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On q-fuzzy Sets in Hilbert Algebras

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Abstract

In this paper, Q-fuzzy ideals and Q-fuzzy subalgebras concepts of Hilbert algebras are introduced and proved some results. Further, we discuss the relation between Q-fuzzy ideals (respectively Q-fuzzy subalgebras) and level subsets of a Q-fuzzy set. Q-fuzzy ideals and Q-fuzzy subalgebras are also applied in the Cartesian product of Hilbert algebras.

Keywords: Hilbert algebra; Q-fuzzy subalgebra; Q-fuzzy ideal.
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1. Introduction

The concept of fuzzy sets was proposed by Zadeh [20]. The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After the introduction of the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The integration between fuzzy sets and some uncertainty approaches such as soft sets and rough sets has been discussed in [1, 4, 7]. The idea of intuitionistic fuzzy sets suggested by Atanassov [2] is one of the extensions of fuzzy sets with better applicability. Applications of intuitionistic fuzzy sets appear in various fields, including medical diagnosis, optimization problems, and multicriteria decision making [12–14]. The concept of Hilbert algebra was introduced in early 50-ties by L. Henkin and T. Skolem for some investigations of implication in intuicionistic and other non-classical logics. In 60-ties, these algebras were studied especially by A. Horn and A. Diego from algebraic point of view. A. Diego proved (cf. [9] that Hilbert algebras form a variety which is locally finite. Hilbert algebras were treated by D. Busneag (cf. [5], [6]) and Y. B. Jun (cf. [15]) and some of their filters forming deductive systems were recognized. W. A. Dudek (cf. [11]) considered the fuzzification of subalgebras and deductive systems in Hilbert algebras. In this paper, Q-fuzzy ideals and Q-fuzzy subalgebras concepts of Hilbert algebras are introduced and proved some results. Further, we discuss the relation between Q-fuzzy ideals (respectively Q-fuzzy subalgebras) and level subsets of a Q-fuzzy set. Q-fuzzy ideals and Q-fuzzy subalgebras are also applied in the Cartesian product of Hilbert algebras.

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2. Preliminaries

Definition 2.1 ([9]). *A Hilbert algebra is a triplet $H = (H, *, 1)$, where H is a nonempty set, $*$ is a binary operation and 1 is fixed element of H such that the following axioms hold for each $x, y, z \in H$.*

1. $x * (y * x) = 1$,
2. $(x * (y * z)) * ((x * y) * (x * z)) = 1$,
3. $x * y = 1$ *and* $y * x = 1$ *imply* $x = y$.

The following result was proved in [11].

Lemma 2.2. *Let $H = (H, *, 1)$ be a Hilbert algebra and $x, y, z \in H$. Then*

1. $x * x = 1$,
2. $1 * x = x$,
3. $x * 1 = 1$,
4. $x * (y * z) = y * (x * z)$.

It is easily checked that in a Hilbert algebra H the relation $\leq y$ defined by $x \leq y \Leftrightarrow x * y = 1$ is a partial order on H with 1 as the largest element.

Definition 2.3 ([16]). *A nonempty subset S of a Hilbert algebra $H = (H, *, 1)$ is called a subalgebra of H if $(\forall x, y \in H)(x \in S, y \in S \Rightarrow x * y \in S)$.*

Definition 2.4 ([8]). *A nonempty subset I of a Hilbert algebra $H = (H, *, 1)$ is called an ideal of H if*

1. $1 \in I$,
2. $x * y \in I$ *for all* $x \in H$, $y \in I$,
3. $(y_2 * (y_1 * x)) * x \in I$ *for all* $x \in H$, $y_1, y_2 \in I$.

Definition 2.5 ([10]). *A fuzzy set μ in a Hilbert algebra H is said to be a fuzzy ideal of H if the following conditions are hold:*

1. $\mu(1) \geq \mu(x)$ *for all* $x \in H$,
2. $\mu(x * y) \geq \mu(y)$ *for all* $x, y \in H$,
3. $\mu((y_1 * (y_2 * x)) * x) \geq \min\{\mu(y_1), \mu(y_2)\}$ *for all* $x, y_1, y_2 \in H$.

Lemma 2.6. *Let μ be a fuzzy set in A. Then the following statements hold: for any $x, y \in A$,*

1. $1 - \max\{\mu(x), \mu(y)\} = \min\{1 - \mu(x), 1 - \mu(y)\}$,

2. $1 - \min\{\mu(x), \mu(y)\} = \max\{1 - \mu(x), 1 - \mu(y)\}$.

Definition 2.7 ([17]). *A Q-fuzzy set in a nonempty set X (or a Q-fuzzy subset of X) is an arbitrary function $\mu : X \times Q \to [0,1]$, where Q is a nonempty set and $[0,1]$ is the unit segment of the real line.*

Definition 2.8 ([17]). *Let μ be a Q-fuzzy set in A. The Q-fuzzy set μ defined by $\overline{\mu}(x,q) = 1 - \mu(x,q)$ for all $x \in A$ and $q \in Q$ is called the complement of μ in A.*

Remark 2.9. *For a Q-fuzzy set μ in A, we have $\mu = \overline{\overline{\mu}}$.*

Definition 2.10 ([18]). *Let $f : A \to B$ be a function and μ be a Q-fuzzy set in B. We define a new Q-fuzzy set in A by μ_f as $\mu(f(x), q)$ for all $x \in A$ and $q \in Q$.*

Definition 2.11 ([18]). *Let $f : A \to B$ be a bijection and μ_f be a Q-fuzzy set in A. We define a new Q-fuzzy set in B by μ as $\mu(y,q) = \mu_f(x,q)$, where $f(x) = y$ for all $y \in B$ and $q \in Q$.*

Definition 2.12 ([18]). *Let μ be a Q-fuzzy set in A and δ be a Q-fuzzy set in B. The Cartesian product $\mu \times \delta : (A \times B) \times Q \to [0,1]$ is defined by $(\mu \times \delta)((x,y),q) = \max\{\mu(x,q), \delta(y,q)\}$ for all $x \in A$, $y \in B$ and $q \in Q$. The dot product $\mu \cdot \delta : (A \times B) \times Q \to [0,1]$ is defined by $(\mu \cdot \delta)((x,y),q) = \min\{\mu(x,q), \delta(y,q)\}$ for all $x \in A$, $y \in B$ and $q \in Q$.*

Lemma 2.13. *For any $a, b \in \mathbb{R}$ such that $a < b$, $a < \frac{b+a}{2} < b$.*

Lemma 2.14 ([19]). *Let μ be a fuzzy set in A and for any $t \in [0,1]$. Then the following properties hold:*

1. $L(\mu, t) = U(\overline{\mu}, 1 - t)$,
2. $L^-(\mu, t) = U^+(\overline{\mu}, 1 - t)$,
3. $U(\mu, t) = L(\overline{\mu}, 1 - t)$,
4. $U^+(\mu, t) = L^-(\overline{\mu}, 1 - t)$.

Lemma 2.15 ([19]). *Let μ be a Q-fuzzy set in A and for any $t \in [0,1]$ and $q \in Q$. Then the following properties hold:*

1. $L(\mu, t, q) = U(\overline{\mu}, 1 - t, q)$,
2. $L^-(\mu, t, q) = U^+(\overline{\mu}, 1 - t, q)$,
3. $U(\mu, t, q) = L(\overline{\mu}, 1 - t, q)$,
4. $U^+(\mu, t, q) = L^-(\overline{\mu}, 1 - t, q)$.

Lemma 2.16 ([19]). *Let μ be a Q-fuzzy set in A and for any $t \in [0,1]$ and $q \in Q$. Then the following properties hold:*

1. $L(\mu, t) = \bigcap_{q \in Q} L(\mu, t, q)$,

2. $L^-(\mu,t) = \bigcap_{q\in Q} L^-(\mu,t,q)$,
3. $U(\mu,t) = \bigcap_{q\in Q} U(\mu,t,q)$,
4. $U^+(\mu,t) = \bigcap_{q\in Q} U^+(\mu,t,q)$.

3. On Q-fuzzy Subalgebra of Hilbert Algebra

Definition 3.1. *A Q-fuzzy set μ in a Hilbert algebra H is called a q-fuzzy subalgebra of H if*

$$(\forall x,y \in H)\left(\ \mu(x*y,q) \geq \min\{\mu(x,q),\mu(y,q)\} \ \right). \tag{1}$$

A Q-fuzzy set μ in a Hilbert algebra H is called a Q-fuzzy subalgebra of H if it is a Q-fuzzy subalgebra of H for all $q \in Q$.

Example 3.2. *Let $H = \{1,x,y,z,0\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$*$	1	x	y	z	0
1	1	x	y	z	0
x	1	1	y	z	0
y	1	x	1	z	z
z	1	1	y	1	y
0	1	1	1	1	1

Then $(H,*,1)$ is a Hilbert algebra. Let $Q = \{q\}$. We define a q-fuzzy set μ in H as follows: $\mu(1,q) = 1, \mu(x,q) = 0.8, \mu(y,q) = 0.8, \mu(z,q) = 0.7, \mu(0,q) = 0.4$. Then μ is a Q-fuzzy subalgebra of H.

Proposition 3.3. *Every Q-fuzzy subalgebra μ of a Hilbert algebra H satisfies $\mu(1,q) \geq \mu(x,q)$ for all $x \in H$ and $q \in Q$.*

Proof. For any $x \in H$ and $q \in Q$, we have $\mu(1,q) = \mu(x*x,q) \geq \min\{\mu(x,q),\mu(x,q)\} = \mu(x,q)$. □

Proposition 3.4. *Let μ be a Q-fuzzy subalgebra of a Hilbert algebra H. Define a Q-fuzzy set γ in H by $\gamma(x,q) = \dfrac{\mu(x,q)}{\mu(1,q)}$ for all $x \in H$ and $q \in Q$. Then γ is a Q-fuzzy subalgebra of H.*

Proof. Let $x,y \in H$ and $q \in Q$. Then

$$\begin{aligned}\gamma(x*y,q) &= \frac{\mu(x*y,q)}{\mu(1,q)} \\ &\geq \left(\frac{1}{\mu(1,q)}\min\{\mu(x,q),\mu(y,q)\}\right) \\ &= \min\left\{\frac{\mu(x,q)}{\mu(1,q)},\frac{\mu(y,q)}{\mu(1,q)}\right\} \\ &= \min\{\gamma(x,q),\gamma(y,q)\}.\end{aligned}$$

□

Definition 3.5. *A Q-fuzzy set μ in a Hilbert algebra H is said to be a Q-fuzzy ideal of H if the following conditions are hold:*

$$(\forall x \in H)\left(\ \mu(1,q) \geq \mu(x,q)\ \right), \tag{2}$$

$$(\forall\ x,y \in H)\left(\ \mu(x*y,q) \geq \mu(y,q)\ \right), \tag{3}$$

$$(\forall\ x,y_1,y_2 \in H)\left(\ \mu((y_1*(y_2*x,q),q)*x,q) \geq \min\{\mu(y_1,q),\mu(y_2,q)\}\ \right). \tag{4}$$

A Q-fuzzy set μ in a Hilbert algebra H is called a Q-fuzzy ideal of H if it is a q-fuzzy ideal of H for all $q \in Q$.

Example 3.6. *Let $H = \{1,x,y,z,0\}$ be a set with a binary operation $*$ defined by the following Cayley table:*

$*$	1	x	y	z
1	1	x	y	z
x	1	1	y	z
y	1	x	1	z
z	1	1	y	1

Then $(H,*,1)$ is a Hilbert algebra. Let $Q = \{q\}$. We define a q-fuzzy set μ as follows: $\mu(1,q) = 0.9, \mu(x,q) = 0.3, \mu(y,q) = 0.1, \mu(z,q) = 0.6$. Then μ is a Q-fuzzy ideal of H.

Proposition 3.7. *If μ is a Q-fuzzy ideal of a Hilbert algebra H, then*

$$(\forall x,y \in H)\left(\ \mu((y*x,q)*x,q) \geq \mu(y,q)\ \right). \tag{5}$$

Proof. Putting $y_1 = y$ and $y_2 = 1$ in (4), for any $q \in Q$, we have

$$\mu((y*x,q)*x,q) \geq \min\{\mu(y,q),\mu(1,q)\} = \mu(y,q).$$

□

Lemma 3.8. *If μ is a Q-fuzzy ideal of a Hilbert algebra H, then we have the following*

$$(\forall\ x,y \in H)\left(\ x \leq y\ \Rightarrow \mu(x,q) \leq \mu(y,q)\ \right). \tag{6}$$

Proof. Let $x,y \in H$ be such that $x \leq y$ and $q \in Q$. Then $x*y = 1$ and so

$$\begin{aligned}
\mu(y,q) &= \mu(1*y,q) \\
&= \mu(((x*y,q)*(x*y,q))*y) \\
&\geq \min\{\mu(x*y,q),\mu(x,q)\} \\
&\geq \min\{\mu(1,q),\mu(x,q)\} \\
&= \mu(x,q).
\end{aligned}$$

□

Theorem 3.9. *Every Q-fuzzy ideal of a Hilbert algebra H is a Q-fuzzy subalgebra of H.*

Proof. Let μ be a Q-fuzzy ideal of H. Since $y \leq x * y$ for all $x, y \in H$, from Lemma 3.8 that, for any $q \in Q$, $\mu(y, q) \geq \mu(x * y, q)$. It follows from (3) that

$$\begin{aligned} \mu(x * y, q) &\geq \mu(y, q) \\ &\geq \min\{\mu(x * y, q), \mu(x, q)\} \\ &\geq \min\{\mu(x, q), \mu(y, q)\}. \end{aligned}$$

Hence μ is a Q-fuzzy subalgebra of H. □

Proposition 3.10. *If $\{\mu_i : i \in \Delta\}$ is a family of Q-fuzzy ideals of a Hilbert algebra H, then $\bigwedge\limits_{i \in \Delta} \mu_i$ is a Q-fuzzy ideal of H.*

Proof. Let $\{\mu_i : i \in \Delta\}$ be a family of Q-fuzzy ideals of a Hilbert algebra H.
Let $x \in H$ and $q \in Q$, we have

$$\left(\bigwedge_{i \in \Delta} \mu_i\right)(1, q) = \inf_{i \in \Delta}\{\mu_i(1, q)\} \geq \inf_{i \in \Delta}\{\mu_i(x, q)\} = \left(\bigwedge_{i \in \Delta} \mu_i\right)(x, q).$$

Let $x, y \in H$ and $q \in Q$, we have

$$\left(\bigwedge_{i \in \Delta} \mu_i\right)(x * y, q) = \inf_{i \in \Delta}\{\mu_i(x * y, q)\} \geq \inf_{i \in \Delta}\{\mu_i(y, q)\} = \left(\bigwedge_{i \in \Delta} \mu_i\right)(y, q).$$

Let $x, y_1, y_2 \in H$ and $q \in Q$, we have

$$\begin{aligned} \left(\bigwedge_{i \in \Delta} \mu_i\right)((y_1 * (y_2 * x, q), q) * x, q) &= \inf_{i \in \Delta}\{\mu_i((y_1 * (y_2 * x, q), q) * x, q)\} \\ &\geq \inf_{i \in \Delta}\{\min\{\mu_i(y_1, q), \mu_i(y_2, q)\}\} \\ &= \min\{\inf_{i \in \Delta} \mu_i(y_1, q), \inf_{i \in \Delta} \mu_i(y_2, q)\} \\ &= \min\left\{\left(\bigwedge_{i \in \Delta} \mu_i\right)(y_1, q), \left(\bigwedge_{i \in \Delta} \mu_i\right)(y_2, q)\right\}. \end{aligned}$$

Hence $\bigwedge\limits_{i \in \Delta} A_i$ is a Q-fuzzy ideal of a Hilbert algebra H. □

Definition 3.11. *A Q-fuzzy set μ in a Hilbert algebra H is said to be a Q-fuzzy deductive system of H if the following conditions are hold:*

$$(\forall x \in H)\left(\ \mu(1, q) \geq \mu(x, q)\ \right), \tag{7}$$

$$(\forall\ x, y \in H)\left(\ \mu(y, q) \geq \min\{\mu(x * y, q), \mu(x, q)\}\ \right). \tag{8}$$

A Q-fuzzy set μ in a Hilbert algebra H is called a Q-fuzzy deductive system of H if it is a Q-fuzzy deductive system of H for all $q \in Q$.

Proposition 3.12. *Every Q-fuzzy ideal of a Hilbert algebra H is a Q-fuzzy deductive system of H.*

Proof. Let μ be a Q-fuzzy ideal of H. If $y_1 = x * y$, $y_2 = x$, where $x, y \in H$ and $q \in Q$, then by (1), (2) of Lemma 2.2 and (4), we have

$$\mu(y,q) = \mu(1 * y, q) = \mu(((x * y, q) * (x * y, q)) * y, q) \geq \min\{\mu(x * y, q), \mu(x, q)\}.$$

Hence μ be a Q-fuzzy deductive system of H. □

Definition 3.13. *Let μ be a q-fuzzy set of a Hilbert algebra H and $t \in [0,1]$. Then we define the sets $U(\mu,t) = \{x \in H : \mu(x,q) \geq t, \forall q \in Q\}$ and $U^+(\mu,t) = \{x \in H : \mu(x,q) > t, \forall q \in Q\}$ are called an upper α-level subset and an upper α-strong level subset of μ, respectively. The sets $L(\mu,t) = \{x \in H : \mu(x,q) \leq t, \forall q \in Q\}$ and $L^-(\mu,t) = \{x \in H : \mu(x,q) < t, \forall q \in Q\}$ are called a lower t-level subset and a lower t-strong level subset of μ, respectively. For any $q \in Q$, the sets $U(\mu,t,q) = \{x \in H : \mu(x,q) \geq t\}$ and $U^+(\mu,t,q) = \{x \in H : \mu(x,q) > t\}$ are called a q-upper t-level subset and a q-upper t-strong level subset of μ, respectively. The sets $L(\mu,t,q) = \{x \in H : \mu(x,q) \leq t\}$ and $L^+(\mu,t,q) = \{x \in H : \mu(x,q) < t\}$ are called a q-lower t-level subset and a q-lower t-strong level subset of μ, respectively.*

Theorem 3.14. *Let μ be a Q-fuzzy set in H. Then the following statements hold:*

1. *$\overline{\mu}$ is a Q-fuzzy ideal of H if and only if for any $t \in [0,1]$ and $q \in Q$, $L(\mu,t,q)$ is either empty or an ideal of H.*
2. *$\overline{\mu}$ is a Q-fuzzy ideal of H if and only if for any $t \in [0,1]$ and $q \in Q$, $L^-(\mu,t,q)$ is either empty or an ideal of H.*
3. *μ is a Q-fuzzy ideal of H if and only if for any $t \in [0,1]$ and $q \in Q$, $U(\mu,t,q)$ is either empty or an ideal of H.*
4. *μ is a Q-fuzzy ideal of H if and only if for any $t \in [0,1]$ and $q \in Q$, $U^+(\mu,t,q)$ is either empty or an ideal of H.*

Proof. (1). Assume that $\overline{\mu}$ is a Q-fuzzy ideal of H. Then $\overline{\mu}$ is a Q-fuzzy ideal of H for all $q \in Q$. Let $q \in Q$ and $t \in [0,1]$ be such that $L(\mu,t,q) \neq \emptyset$ and let $x \in L(\mu,t,q)$. Then $\mu(x,q) \leq t$. Thus $\overline{\mu}(1,q) = \overline{\mu}(x * x, q) \geq \overline{\mu}(x,q)$. Then $1 - \mu(1,q) \geq 1 - \mu(x,q)$, so $\mu(1,q) \leq \mu(x,q) \leq t$. Hence $1 \in L(\mu,t,q)$. Let $x, y \in H$ be such that $y \in L(\mu,t,q)$. Then $\mu(y,q) \leq t$. By Definition 1.10 (2), we have $\overline{\mu}(x * y, q) \geq \overline{\mu}(y,q)$. Then

$$\begin{aligned} 1 - \mu(x * y, q) &\geq 1 - \mu(y,q) \\ \mu(x * y, q) &\leq \mu(y,q) \\ \mu(x * y, q) &\leq \mu(y,q) \leq t \end{aligned}$$

Thus $x * y \in L(\mu, t, q)$. Let $x, y_1, y_2 \in H$ be such that $y_1, y_2 \in L(\mu, t, q)$. Then $\mu(y_1, q) \leq t$ and $\mu(y_2, q) \leq t$. Then we have $\overline{\mu}((y_1 * (y_2 * x, q), q) * x, q) \geq \min\{\overline{\mu}(y_1, q), \overline{\mu}(y_2, q)\}$. Then

$$\begin{aligned}
\overline{\mu}((y_1 * (y_2 * x, q), q) * x, q) &\geq \min\{\overline{\mu}(y_1, q), \overline{\mu}(y_2, q)\} \\
1 - \mu((y_1 * (y_2 * x, q), q) * x, q) &\geq \min\{1 - \mu(y_1, q), 1 - \mu(y_2, q)\} \\
1 - \mu((y_1 * (y_2 * x, q), q) * x, q) &\geq 1 - \max\{\mu(y_1, q), \mu(y_2, q)\} \\
\mu((y_1 * (y_2 * x, q), q) * x, q) &\leq \max\{\mu(y_1, q), \mu(y_2, q)\} \leq t
\end{aligned}$$

Thus $(y_1 * (y_2 * x)) * x \in L(\mu, t, q)$. Hence $L(\mu, t, q)$ is an ideal of H. Conversely, assume that every nonempty set $L(\mu, t, q)$ is ideal in H. If $\overline{\mu}(1, q) \geq \overline{\mu}(x, q)$ is not true for all $x \in H$ and $q \in Q$. Then there exist $x_0 \in H$ and $q \in Q$ such that $\overline{\mu}(1, q) < \overline{\mu}(x_0, q)$. Then

$$\begin{aligned}
\overline{\mu}(1, q) &< \overline{\mu}(x_0, q) \\
1 - \mu(1, q) &< 1 - \mu(x_0, q) \\
\mu(1, q) &> \mu(x_0, q)
\end{aligned}$$

Let $t = \frac{1}{2}(\mu(1, q) + \mu(x_0, q))$. Then $t \in [0, 1]$ and by Lemma 2.13, we have $\mu(1, q) > t > \mu(x_0, q)$. Thus $x_0 \in L(\mu, t, q)$, that is $L(\mu, t, q) \neq \emptyset$. By assumption, we have $L(\mu, s, q)$ is an ideal of H. It follows that $1 \in L(\mu, t, q)$, so $\mu(1, q) \leq t$, which is a contradiction. Hence $\overline{\mu}(1, q) \geq \overline{\mu}(x, q)$ for all $x \in H$ and $q \in Q$. If $\overline{\mu}(x * y, q) \geq \overline{\mu}(y, q)$ is not true for all $x, y \in H$ and $q \in Q$. Then there exist $x_0, y_0 \in H$ and $q \in Q$ such that $\overline{\mu}(x_0 * y_0, q) < \overline{\mu}(y_0, q)$. Then

$$\begin{aligned}
\overline{\mu}(x_0 * y_0, q) &< \overline{\mu}(y_0, q) \\
1 - \mu(x_0 * y_0, q) &< 1 - \mu(y_0, q) \\
\mu(x_0 * y_0, q) &> \mu(y_0, q)
\end{aligned}$$

Let $t_0 = \frac{1}{2}(\mu(x_0 * y_0, q) + \mu(y_0, q))$. Then $t_0 \in [0, 1]$ and $\mu(x_0 * y_0, q) < t_0 < \mu(y_0, q)$, which prove that $y_0 \in L(\mu, t_0, q)$. Since $L(\mu, t_0, q)$ is an ideal of H, $x_0 * y_0 \in L(\mu, t_0, q)$. Hence $\mu(x_0 * y_0, q) \geq t_0$, a contradiction. Thus $\mu(x * y, q) \geq \mu(y, q)$ is true for all $x, y \in H$ and $q \in Q$. Suppose that $\overline{\mu}((y_1 * (y_2 * x, q), q) * x, q) \geq \min\{\overline{\mu}(y_1, q), \overline{\mu}(y_2, q)\}$ is not true for all $x, y_1, y_2 \in H$. Then there exist $u_0, v_0, x_0 \in H$ and $q \in Q$ such that $\overline{\mu}((u_0 * (v_0 * x_0, q), q) * x_0, q) < \min\{\overline{\mu}(u_0, q), \overline{\mu}(v_0, q)\}$. Then

$$\begin{aligned}
\overline{\mu}((u_0 * (v_0 * x_0, q), q) * x_0, q) &< \min\{\overline{\mu}(u_0, q), \overline{\mu}(v_0, q)\} \\
1 - \mu((u_0 * (v_0 * x_0, q), q) * x_0, q) &< \min\{1 - \mu(u_0, q), 1 - \mu(v_0, q)\} \\
1 - \mu((u_0 * (v_0 * x_0, q), q) * x_0, q) &< 1 - \max\{\mu(u_0, q), \mu(v_0, q)\} \\
\mu((u_0 * (v_0 * x_0, q), q) * x_0, q) &> \max\{\mu(u_0, q), \mu(v_0, q)\}
\end{aligned}$$

Taking $p = \frac{1}{2}(\mu((u_0 * (v_0 * x_0, q), q) * x_0, q) + \max\{\mu(u_0, q), \mu(v_0, q)\})$. Then $p \in [0, 1]$ and by Lemma 2.13, we have $\mu((u_0 * (v_0 * x_0, q), q) * x_0, q) > p > \max\{\mu(u_0, q), \mu(v_0, q)\}$. Thus $\mu(u_0, q) < p$ and

$\mu(v_0, q) < p$, so $u_0, v_0 \in L(\mu, p, q)$, so $L(\mu, p, q) \neq \emptyset$. By assumption, we have $L(\mu, p, q)$ is an ideal of H. It follows that $(u_0 * (v_0 * x_0) * x_0 \in L(\mu, p, q)$, so $\mu((u_0 * (v_0 * x_0, q), q) * x_0, q) \leq p$, a contradiction. Hence $\overline{\mu}((y_1 * (y_2 * x, q), q) * x, q) \geq \min\{\overline{\mu}(y_1, q), \overline{\mu}(y_2, q)\}$ is true for all $x, y_1, y_2 \in H$ and $q \in Q$. Hence $\overline{\mu}$ is a Q-fuzzy ideal of H for all $q \in Q$. Consequently $\overline{\mu}$ is a Q-fuzzy ideal of H.

(2). Assume that $\overline{\mu}$ is a Q-fuzzy ideal of H. Then $\overline{\mu}$ is a Q-fuzzy ideal of H for all $q \in Q$. Let $q \in Q$ and $t \in [0,1]$ be such that $L^-(\mu, t, q) \neq \emptyset$ and let $x \in L^-(\mu, t, q)$. Then $\mu(x, q) < t$. Thus $\overline{\mu}(1, q) = \overline{\mu}(x * x, q) \geq \overline{\mu}(x, q)$. Then $1 - \mu(1, q) \geq 1 - \mu(x, q)$, so $\mu(1, q) \leq \mu(x, q) < t$. Hence $1 \in L^-(\mu, t, q)$. Let $x, y \in H$ be such that $y \in L^-(\mu, t, q)$. Then $\mu(y, q) < t$. By Definition 1.10 (2), we have $\overline{\mu}(x * y, q) \geq \overline{\mu}(y, q)$. Then

$$\begin{aligned} 1 - \mu(x * y, q) &\geq 1 - \mu(y, q) \\ \mu(x * y, q) &\leq \mu(y, q) \\ \mu(x * y, q) &\leq \mu(y, q) < t \end{aligned}$$

Thus $x * y \in L^-(\mu, t, q)$. Let $x, y_1, y_2 \in H$ be such that $y_1, y_2 \in L^-(\mu, t, q)$. Then $\mu(y_1, q) < t$ and $\mu(y_2, q) < t$. Then we have $\overline{\mu}((y_1 * (y_2 * x, q), q) * x, q) \geq \min\{\overline{\mu}(y_1, q), \overline{\mu}(y_2, q)\}$. Then

$$\begin{aligned} \overline{\mu}((y_1 * (y_2 * x, q), q) * x, q) &\geq \min\{\overline{\mu}(y_1, q), \overline{\mu}(y_2, q)\} \\ 1 - \mu((y_1 * (y_2 * x, q), q) * x, q) &\geq \min\{1 - \mu(y_1, q), 1 - \mu(y_2, q)\} \\ 1 - \mu((y_1 * (y_2 * x, q), q) * x, q) &\geq 1 - \max\{\mu(y_1, q), \mu(y_2, q)\} \\ \mu((y_1 * (y_2 * x, q), q) * x, q) &\leq \max\{\mu(y_1, q), \mu(y_2, q)\} < t \end{aligned}$$

Thus $(y_1 * (y_2 * x)) * x \in L^-(\mu, t, q)$. Hence $L^-(\mu, t, q)$ is an ideal of H. Conversely, assume that every nonempty set $L^-(\mu, t, q)$ is ideal in H. If $\overline{\mu}(1, q) \geq \overline{\mu}(x, q)$ is not true for all $x \in H$ and $q \in Q$. Then there exist $x_0 \in H$ and $q \in Q$ such that $\overline{\mu}(1, q) < \overline{\mu}(x_0, q)$. Then

$$\begin{aligned} \overline{\mu}(1, q) &< \overline{\mu}(x_0, q) \\ 1 - \mu(1, q) &< 1 - \mu(x_0, q) \\ \mu(1, q) &> \mu(x_0, q) \end{aligned}$$

Let $t = \frac{1}{2}(\mu(1, q) + \mu(x_0, q))$. Then $t \in [0,1]$ and by Lemma 2.13, we have $\mu(1, q) > t > \mu(x_0, q)$. Thus $x_0 \in L^-(\mu, t, q)$, that is $L^-(\mu, t, q) \neq \emptyset$. By assumption, we have $L^-(\mu, s, q)$ is an ideal of H. It follows that $1 \in L^-(\mu, t, q)$, so $\mu(1, q) < t$, which is a contradiction. Hence $\overline{\mu}(1, q) \geq \overline{\mu}(x, q)$ for all $x \in H$ and $q \in Q$. If $\overline{\mu}(x * y, q) \geq \overline{\mu}(y, q)$ is not true for all $x, y \in H$ and $q \in Q$. Then there exist $x_0, y_0 \in H$ and $q \in Q$ such that $\overline{\mu}(x_0 * y_0, q) < \overline{\mu}(y_0, q)$. Then

$$\begin{aligned} \overline{\mu}(x_0 * y_0, q) &< \overline{\mu}(y_0, q) \\ 1 - \mu(x_0 * y_0, q) &< 1 - \mu(y_0, q) \\ \mu(x_0 * y_0, q) &> \mu(y_0, q) \end{aligned}$$

Let $t_0 = \frac{1}{2}(\mu(x_0 * y_0, q) + \mu(y_0, q))$. Then $t_0 \in [0,1]$ and $\mu(x_0 * y_0, q) < t_0 < \mu(y_0, q)$, which prove that $y_0 \in L^-(\mu, t_0, q)$. Since $L^-(\mu, t_0, q)$ is an ideal of H, $x_0 * y_0 \in L^-(\mu, t_0, q)$. Hence $\mu(x_0 * y_0, q) \geq t_0$, a contradiction. Thus $\mu(x * y, q) \geq \mu(y, q)$ is true for all $x, y \in H$ and $q \in Q$. Suppose that $\overline{\mu}((y_1 * (y_2 * x, q), q) * x, q) \geq \min\{\overline{\mu}(y_1, q), \overline{\mu}(y_2, q)\}$ is not true for all $x, y_1, y_2 \in H$. Then there exist $u_0, v_0, x_0 \in H$ and $q \in Q$ such that $\overline{\mu}((u_0 * (v_0 * x_0, q), q) * x_0, q) < \min\{\overline{\mu}(u_0, q), \overline{\mu}(v_0, q)\}$. Then

$$
\begin{aligned}
\overline{\mu}((u_0 * (v_0 * x_0, q), q) * x_0, q) &< \min\{\overline{\mu}(u_0, q), \overline{\mu}(v_0, q)\} \\
1 - \mu((u_0 * (v_0 * x_0, q), q) * x_0, q) &< \min\{1 - \mu(u_0, q), 1 - \mu(v_0, q)\} \\
1 - \mu((u_0 * (v_0 * x_0, q), q) * x_0, q) &< 1 - \max\{\mu(u_0, q), \mu(v_0, q)\} \\
\mu((u_0 * (v_0 * x_0, q), q) * x_0, q) &> \max\{\mu(u_0, q), \mu(v_0, q)\}
\end{aligned}
$$

Taking $p = \frac{1}{2}(\mu((u_0 * (v_0 * x_0, q), q) * x_0, q) + \max\{\mu(u_0, q), \mu(v_0, q)\})$. Then $p \in [0,1]$ and by lemma 2.8, we have $\mu((u_0 * (v_0 * x_0, q), q) * x_0, q) > p > \max\{\mu(u_0, q), \mu(v_0, q)\}$. Thus $\mu(u_0, q) < p$ and $\mu(v_0, q) < p$, so $u_0, v_0 \in L^-(\mu, p, q)$, so $L^-(\mu, p, q) \neq \emptyset$. By assumption, we have $L^-(\mu, p, q)$ is an ideal of H. It follows that $(u_0 * (v_0 * x_0) * x_0 \in L^-(\mu, p, q)$, so $\mu((u_0 * (v_0 * x_0, q), q) * x_0, q) \leq p$, a contradiction. Hence $\overline{\mu}((y_1 * (y_2 * x, q), q) * x, q) \geq \min\{\overline{\mu}(y_1, q), \overline{\mu}(y_2, q)\}$ is true for all $x, y_1, y_2 \in H$ and $q \in Q$. Hence $\overline{\mu}$ is a Q-fuzzy ideal of H for all $q \in Q$. Consequently $\overline{\mu}$ is a Q-fuzzy ideal of H.

(3). Assume that μ is a Q-fuzzy ideal of H. Then μ is a Q-fuzzy ideal of H for all $q \in Q$. Let $q \in Q$ and $t \in [0,1]$ be such that $U(\mu, t, q) \neq \emptyset$ and let $x \in H$ be such that $x \in U(\mu, t, q)$. Then $\mu(x, q) \geq t$. Thus $\mu(1, q) = \mu(x * x, q) \geq \mu(x, q)$. Hence $\mu(1, q) \geq \mu(x, q) \geq t$, so $1 \in U(\mu, t, q)$. Let $x, y \in H$ and $q \in Q$ be such that $y \in U(\mu, t, q)$. Then $\mu(y, q) \geq t$. Then we have $\mu(x * y) \geq \mu(y, q)$. Then $\mu(x * y, q) \geq \mu(y, q)$. Thus, $\mu(x * y, q) \geq \mu(y, q) \geq t$, so $x * y \in U(\mu, t, q)$. Let $x, y_1, y_2 \in H$ and $q \in Q$ be such that $y_1, y_2 \in U(\mu, t, q)$. Then $\mu(y_1, q) \geq t$ and $\mu(y_2, q) \geq t$. Then we have $\mu((y_1 * (y_2 * x, q), q) * x, q) \geq \min\{\mu(y_1, q), \mu(y_2, q)\}$. Then $\mu((y_1 * (y_2 * x, q), q) * x, q) \geq \min\{\mu(y_1, q), \mu(y_2, q)\} \geq t$, so $(y_1 * (y_2 * x)) * x \in U(\mu, t, q)$. Hence $U(\mu, t, q)$ is an ideal of H. Conversely, assume that every nonempty set $U(\mu, t, q)$ is an ideal in H. If $\mu(1, q) \geq \mu(x, q)$ is not true for all $x \in H$ and $q \in Q$. Then there exist $x_0 \in H$ and $q \in Q$ such that $\mu(1, q) < \mu(x_0, q)$. Let $t = \frac{1}{2}(\mu(1, q) + \mu(x_0, q))$. Then $t \in [0,1]$ and by Lemma 2.13, we have $\mu(1, q) < t < \mu(x_0, q)$. Thus $x_0 \in U(\mu, t, q)$, so, $U(\mu, t, q) \neq \emptyset$. By assumption, we have $U(\mu, t, q)$ is an ideal of H. It follows that $1 \in U(\mu, t, q)$, so $\mu(1, q) \geq t$, which is a contradicition. Hence $\mu(1, q) \geq \mu(x, q)$ for all $x \in H$ and $q \in Q$. If $\mu(x * y, q) \geq \mu(y, q)$ is not true for all $x, y \in H$ and $q \in Q$. Then there exist $x_0, y_0 \in H$ and $q \in Q$ such that $\mu(x_0 * y_0, q) < \mu(y_0, q)$. Let $t_0 = \frac{1}{2}(\mu(x_0 * y_0, q) + \mu(y_0, q))$. Then $t_0 \in [0,1]$ and by Lemma 2.13, we have $\mu(x_0 * y_0, q) < t_0 < \mu(y_0, q)$. Then $y_0 \in U(\mu, t_0, q)$, so, $U(\mu, t_0, q) \neq \emptyset$. By assumption, we have $U(\mu, t_0, q)$ is an ideal of H. It follows that $x_0 * y_0 \in U(\mu, t_0, q)$. Hence $\mu(x_0 * y_0, q) \geq t_0$, which is a contradiction. Hence $\mu(x * y, q) \geq \mu(y, q)$ is true for all $x, y \in H$ and $q \in Q$. Suppose that $\mu((y_1 * (y_2 * x, q), q) * x, q) \geq \min\{\mu(y_1, q), \mu(y_2, q)\}$ is not true for all $x, y_1, y_2 \in H$ and $q \in Q$. Then there exist $u_0, v_0, x_0 \in H$ and $q \in Q$ such that $\mu((u_0 * (v_0 * x_0, q), q) * x_0, q) < \min\{\mu(u_0, q), \mu(v_0, q)\}$. Let $p = \frac{1}{2}(\mu((u_0 * (v_0 * x_0, q), q) * x_0, q) + \min\{\mu(u_0, q), \mu(v_0, q)\})$. Then $p \in$

$[0,1]$ and by Lemma 2.13, we have $\mu((u_0 * (v_0 * x_0, q), q) * x_0, q) < p < \min\{\mu(u_0, q), \mu(v_0, q)\}$. Then $u_0, v_0 \in U(\mu, p, q)$, so, $U(\mu, p, q) \neq \emptyset$. By assumption, we have $U(\mu, p, q)$ is an ideal of H. It follows that $(u_0 * (v_0 * x_0)) * x_0 \in U(\mu, p, q)$. Hence $\mu((u_0 * (v_0 * x_0, q), q) * x_0, q) \geq p$, which is a contradiction. Hence $\mu((y_1 * (y_2 * x, q), q) * x, q) \geq \min\{\mu(y_1, q), \mu(y_2, q)\}$ is true for all $x, y_1, y_2 \in H$ and $q \in Q$. Hence μ is a Q-fuzzy ideal of H for all $q \in Q$. Consequently μ is a Q-fuzzy ideal of H.

(4). Assume that μ is a Q-fuzzy ideal of H. Then μ is a Q-fuzzy ideal of H for all $q \in Q$. Let $q \in Q$ and $t \in [0,1]$ be such that $U^+(\mu, t, q) \neq \emptyset$ and let $x \in H$ be such that $x \in U^+(\mu, t, q)$. Then $\mu(x, q) > t$. Thus $\mu(1, q) = \mu(x * x, q) \geq \mu(x, q)$. Hence $\mu(1, q) \geq \mu(x, q) > t$, so $1 \in U^+(\mu, t, q)$. Let $x, y \in H$ and $q \in Q$ be such that $y \in U^+(\mu, t, q)$. Then $\mu(y, q) > t$. Then we have $\mu(x * y) \geq \mu(y, q)$. Then $\mu(x * y, q) \geq \mu(y, q)$. Thus, $\mu(x * y, q) \geq \mu(y, q) > t$, so $x * y \in U^+(\mu, t, q)$. Let $x, y_1, y_2 \in H$ and $q \in Q$ be such that $y_1, y_2 \in U^+(\mu, t, q)$. Then $\mu(y_1, q) > t$ and $\mu(y_2, q) > t$. Then we have $\mu((y_1 * (y_2 * x, q), q) * x, q) \geq \min\{\mu(y_1, q), \mu(y_2, q)\}$. Then $\mu((y_1 * (y_2 * x, q), q) * x, q) \geq \min\{\mu(y_1, q), \mu(y_2, q)\} > t$, so $(y_1 * (y_2 * x)) * x \in U^+(\mu, t, q)$. Hence $U^+(\mu, t, q)$ is an ideal of H. Conversely, assume that every nonempty set $U^+(\mu, t, q)$ is an ideal in H. If $\mu(1, q) \geq \mu(x, q)$ is not true for all $x \in H$ and $q \in Q$. Then there exist $x_0 \in H$ and $q \in Q$ such that $\mu(1, q) < \mu(x_0, q)$. Let $t' = \frac{1}{2}(\mu(1, q) + \mu(x_0, q))$. Then $t' \in [0,1]$ and by Lemma 2.13, we have $\mu(1, q) < t' < \mu(x_0, q)$. Thus $x_0 \in U^+(\mu, t', q)$, so, $U^+(\mu, t', q) \neq \emptyset$. By assumption, we have $U^+(\mu, t', q)$ is an ideal of H. It follows that $1 \in U^+(\mu, t', q)$, so $\mu(1, q) > t'$, which is a contradicition. Hence $\mu(1, q) \geq \mu(x, q)$ for all $x \in H$ and $q \in Q$. If $\mu(x * y, q) \geq \mu(y, q)$ is not true for all $x, y \in H$ and $q \in Q$. Then there exist $x_0, y_0 \in H$ and $q \in Q$ such that $\mu(x_0 * y_0, q) < \mu(y_0, q)$. Let $t'_0 = \frac{1}{2}(\mu(x_0 * y_0, q) + \mu(y_0, q))$. Then $t'_0 \in [0,1]$ and by Lemma 2.13, we have $\mu(x_0 * y_0, q) < t'_0 < \mu(y_0, q)$. Then $y_0 \in U^+(\mu, t'_0, q)$, so, $U^+(\mu, t'_0, q) \neq \emptyset$. By assumption, we have $U^+(\mu, t'_0, q)$ is an ideal of H. It follows that $x_0 * y_0 \in U^+(\mu, t'_0, q)$. Hence $\mu(x_0 * y_0, q) > t'_0$, which is a contradiction. Hence $\mu(x * y, q) \geq \mu(y, q)$ is true for all $x, y \in H$ and $q \in Q$. Suppose that $\mu((y_1 * (y_2 * x, q), q) * x, q) \geq \min\{\mu(y_1, q), \mu(y_2, q)\}$ is not true for all $x, y_1, y_2 \in H$ and $q \in Q$. Then there exist $u_0, v_0, x_0 \in H$ and $q \in Q$ such that $\mu((u_0 * (v_0 * x_0, q), q) * x_0, q) < \min\{\mu(u_0, q), \mu(v_0, q)\}$. Let $p' = \frac{1}{2}(\mu((u_0 * (v_0 * x_0, q), q) * x_0, q) + \min\{\mu(u_0, q), \mu(v_0, q)\})$. Then $p' \in [0,1]$ and by Lemma 2.13, we have $\mu((u_0 * (v_0 * x_0, q), q) * x_0, q) < p' < \min\{\mu(u_0, q), \mu(v_0, q)\}$. Then $u_0, v_0 \in U(\mu, p', q)$, so, $U^+(\mu, p', q) \neq \emptyset$. By assumption, we have $U^+(\mu, p', q)$ is an ideal of H. It follows that $(u_0 * (v_0 * x_0)) * x_0 \in U^+(\mu, p', q)$. Hence $\mu((u_0 * (v_0 * x_0, q), q) * x_0, q) > p'$, which is a contradiction. Hence $\mu((y_1 * (y_2 * x, q), q) * x, q) \geq \min\{\mu(y_1, q), \mu(y_2, q)\}$ is true for all $x, y_1, y_2 \in H$ and $q \in Q$. Hence μ is a Q-fuzzy ideal of H for all $q \in Q$. Consequently μ is a Q-fuzzy ideal of H.

□

Corollary 3.15. *Let μ be a Q-fuzzy set in H. Then the following statements hold:*

1. *$\overline{\mu}$ is a Q-fuzzy ideal of H, then for any $t \in [0,1]$, $L(\mu, t)$ is either empty or an ideal of H.*

2. *$\overline{\mu}$ is a Q-fuzzy ideal of H, then for any $t \in [0,1]$, $L^-(\mu, t)$ is either empty or an ideal of H.*

3. *μ is a Q-fuzzy ideal of H, then for any $t \in [0,1]$, $U(\mu,t)$ is either empty or an ideal of H.*

4. *μ is a Q-fuzzy ideal of H, for any $t \in [0,1]$, $U^{+}(\mu,t)$ is either empty or an ideal of H.*

Proof. (1). Assume that $\overline{\mu}$ is a Q-fuzzy ideal of H. By Theorem 3.14 (1), we have for any $t \in [0,1]$ and $q \in Q$. Let $L(\mu,t,q)$ is either empty or an ideal of H. Let $t \in [0,1]$. If $L(\mu,t,q) = \emptyset$ for some $q \in Q$, it follows Lemma 2.16 (1) that $L(\mu,t) = \bigcap_{q\in Q} L(\mu,t,q)$. If $L(\mu,t,q) \neq \emptyset$ for all $q \in Q$, it follows from Theorem 3.14 (1) that $L(\mu,t,q)$ is an ideal of H for all $q \in Q$. By Lemma 2.16 (1) and the intersection of any ideals of H is also ideal in H, we have $L(\mu,t) = \bigcap_{q\in Q} L(\mu,t,q)$ is an ideal of H.
(2). Similarly to as in the proof of (1).
(3). Assume that μ is a Q-fuzzy ideal of H. By Theorem 3.14 (3), we have for any $t \in [0,1]$ and $q \in Q$. Let $U(\mu,t,q)$ is either empty or an ideal of H. Let $t \in [0,1]$. If $U(\mu,t,q) = \emptyset$ for some $q \in Q$, it follows Lemma 2.16 (3) that $U(\mu,t) = \bigcap_{q\in Q} U(\mu,t,q)$. If $U(\mu,t,q) \neq \emptyset$ for all $q \in Q$, it follows from Theorem 3.14 (3) that $U(\mu,t,q)$ is an ideal of H for all $q \in Q$. By Lemma 2.16 (3) and the intersection of any ideals of H is also ideal in H, we have $U(\mu,t) = \bigcap_{q\in Q} U(\mu,t,q)$ is an ideal of H.
(4). Similarly to as in the proof of (3).

□

Corollary 3.16. *Let S be a subalgebra of H. Then the following statements hold:*

1. *for any $k \in (0,1]$, there exists a Q-fuzzy subalgebra γ of H such that $L(\overline{\gamma},t) = S$ for all $t < k$ and $L(\overline{\gamma},t) = H$ for all $t \geq k$,*

2. *for any $k \in [0,1)$, there exists a Q-fuzzy subalgebra μ of H such that $U(\mu,t) = S$ for all $t > k$ and $U(\mu,t) = H$ for all $t \leq k$.*

Proof. (1). Let μ be a Q-fuzzy set in H defined by, for all $q \in Q$

$$\mu(x,q) = \begin{cases} 0 & \text{if } x \in S \\ k & \text{if } x \notin S. \end{cases}$$

In the proof of Corollary 3.17 (1), we have $L(\mu,t) = S$ for all $t < k$ and $L(\mu,t) = A$ for all $t \leq k$, $L(\mu,t,q) = L(\mu,t,q')$ for all $q,q' \in Q$. By Lemma 2.16 (1), we have $L(\mu,t) = \bigcap_{q\in Q} L(\mu,t,q)$. By the claim, we have $L(\mu,t) = L(\mu,t,q)$ for all $q \in Q$. Since $L(\mu,t,q) = L(\mu,t) = S$ for all $t < k$ and $L(\mu,t) = L(\mu,t) = H$ for all $t \geq k$, it follows that $\overline{\mu}$ is a Q-fuzzy subalgebra of H. By Remark 2.9, we have $L(\overline{\overline{\mu}},t) = L(\mu,t) = S$ for all $t < k$ and $L(\overline{\overline{\mu}},t) = L(\mu,t) = H$ for all $t \geq k$. Let $\overline{\mu} = \theta$. Then θ is a Q-fuzzy subalgebra of H such that $L(\overline{\mu},t) = L(\mu,t) = S$ for all $t < k$ and $L(\overline{\mu},t) = L(\mu,t) = H$ for all $t \geq k$.

(2). Let μ be a Q-fuzzy set in H defined by, for all $q \in Q$

$$\mu(x) = \begin{cases} 1 & \text{if } x \in S \\ k & \text{if } x \notin S. \end{cases}$$

In the proof of Corollary 3.17 (2), we have $U(\mu, t) = S$ for all $t > k$ and $U(\mu, t) = H$ for all $t \leq k$ and $U(\mu, t, q) = U(\mu, t, q')$ for all $q, q' \in Q$. By Lemma 2.16 (3), we have $U(\mu, t) = \bigcap_{q \in Q} U(\mu, t, q)$. By the claim, we have $U(\mu, t) = U(\mu, t, q)$ for all $q \in Q$. Since $U(\mu, t, q) = U(\mu, t) = S$ for all $t > k$ and $U(\mu, t) = U(\mu, t) = H$ for all $t \leq k$, it follows that μ is a Q-fuzzy subalgebra of H. □

Corollary 3.17. *Let I be an ideal of H. Then the following statements hold:*

1. *for any $k \in (0, 1]$, there exists a Q-fuzzy ideal μ of H such that $L(\overline{\gamma}, t) = I$ for all $t < k$ and $L(\overline{\gamma}, t) = H$ for all $t \geq k$,*
2. *for any $k \in [0, 1)$, there exists a Q-fuzzy ideal γ of H such that $U(\gamma, t) = I$ for all $t > k$ and $U(\gamma, t) = H$ for all $t \leq k$.*

Proof. (1). Let μ be a Q-fuzzy set in H defined by, for all $q \in Q$

$$\mu(x, q) = \begin{cases} 0 & \text{if } x \in I \\ k & \text{if } x \notin I. \end{cases}$$

Case 1 : To show that $L(\mu, t) = I$ for all $t < k$, let $t \in [0, 1]$ be such that $t < k$. Let $x \in L(\mu, t)$. Then $\mu(x, q) \leq t < k$. Thus $\mu(x, q) \neq k$ for all $q \in Q$, so $\mu(x, q) = 0$ for all $q \in Q$. Then $x \in I$, so $L(\mu, t) \subseteq I$. Now, let $x \in I$. Then $\mu(x, q) = 0 \leq t$ for all $q \in Q$. Thus $x \in L(\mu, t)$, so $I \subseteq L(\mu, t)$. Hence $L(\mu, t) = I$ for all $t < k$.
Case 2 : To show that $L(\mu, t) = A$ for all $t \geq k$, let $t \in [0, 1]$ be such that $t \geq k$. Clearly, $L(\mu, t) \subseteq H$. Let $x \in H$. Then for all $q \in Q$, we define

$$\mu(x, q) = \begin{cases} 0 < t & \text{if } x \in I \\ k \leq t & \text{if } x \notin I. \end{cases}$$

Then $x \in L(\mu, t)$, so $H \subseteq L(\mu, t)$. Hence $L(\mu, t) = H$ for all $t \leq k$. We claim that $L(\mu, t, q) = L(\mu, t, q')$ for all $q, q' \in Q$. For $q, q' \in Q$, we obtain

$$\begin{aligned} x \in L(\mu, t, q) &\Leftrightarrow \mu(x, q) \leq t \\ &\Leftrightarrow \mu(x, q') \leq t \\ &\Leftrightarrow x \in L(\mu, t, q'). \end{aligned}$$

Hence $L(\mu, t, q) = L(\mu, t, q')$ for all $q, q' \in Q$. By Lemma 2.16 (1), we have $L(\mu, t) = \bigcap_{q \in Q} L(\mu, t, q)$. By the claim, we have $L(\mu, t) = L(\mu, t, q)$ for all $q \in Q$. Since $L(\mu, t, q) = L(\mu, t) = I$ for all $t < k$ and

$L(\mu, t) = L(\mu, t) = H$ for all $t \geq k$, it follows from Theorem 3.14 (1) that $\overline{\mu}$ is a Q-fuzzy ideal of H. By Remark 2.9, we have $L(\overline{\overline{\mu}}, t) = L(\mu, t) = I$ for all $t < k$ and $L(\overline{\overline{\mu}}, t) = L(\mu, t) = H$ for all $t \geq k$. Let $\overline{\mu} = \theta$. Then θ is a Q-fuzzy ideal of H such that $L(\overline{\mu}, t) = L(\mu, t) = I$ for all $t < k$ and $L(\overline{\mu}, t) = L(\mu, t) = H$ for all $t \geq k$.

(2). Let μ be a Q-fuzzy set in H defined by, for all $q \in Q$

$$\mu(x, q) = \begin{cases} 1 & \text{if } x \in I \\ k & \text{if } x \notin I. \end{cases}$$

Case 1 : To show that $U(\mu, t) = I$ for all $t > k$, let $t \in [0, 1]$ be such that $t > k$. Let $x \in U(\mu, t)$. Then $\mu(x, q) \geq t > k$ for all $q \in Q$. Then $\mu(x, q) \neq k$ for all $q \in Q$, so $\mu(x, q) = 1$ for all $q \in Q$. Thus $x \in I$, so $U(\mu, t) \subseteq I$. Now, let $x \in I$. Then $\mu(x, q) = 1 \geq t$ for all $q \in Q$. Then $x \in U(\mu, t)$, so $I \subseteq U(\mu, t)$. Hence $U(\mu, t) = I$ for all $t > k$.

Case 2 : To show that $U(\mu, t) = H$ for all $t \leq k$, let $t \in [0, 1]$ be such that $t \leq k$. Clearly, $U(\mu, t) \subseteq A$. Let $x \in H$. Then for all $q \in Q$

$$\mu(x) = \begin{cases} 1 > t & \text{if } x \in I \\ k \geq t & \text{if } x \notin I. \end{cases}$$

Then $x \in U(\mu, t)$, so $A \subseteq U(\mu, t)$. Hence $U(\mu, t) = H$ for all $t \leq k$. We claim that $U(\mu, t, q) = U(\mu, t, q')$ for all $q, q' \in Q$. For $q, q' \in Q$, we obtain

$$\begin{aligned} x \in U(\mu, t, q) &\Leftrightarrow \mu(x, q) \leq t \\ &\Leftrightarrow \mu(x, q') \leq t \\ &\Leftrightarrow x \in U(\mu, t, q'). \end{aligned}$$

Hence $U(\mu, t, q) = U(\mu, t, q')$ for all $q, q' \in Q$. By Lemma 2.16 (1), we have $U(\mu, t) = \bigcap_{q \in Q} U(\mu, t, q)$. By the claim, we have $U(\mu, t) = U(\mu, t, q)$ for all $q \in Q$. Since $U(\mu, t, q) = U(\mu, t) = I$ for all $t > k$ and $U(\mu, t) = U(\mu, t) = H$ for all $t \leq k$, it follows from Theorem 3.14 (3) that μ is a Q-fuzzy ideal of H. □

Theorem 3.18. *Let μ be a Q-fuzzy set in H. Then the following statements hold:*

1. *$\overline{\mu}$ is a Q-fuzzy subalgebra of H if and only if for any $t \in [0, 1]$ and $q \in Q$, $L(\mu, t, q)$ is either empty or a subalgebra of H.*

2. *$\overline{\mu}$ is a Q-fuzzy subalgebra of H if and only if for any $t \in [0, 1]$ and $q \in Q$, $L^{-}(\mu, t, q)$ is either empty or a subalgebra of H.*

3. *μ is a Q-fuzzy subalgebra of H if and only if for any $t \in [0, 1]$ and $q \in Q$, $U(\mu, t, q)$ is either empty or a subalgebra of H.*

4. *μ is a Q-fuzzy subalgebra of H if and only if for any $t \in [0, 1]$ and $q \in Q$, $U^{+}(\mu, t, q)$ is either empty or a subalgebra of H.*

Proof. (1). Assume that $\overline{\mu}$ is a Q-fuzzy subalgebra of H. Then $\overline{\mu}$ is a Q-fuzzy subalgebra of H for all $q \in Q$. Let $q \in Q$ and $t \in [0,1]$ be such that $L(\mu,t,q) \neq \emptyset$ and let $x, y \in L(\mu,t,q)$. Then $\mu(x,q) \leq t$ and $\mu(t,q) \leq t$ and let $x, y \in L(\mu,t,q)$. Then $\mu(x,q) \leq t$ and $\mu(y,q) \leq t$. Now

$$\begin{aligned}\overline{\mu}(x*y,q) &\geq \min\{\overline{\mu}(x,q), \overline{\mu}(y,q)\}\\ &= \min\{1-\mu(x,q), 1-\mu(y,q)\}\\ &= 1-\max\{\mu(x,q), \mu(y,q)\}.\end{aligned}$$

Then $\overline{\mu}(x*y,q) \leq \max\{\mu(x,q), \mu(y,q)\} \leq t$, so, $x*y \in L(\mu,t,q)$. Hence $L(\mu,t,q)$ is a subalgebra of H. Conversely, let $x, y \in H$ and $q \in Q$ and let $t = \max\{\mu(x,q), \mu(y,q)\}$. Thus $\mu(x,q) \leq t$ and $\mu(y,q) \leq t$, so $x, y \in L(\mu,t,q) \neq \emptyset$. By assumption we have $L(\mu,t,q)$ is a subalgebra of H. It follows that $x*y \in L(\mu,t,q)$. Thus, $\mu(x*y,q) \leq t = \max\{\mu(x,q), \mu(y,q)\}$, so

$$1-\mu(x*y,q) \geq 1-\max\{\mu(x,q), \mu(y,q)\} = \min\{1-\mu(x,q), 1-\mu(y,q)\}.$$

Hence $\mu(x*y,q) \geq \{1-\mu(x,q), 1-\mu(y,q)\}$. Therefore, μ is a Q-fuzzy subalgebra of H for all $q \in Q$. Consequently, $\overline{\mu}$ is a Q-fuzzy subalgebra of H.

(2). Similarly to as in the proof of the necessity of (1). Conversely, there exist $x, y \in H$ and $q \in Q$ such that $\overline{\mu}(x*y,q) < \min\{\overline{\mu}(x,q), \overline{\mu}(y,q)\}$. Then we have $1-\overline{\mu}(x*y,q) < \min\{\overline{\mu}(x,q), \overline{\mu}(y,q)\} = 1-\max\{\mu(x,q), \mu(y,q)\}$. Thus, $\mu(x*y,q) > \max\{\mu(x,q), \mu(y,q)\}$. Now $\mu(x*y,q) \in [0,1]$, we choose $t = \mu(x*y,q)$. Thus $\mu(x,q) < t$ and $\mu(y,q) < t$, so $x, y \in L^{-}(\mu,t,q)$. By assumption, we have $L^{-}(\mu,t,q)$ is a subalgebra of H and so $x*y \in L^{-}(\mu,t,q) \neq \emptyset$. Thus, $\mu(x*y,q) < t = \mu(x*y,q)$, which is a contradiction. Hence $\overline{\mu}(x*y,q) \geq \min\{\overline{\mu}(x,q), \overline{\mu}(y,q)\}$ for all $x, y \in H$ and $q \in Q$. Therefore, $\overline{\mu}$ is a Q-fuzzy subalgebra of H for all $q \in Q$. Consequently, $\overline{\mu}$ is a Q-fuzzy subalgebra of H.

(3). Assume that μ is a Q-fuzzy subalgebra of H. Then μ is a Q-fuzzy subalgebra of H for all $q \in Q$. Let $q \in Q$ and $t \in [0,1]$ be such that $U(\mu,t,q) \neq \emptyset$ and let $x, y \in U(\mu,t,q)$. Then $\mu(x,q) \geq t$ and $\mu(t,q) \geq t$ and let $x, y \in U(\mu,t,q)$. Then $\mu(x,q) \geq t$ and $\mu(y,q) \geq t$. Now $\mu(x*y,q) \geq \min\{\mu(x,q), \mu(y,q)\} \geq t$. Then $x*y \in U(\mu,t,q)$. Hence $U(\mu,t,q)$ is a subalgebra of H. Conversely, let $x, y \in H$ and $q \in Q$ and let $t = \min\{\mu(x,q), \mu(y,q)\}$. Thus $\mu(x,q) \geq t$ and $\mu(y,q) \geq t$, so $x, y \in U(\mu,t,q) \neq \emptyset$. By assumption we have $U(\mu,t,q)$ is a subalgebra of H. It follows that $x*y \in U(\mu,t,q)$. Thus, $\mu(x*y,q) \geq t = \min\{\mu(x,q), \mu(y,q)\}$. Hence μ is a Q-fuzzy subalgebra of H for all $q \in Q$. Consequently, μ is a Q-fuzzy subalgebra of H.

(4). Similarly to as in the proof of the necessity of (3). Conversely, there exist $x, y \in H$ and $q \in Q$ such that $\mu(x*y,q) < \min\{\mu(x,q), \mu(y,q)\}$. Then $\mu(x*y,q) \in [0,1]$, we choose $t = \mu(x*y,q)$. Thus $\mu(x,q) > t$ and $\mu(y,q) > t$, so $x, y \in U^{+}(\mu,t,q)$. By assumption, we have $U^{+}(\mu,t,q)$ is a subalgebra of H and so $x*y \in U^{+}(\mu,t,q) \neq \emptyset$. Thus, $\mu(x*y,q) > t = \mu(x*y,q)$, which is a contradiction. Hence $\mu(x*y,q) \geq \min\{\mu(x,q), \mu(y,q)\}$ for all $x, y \in H$ and $q \in Q$. Therefore, μ is a Q-fuzzy subalgebra of H for all $q \in Q$. Consequently, μ is a Q-fuzzy subalgebra of H. □

Corollary 3.19. *Let μ be a Q-fuzzy set in H. Then the following statements hold:*

1. *$\overline{\mu}$ is a Q-fuzzy subalgebra of H, then for any $t \in [0,1]$, $L(\mu,t)$ is either empty or a subalgebra of H.*
2. *$\overline{\mu}$ is a Q-fuzzy subalgebra of H, then for any $t \in [0,1]$, $L^{-}(\mu,t)$ is either empty or a subalgebra of H.*
3. *μ is a Q-fuzzy subalgebra of H, then for any $t \in [0,1]$, $U(\mu,t)$ is either empty or a subalgebra of H.*
4. *μ is a Q-fuzzy subalgebra of H, for any $t \in [0,1]$, $U^{+}(\mu,t)$ is either empty or a subalgebra of H.*

Proof. (1). Assume that $\overline{\mu}$ is a Q-fuzzy subalgebra of H. By Theorem 3.18 (1), we have for any $t \in [0,1]$ and $q \in Q$. Let $L(\mu,t,q)$ is either empty or a subalgebra of H. Let $t \in [0,1]$. If $L(\mu,t,q) = \emptyset$ for some $q \in Q$, it follows Lemma 2.16 (1) that $L(\mu,t) = \bigcap_{q \in Q} L(\mu,t,q)$. If $L(\mu,t,q) \neq \emptyset$ for all $q \in Q$, it follows from Theorem 3.18 (1) that $L(\mu,t,q)$ is a subalgebra of H for all $q \in Q$. By Lemma 2.16 (1) and the intersection of any ideals of H is also ideal in H, we have $L(\mu,t) = \bigcap_{q \in Q} L(\mu,t,q)$ is a subalgebra of H.
(2). Similarly to as in the proof of (1).
(3). Assume that μ is a Q-fuzzy subalgebra of H. By Theorem 3.18 (3), we have for any $t \in [0,1]$ and $q \in Q$. Let $U(\mu,t,q)$ is either empty or a subalgebra of H. Let $t \in [0,1]$. If $U(\mu,t,q) = \emptyset$ for some $q \in Q$, it follows Lemma 2.16 (3) that $U(\mu,t) = \bigcap_{q \in Q} U(\mu,t,q)$. If $U(\mu,t,q) \neq \emptyset$ for all $q \in Q$, it follows from Theorem 3.18 (3) that $U(\mu,t,q)$ is a subalgebra of H for all $q \in Q$. By Lemma 2.16 (3) and the intersection of any ideals of H is also ideal in H, we have $U(\mu,t) = \bigcap_{q \in Q} U(\mu,t,q)$ is a subalgebra of H.
(4). Similarly to as in the proof of (3).

□

Let $(A,*,1_A)$ and $(B,\star,1_B)$ be Hilbert algebras A mapping $f : A \to B$ is called a homomorphism if $f(x*y) = f(x) \star f(y)$ for all $x,y \in A$. Note that if $f : X \to Y$ is a homomorphism of Hilbert algebras, then $f(1_A) = 1_B$. Let $f : X \to Y$ be a homomorphism of Hilbert algebras.

Theorem 3.20. *Let $(A,*,1_A)$ and $(B,\star,1_B)$ be Hilbert algebras and let $f : A \to B$ be a homomorphism. Then the following statements hold:*

1. *if μ is a Q-fuzzy ideal of B, then μ_f is also a Q-fuzzy ideal of A.*
2. *if μ is a Q-fuzzy subalgebra of B, then μ_f is also a Q-fuzzy subalgebra of A.*

Proof. (1). Assume that μ be a Q-fuzzy ideal of B. Let $x \in A$. Then

$$\begin{aligned}
\mu_f(1_A,q) &= \mu(f(1_A),q) \\
&= \mu(1_B,q) \\
&\geq \mu(f(x),q) \\
&= \mu_f(x,q).
\end{aligned}$$

Let $x, y \in A$. Then

$$\begin{aligned} \mu_f(x * y, q) &= \mu(f(x * y), q) \\ &= \mu(f(x) \star f(y), q) \\ &\geq \mu(f(y), q) \\ &= \mu_f(y, q). \end{aligned}$$

Let $x, y_1, y_2 \in A$. Then

$$\begin{aligned} \mu_f(((y_1 * (y_2 * x)) * x), q) &= \mu(f(((y_1 * (y_2 * x)) * x), q)) \\ &= \mu((f(y_1) \star (f(y_2) \star f(x)) \star f(x), q) \\ &\geq \min\{\mu(f(y_1), q), \mu(f(y_2), q)\} \\ &= \min\{\mu_f(y_1, q), \mu_f(y_2, q)\}. \end{aligned}$$

Hence μ_f is a Q-fuzzy ideal of A.

(2). Assume that μ be a Q-fuzzy subalgebra of B. Let $x, y \in A$. Then

$$\begin{aligned} \mu_f(x * y, q) &= \mu(f(x * y), q) \\ &= \mu(f(x) \star f(y), q) \\ &\geq \min\{\mu(f(x), q), \mu(f(y), q)\} \\ &= \min\{\mu_f(x, q), \mu_f(y, q)\}. \end{aligned}$$

Hence μ_f is a Q-fuzzy subalgebra of A. □

Theorem 3.21. *Let $(A, *, 1_A)$ and $(B, \star, 1_B)$ be Hilbert algebras and let $f : A \to B$ be an isomorphism. Then the following statements hold:*

1. *if μ_f is a Q-fuzzy ideal of A, then μ is also a Q-fuzzy ideal of B.*

2. *if μ_f is a Q-fuzzy subalgebra of A, then μ is also a Q-fuzzy subalgebra of B.*

Proof. (1). Assume that μ_f be a Q-fuzzy ideal of A.

Let $y \in B$. Then there exists $x \in A$ such that $f(x) = y$, we have

$$\begin{aligned} \mu(1_B, q) &= \mu(y \star 1_B, q) \\ &= \mu(f(x) \star f(1_A), q) \\ &= \mu(f(x * 1_A), q) \\ &= \mu_f(x * 1_A, q) \\ &= \mu_f(1_A, q) \\ &\geq \mu_f(x, q) \\ &\geq \mu(f(x), q) \\ &= \mu(y, q). \end{aligned}$$

Let $x, y \in B$. Then there exist $a, b \in X$ such that $f(a) = x$ and $f(b) = y$. It follows that

$$\begin{aligned}
\mu(x \star y, q) &= \mu(f(a) \star f(b), q) \\
&= \mu(f(a * b), q) \\
&= \mu_f(a * b, q) \\
&\geq \mu_f(b, q) \\
&= \mu(f(b), q) \\
&= \mu(y, q).
\end{aligned}$$

Let $x, y_1, y_2 \in B$. Then there exist $y, x_1, x_2 \in A$ such that $f(y) = x$, $f(x_1) = y_1$ and $f(x_2) = y_2$. It follows that

$$\begin{aligned}
\mu((y_1 \star (y_2 \star x)) \star x, q) &= \mu((f(x_1) \star (f(x_2) \star f(y)) \star f(y), q) \\
&= \mu(f((x_1 * (x_2 * y)) * y), q) \\
&= \mu_f((x_1 * (x_2 * y)) * y), q) \\
&\geq \min\{\mu_f(x_1, q), \mu_f(x_2, q)\} \\
&= \min\{\mu(f(x_1), q), \mu(f(x_2), q)\} \\
&= \min\{\mu(y_1, q), \mu(y_2, q)\}.
\end{aligned}$$

Hence μ is a Q-fuzzy ideal of B.

(2). Assume that μ_f be a Q-fuzzy ideal of A.

Let $x, y \in B$. Then there exist $a, b \in A$ such that $f(a) = x$ and $f(b) = y$. It follows that

$$\begin{aligned}
\mu(x \star y, q) &= \mu(f(a) \star f(b), q) \\
&= \mu(f(a \star b), q) \\
&= \mu_f(a * b, q) \\
&\geq \min\{\mu_f(a, q), \mu_f(b, q)\} \\
&= \min\{\mu(f(a), q), \mu(f(b), q)\} \\
&= \min\{\mu(x, q), \mu(y, q)\}.
\end{aligned}$$

Hence μ is a Q-fuzzy subalgebra of B. □

Remark 3.22. *Let $(A, *, 1_A)$ and $(B, \star, 1_B)$ be Hilbert algebras. Then $A \times B$ is a Hilbert algebra defined by $(x, y) \diamond (u, v) = (x * u, y \star v)$ for every $x, u \in A$ and $y, v \in B$, then clearly $(A \times B, \diamond, (1_A, 1_B))$ is a Hilbert algebra.*

Theorem 3.23. *Let $(A, *, 1_A)$ and $(B, \star, 1_B)$ be Hilbert algebras. Then the following statements hold:*

1. *if μ is a Q-fuzzy ideal of A and δ is a Q-fuzzy ideal of B, then $\mu \cdot \delta$ is a Q-fuzzy ideal of $A \times B$.*
2. *if μ is a Q-fuzzy subalgebra of A and δ is a Q-fuzzy subalgebra of B, then $\mu \cdot \delta$ is a Q-fuzzy subalgebra of $A \times B$.*

Proof. (1). Assume that μ is a Q-fuzzy ideal of A and δ is a Q-fuzzy ideal B. Let $(x,y) \in A \times B$. Then

$$\begin{aligned}(\mu \cdot \delta)((1_A, 1_B), q) &= \min\{\mu(1_A, q), \delta(1_B, q)\} \\ &\geq \min\{\mu(x,q), \delta(y,q)\}\} \\ &= (\mu \cdot \delta)((x,y), q).\end{aligned}$$

Let $(x_1, y_1), (x_2, y_2) \in A \times B$. Then

$$\begin{aligned}(\mu \cdot \delta)((x_1, y_1) \diamond (x_2, y_2), q) &= (\mu \cdot \delta)((x_1 \cdot y_1, x_2 \star y_2), q) \\ &= \min\{\mu(x_1 \cdot y_1, q), \delta(x_2 \star y_2, q)\} \\ &\geq \min\{\mu(y_1, q), \delta(y_2, q)\} \\ &= (\mu \cdot \delta)((y_1, y_2), q).\end{aligned}$$

Let $(x,y), (x_1, y_1), (x_2, y_2) \in A \times B$. Then
$(\mu \cdot \delta)((x_1, y_1) \diamond ((x_2, y_2) \diamond (x,y))) \diamond (x,y), q)$

$$\begin{aligned}&= (\mu \cdot \delta)((x_1 * (x_2 * x)) * x), (y_1 \star (y_2 \star y)) \star y), q) \\ &= \min\{\mu((x_1 * (x_2 * x)) * x, q), \delta((y_1 \star (y_2 \star y)) \star y, q)\} \\ &\geq \min\{\min\{\mu(x_1, q), \mu(x_2, q)\}, \min\{\delta(y_1, q), \delta(y_2, q)\}\} \\ &= \min\{\min\{\mu(x_1, q), \delta(y_1, q)\}, \min\{\mu(x_2, q), \delta(y_2, q)\}\} \\ &= \min\{(\mu \cdot \delta)((x_1, y_1), q), (\mu \cdot \delta)((x_2, y_2), q)\}.\end{aligned}$$

Hence $\mu \cdot \delta$ is a Q-fuzzy ideal of $A \times B$.
(2). Assume that μ is a Q-fuzzy subalgebra of A and δ is a Q-fuzzy subalgebra B. Let $(x_1, y_1), (x_2, y_2) \in A \times B$. Then
$(\mu \cdot \delta)((x_1, y_1) \diamond (x_2, y_2), q)$

$$\begin{aligned}&= (\mu \cdot \delta)((x_1 * y_1, x_2 \star y_2), q) \\ &= \min\{\mu(x_1 * y_1, q), \delta(x_2 \star y_2, q)\} \\ &\geq \min\{\min\{\mu(x_1, q), \mu(y_1, q)\}, \min\{\delta(x_2, q), \delta(y_2, q)\}\} \\ &= \min\{\min\{\mu(x_1, q), \delta(x_2, q)\}, \min\{\mu(y_1, q), \delta(y_2, q)\}\} \\ &= \min\{(\mu \cdot \delta)((x_1, x_2), q), (\mu \cdot \delta)((y_1, y_2), q)\}.\end{aligned}$$

Hence $\mu \cdot \delta$ is a Q-fuzzy subalgebra of $A \times B$. □

Theorem 3.24. *If μ is a Q-fuzzy set of A and δ is a Q-fuzzy set of B such that $\mu \cdot \delta$ is a Q-fuzzy ideal of $A \times B$, then the following statements hold:*

1. *for all $q \in Q$, either $\mu(1_A, q) \geq \mu(x,q)$ for all $x \in A$ or $\delta(1_B, q) \geq \delta(x,q)$ for all $x \in B$,*

2. *for all $q \in Q$, if $\mu(1_A, q) \geq \mu(x,q)$ for all $x \in A$, then either $\delta(1_B, q) \geq \mu(x,q)$ for all $x \in A$ or $\delta(1_B, q) \geq \delta(x,q)$ for all $x \in B$,*

3. *for all $q \in Q$, if $\delta(1_A, q) \geq \delta(x, q)$ for all $x \in B$, then either $\mu(1_A, q) \geq \mu(x, q)$ for all $x \in A$ or $\mu(1_A, q) \geq \delta(x, q)$ for all $x \in B$.*

Proof. (1). Suppose that there exist $x \in A$ and $y \in B$ such that $\mu(1_A, q) < \mu(x, q)$ and $\delta(1_B, q) < \delta(y, q)$. Then

$$\begin{aligned} (\mu \cdot \delta)((x, y), q) &= \min\{\mu(x, q), \delta(y, q)\} \\ &> \min\{\mu(1_A, q), \delta(1_B, q)\}\} \\ &= (\mu \cdot \delta)((1_A, 1_B), q), \end{aligned}$$

which is a contradiction. Hence $\mu(1_A, q) \geq \mu(x, q)$ for all $x \in A$ or $\delta(1_B, q) \geq \delta(x, q)$ for all $x \in B$.
(2). Assume that $\mu(1_A, q) \geq \mu(x, q)$ for all $x \in A$. Suppose that there exist $x \in A$ and $y \in B$ such that $\mu(1_A, q) < \mu(x, q)$ and $\delta(1_B, q) < \delta(y, q)$. Then $\mu(1_A, q) \geq \mu(x, q) > \delta(1_B, q)$. Thus

$$\begin{aligned} (\mu \cdot \delta)((x, y), q) &= \min\{\mu(x, q), \delta(y, q)\} \\ &> \min\{\mu(1_A, q), \delta(1_B, q)\}\} \\ &= \delta(1_B, q) \\ &= \min\{\mu(1_A, q), \delta(1_B, q)\} \\ &= (\mu \cdot \delta)((1_A, 1_B), q), \end{aligned}$$

which is a contradiction. Hence $\delta(1_A, q) \geq \mu(x, q)$ for all $x \in A$ or $\delta(1_B, q) \geq \delta(x, q)$ for all $x \in B$.
(3). Assume that $\delta(1_A, q) \geq \delta(x, q)$ for all $x \in B$. Suppose that there exist $x \in A$ and $y \in B$ such that $\mu(1_A, q) < \mu(x, q)$ and $\mu(1_A, q) < \delta(y, q)$. Then $\delta(1_B, q) \geq \delta(x, q) > \mu(1_A, q)$. Thus

$$\begin{aligned} (\mu \cdot \delta)((x, y), q) &= \min\{\mu(x, q), \delta(y, q)\} \\ &> \min\{\mu(1_A, q), \mu(1_A, q)\}\} \\ &= \mu(1_A, q) \\ &= \min\{\mu(1_A, q), \delta(1_B, q)\} \\ &= (\mu \cdot \delta)((1_A, 1_B), q), \end{aligned}$$

which is a contradiction. Hence $\mu(1_A, q) \geq \mu(x, q)$ for all $x \in A$ or $\mu(1_B, q) \geq \delta(x, q)$ for all $x \in B$. □

Theorem 3.25. *Let $(A, \cdot, 1_A)$ and $(B, \star, 1_B)$ be Hilbert algebras and let μ be a Q-fuzzy set in A and δ be a Q-fuzzy set in B. Then the following statements hold:*

1. *if $\mu \cdot \delta$ is a Q-fuzzy ideal of $A \times B$, then either μ is a Q-fuzzy ideal of A or δ is a Q-fuzzy ideal of B,*

2. *if $\mu \cdot \delta$ is a Q-fuzzy subalgebra of $A \times B$, then either μ is a Q-fuzzy subalgebra of A or δ is a Q-fuzzy subalgebra of B.*

Proof. Assume that $\mu \cdot \delta$ is a Q-fuzzy ideal of $A \times B$. Suppose that μ is not a Q-fuzzy ideal of A and δ is not a Q-fuzzy ideal of B. Then we have $\mu(1_A, q) \geq \mu(x, q)$ for all $x \in A$ or $\delta(1_B, q) \geq \delta(x, q)$ for all $x \in B$. Suppose that $\mu(1_A, q) \geq \mu(x, q)$ for all $x \in A$. Then either $\delta(1_B, q) \geq \mu(x, q)$ for all

$x \in A$ or $\delta(1_B, q) \geq \delta(x, q)$ for all $x \in B$. If $\delta(1_B, q) \geq \mu(x, q)$ for all $x \in A$, then $(\mu \cdot \delta)((x, 1_B), q) = \min\{\mu(x, q), \delta(1_B, q)\} = \mu(x, q)$. We consider, for all $x, y \in A$,

$$\begin{aligned}
\mu(x \cdot y, q) &= \min\{\mu(x \cdot y, q), \delta(1_B, q)\} \\
&= (\mu \cdot \delta)((x * y, 1_B), q) \\
&= (\mu \cdot \delta)((x * y, 1_B \star 1_B), q) \\
&= (\mu \cdot \delta)((x, 1_B) \diamond (y, 1_B), q) \\
&\geq (\mu \cdot \delta)((y, 1_B), q) \\
&= \min\{\mu(y, q), \delta(1_B, q)\} \\
&= \mu(y, q).
\end{aligned}$$

Let $x, y_1, y_2 \in A$.

$$\begin{aligned}
\mu((y_1 \cdot (y_2 * x)) * x, q) &= \min\{\mu((y_1 * (y_2 * x)) * x, q), \delta(1_B, q)\} \\
&= (\mu \cdot \delta)(((y_1 * (y_2 * x)) * x, 1_B), q) \\
&= (\mu \cdot \delta)((y_1 * (y_2 * x)) * x, (1_B \star (1_B \star 1_B)) \star 1_B, q) \\
&= (\mu \cdot \delta)(((y_1, 1_B) \diamond ((y_2, 1_B) \diamond (x, 1_B))) \diamond (x, 1_B), q) \\
&\geq \min\{(\mu \cdot \delta)((y_1, 1_B), q), (\mu \cdot \delta)((y_2, 1_B), q)\} \\
&= \min\{\min\{\mu(y_1, q), \delta(1_B, q)\}, \min\{\mu(y_2, q), \delta(1_B, q)\}\} \\
&= \min\{\mu(y_1, q), \mu(y_2, q)\}.
\end{aligned}$$

Hence μ is a Q-fuzzy ideal of A, which is a contradiction. Suppose that $\delta(1_B, q) \geq \delta(x, q)$ for all $x \in B$. Then either $\mu(1_A, q) \geq \mu(x, q)$ for all $x \in B$ or $\mu(1_A, q) \geq \delta(x, q)$ for all $x \in B$. If $\mu(1_A, q) \geq \delta(x, q)$ for all $x \in B$, then $(\mu \cdot \delta)((1_A, x), q) = \min\{\mu(1_A, q), \delta(x, q)\} = \delta(x, q)$. We consider, for all $x, y \in B$,

$$\begin{aligned}
\delta(x \cdot y, q) &= \min\{\mu(1_A, q), \delta(x \star y, q)\} \\
&= (\mu \cdot \delta)((1_A, x \star y), q) \\
&= (\mu \cdot \delta)((1_A * 1_A, x \star y), q) \\
&= (\mu \cdot \delta)((1_A, x) \diamond (1_A, y), q) \\
&\geq (\mu \cdot \delta)((1_A, y), q) \\
&= \min\{\mu(1_A, q), \delta(y, q)\} \\
&= \delta(y, q).
\end{aligned}$$

Let $x, y_1, y_2 \in B$.

$$\begin{aligned}
\delta((y_1 * (y_2 * x)) * x, q) &= \min\{\mu(1_A, q), \delta((y_1 \star (y_2 \star x)) \star x, q)\} \\
&= (\mu \cdot \delta)((1_A, (y_1 \star (y_2 \star x)) \star x, q) \\
&= (\mu \cdot \delta)((1_A * (1_A * 1_A)) * 1_A, (y_1 \star (y_2 \star x)) \star x, q) \\
&= (\mu \cdot \delta)((((1_A, y_1) \diamond ((1_A, y_2) \diamond (1_A, x))) \diamond (1_A, x), q) \\
&\geq \min\{(\mu \cdot \delta)((1_A, y_1), q), (\mu \cdot \delta)((1_A, y_2), q)\} \\
&= \min\{\min\{\mu(1_A, q), \delta(y_1, q)\}, \min\{\mu(1_A, q), \delta(y_2, q)\}\} \\
&= \min\{\delta(y_1, q), \delta(y_2, q)\}.
\end{aligned}$$

Hence δ is a Q-fuzzy ideal of B, which is a contradiction. Since μ is not a Q-fuzzy ideal of A and δ is not a Q-fuzzy ideal of B, we have $\mu(1_A, q) \geq \mu(x, q)$ for all $x \in A$ and $\delta(1_B, q) \geq \delta(x, q)$ for all $x \in B$. Let $x_1, x_2 \in A$ and $y_1, y_2 \in B$ be such that $\mu(x_1 \cdot x_2, q) < \mu(x_2, q)$ and $\delta(y_1 \star y_2, q) < \delta(y_2, q)$, so we have $\min\{\mu(x_1 \cdot x_2, q), \delta(y_1 \star y_2, q)\} < \min\{\mu(x_2, q), \delta(y_2, q)\}$. Thus

$$\begin{aligned}
\min\{\mu(x_1 \cdot x_2, q), \delta(y_1 \star y_2, q)\} &= (\mu \cdot \delta)((x_1 * x_2, y_1 \star y_2), q) \\
&= (\mu \cdot \delta)((x_1, y_1) \diamond (x_2, y_2), q) \\
&\geq (\mu \cdot \delta)((x_2, y_2), q) \\
&= \min\{\mu(x_2, q), \delta(y_2, q)\}.
\end{aligned}$$

It follows that $\min\{\mu(x_1 \cdot x_2, q), \delta(y_1 \star y_2, q)\} \not< \min\{\mu(x_2, q), \delta(y_2, q)\}$, which is a contradiction. Let $x, x_1, x_2 \in A$ and $y, y_1, y_2 \in B$ such that $\mu((x_1 * (x_2 * x)) * x, q) < \min\{\mu(x_1, q), \mu(x_2, q)\}$ and $\delta((y_1 \star (y_2 \star y)) \star y, q) < \min\{\delta(y_1, q), \delta(y_2, q)\}$, so $\min\{\mu((x_1 * (x_2 * x)) * x, q), \delta((y_1 \star (y_2 \star y)) \star y, q)\} < \min\{\min\{\mu(x_1, q), \mu(x_2, q)\}, \min\{\delta(y_1, q), \delta(y_2, q)\}\}$. Thus
$\min\{\mu((x_1 * (x_2 * x)) * x, q), \delta((y_1 \star (y_2 \star y)) \star y, q)\}$

$$\begin{aligned}
&= (\mu \cdot \delta)(((x_1 \cdot (x_2 \cdot x)) \cdot x, (y_1 \star (y_2 \star y)) \star y), q) \\
&= (\mu \cdot \delta)((((x_1, y_1) \diamond ((x_2, y_2) \diamond (x, y))) \diamond (x, y)), q) \\
&\geq \min\{(\mu \cdot \delta)((x_1, y_1), q), (\mu \cdot \delta)((x_2, y_2), q)\} \\
&= \min\{\min\{\mu(x_1, q), \delta(y_1, q)\}, \min\{\mu(x_2, q), \delta(y_2, q)\}\} \\
&= \min\{\min\{\mu(x_1, q), \mu(x_2, q)\}, \min\{\delta(y_1, q), \delta(y_2, q)\}\}
\end{aligned}$$

It follows that $\min\{\mu((x_1 * (x_2 * x)) * x, q), \delta((y_1 \star (y_2 \star y)) \star y, q)\} \not<$
$\min\{\min\{\mu(x_1, q), \mu(x_2, q)\}, \min\{\delta(y_1, q), \delta(y_2, q)\}\}$, which is a contradiction. Hence μ is a Q-fuzzy ideal of A or δ is a Q-fuzzy ideal of B.

(2). Assume that $\mu \cdot \delta$ is a Q-fuzzy subalgebra of $A \times B$. Suppose that μ is not a Q-fuzzy subalgebra of A and δ is not a Q-fuzzy subalgebra of B. Then there exist $x, y \in A$ and $a, b \in B$ such that

$$\mu(x * y, q) < \min\{\mu(x, q), \mu(y, q)\} \textit{ and } \delta(a \star b, q) < \min\{\delta(a, q), \delta(b, q)\}.$$

Then $\min\{\mu(x*y,q),\delta(a\star b,q)\} < \min\{\min\{\mu(x,q),\mu(y,q)\},\min\{\delta(a,q),\delta(b,q)\}\}$. Consider,

$$\begin{aligned}
\min\{\mu(x\cdot y,q),\delta(a\star b,q)\} &= (\mu\cdot\delta)((x*y,a\star b),q)\\
&= (\mu\cdot\delta)((x,a)\diamond(y,b),q)\\
&> \min\{(\mu\cdot\delta)((x,a),q),(\mu\cdot\delta)((y,b),q)\}\\
&= \min\{\min\{\mu(x,q),\delta(a,q)\},\min\{\mu(y,q),\delta(b,q)\}\}\\
&= \min\{\min\{\mu(x,q),\mu(y,q)\},\min\{\delta(a,q),\delta(b,q)\}\}.
\end{aligned}$$

Then $\min\{\mu(x\cdot y,q),\delta(a\star b,q)\} \not< \min\{\min\{\mu(x,q),\mu(y,q)\},\min\{\delta(a,q),\delta(b,q)\}\}$, which is a contradiction. Hence μ is a Q-fuzzy subalgebra of A or δ is a Q-fuzzy subalgebra of B. □

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