition:*

$$\|Tx - Sy, a\| \le \varphi'(\|x-y,a\|, \|x-Tx,a\|, \|y-Sy,a\|) \tag{3}$$

for all $x, y, a \in X$. If there exists a point $x_0 \in X$ such that for all $x, a \in X$

$$\|x_0 - Tx_0, a\| \le \|x - Sx, a\| \tag{4}$$

then S and T has a unique common fixed point.

Proof. Let $Tx_0 \ne x_0$. Put $x = x_0$, $y = Tx_0$ in (3), we obtain

$$\|Tx_0 - S(Tx_0), a\| \le \varphi'(\|x_0 - Tx_0, a\|, \|x_0 - Tx_0, a\|, \|Tx_0 - S(Tx_0), a\|)$$

By φ'-contraction, we get $\|Tx_0 - S(Tx_0), a\| \le k\|x_0 - Tx_0, a\| < \|x_0 - Tx_0, a\|$. This is a contradiction to (4). Therefore $Tx_0 = x_0$, which implies x_0 is also a fixed point of S. Let $Sx_0 \ne x_0$, then

$$\|x_0 - Sx_0, a\| = \|Tx_0 - Sx_0, a\| \le \varphi'(\|x_0 - x_0, a\|, \|x_0 - Tx_0, a\|, \|x_0 - Sx_0, a\|)$$

That is,

$$\|x_0 - Sx_0, a\| \le \varphi'(0, 0, \|x_0 - Sx_0, a\|)$$
$$\|x_0 - Sx_0, a\| \le 0$$

Hence, $Sx_0 = x_0$.

For uniqueness, let y_0 be another fixed point of S and T. That is, $Sy_0 = Ty_0 = y_0$. Then,

$$\|x_0 - y_0, a\| = \|Tx_0 - Ty_0, a\| \le \varphi'(\|x_0 - y_0, a\|, \|x_0 - Tx_0, a\|, \|y_0 - Ty_0, a\|)$$

$$\leq \varphi'(\|x_0 - y_0, a\|, \|x_0 - x_0, a\|, \|y_0 - y_0, a\|)$$
$$\leq \varphi'(\|x_0 - y_0, a\|, 0, 0)$$

Therefore by φ'-contraction, we obtain $\|x_0 - y_0, a\| \leq 0$ or $\|x_0 - y_0, a\| = 0 \Rightarrow x_0 = y_0$. Which implies that the fixed point is unique. □

Corollary 2.6. *Let $(X, \|.,.\|)$ be a quasi semi linear 2-normed space and let S and T be self maps of $(X, \|.,.\|)$ satisfies the following conditions:*

(e). there exists an integer m and n such that $\|T^n x - S^m y, a\| \leq \varphi'(\|x-y, a\|, \|x - T^n x, a\|, \|y - S^m y, a\|)$ for all $x, y, a \in X$.

(f). if there exists a point $x_0 \in X$ such that for all $x, a \in X \|x_0 - T^n x_0, a\| \leq \|x - S^m x, a\|$, then S and T has a unique fixed point.

Corollary 2.7. *Let $(X, \|.,.\|)$ be a quasi semi linear 2-normed space and T be a self map of $(X, \|.,.\|)$ satisfying the following conditions:*

(g). There exists an integer m and n such that $\|T^n x - T^m y, a\| \leq \varphi'(\|x-y, a\|, \|x - T^n x, a\|, \|y - T^m y, a\|)$ for all $x, y, a \in X$.

(h). For all $x, a \in X$, there exists a point $x_0 \in X$ such that $\|x_0 - T^n x_0, a\| \leq \|x - T^m x, a\|$, then T has a unique fixed point.

Theorem 2.8. *Let $(X, \|.,.\|)$ be a quasi semi linear 2-normed space and T be a self map of X.*

$$\|Tx - Ty, a\| < \{\|x - Tx, a\| \ \|x - y, a\|\}^{1/2} \quad \forall \quad x, y, a \in X. \tag{5}$$

if there exists a real valued function F defined by $F(x) = \|x - Tx, a\|$ for all $x \in X$such that $F(x) < F(Tx)$, then T has a unique fixed point in X.

Proof. Suppose for some x$_0 \in X, x_0 \neq Tx_0$. Then $F(Tx_0) = \|Tx_0 - T(Tx_0), a\| < \{\|x_0 - Tx_0, a\| \ \|x_0 - Tx_0, a\|\}^{1/2} \Rightarrow F(Tx_0) < \|x_0 - Tx_0, a\| \Rightarrow F(Tx_0) < F(x_0)$, which is a contradiction. Hence, $Tx_0 = x_0$.
Uniqueness: Let y_0 be another point of X different from x_0 such that $Ty_0 = y_0$. Then

$$\|x_0 - y_0, a\| = \|Tx_0 - Ty_0, a\| < \{\|x_0 - Tx_0, a\| \ \|x_0 - y_0, a\|\}^{1/2}$$
$$= \{\|x_0 - x_0, a\| \ \|x_0 - y_0, a\|\}^{1/2}$$
$$= 0$$

Hence $\|x_0 - y_0, a\| < 0$ which implies that $\|x_0 - y_0, a\| = 0 or x_0 = y_0$. □

2.1. Expansion Mappings in Quasi Semi Linear 2-normed Space

In the case of expansion mappings, we have the following theorem:

Theorem 2.9. *Let $(X, \|.,.\|)$ be a quasi semi linear 2-normed space and let T be a surjective self map of X such that for all $x, y, a \in X$.*

$$\|Tx - Ty, a\| \geq min\{\|x - y, a\| \ \|x - Ty, a\|\}^{1/2} \tag{6}$$

for all $x, y, a \in X$. If there exists a real valued function F defined by $F(x) = \|x - Tx, a\|$ such that $F(x) < F(Tx)$, then T has a unique fixed point of X.

Proof. Suppose for some $x_0 \in X, x_0 \neq Tx_0$. Then

$$\begin{aligned} F(Tx_0) &= \|Tx_0 - T^2x_0, a\| = \|Tx_0 - T(Tx_0), a\| \\ &\geq \min\{\|x_0 - Tx_0, a\|, \|x_0 - Tx_0, a\|\}^{1/2} \\ &= \|x_0 - Tx_0, a\| \\ &= F(x_0) \end{aligned}$$

Thus $F(Tx_0) \geq F(x_0)$, which is a contradiction. Hence, $Tx_0 = x_0$. Thus T has a fixed point of X.

Uniqueness: Let y_0 be another point of X different from x_0 such that $Ty_0 = y_0$. Then $F(Ty_0) = \|Ty_0 - T^2y_0, a\| = \|Ty_0 - T(Ty_0\| \geq F(y_0)$. Thus T has a unique fixed point of X. □

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