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Existence of Solutions for Nabla-differentiable Systems of the Second Order

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Abstract

We establish the existence of solutions to systems of second-order differential equations in time scales. The problem is of type nabla differential of order two with the member f being a $\nabla-$caratheodory function. We consider differential systems in which the nonlinearity f depends on the derivative nabla u^{∇}. The existence results are based on the notion of solution-tube and the fixed point theorem.

Keywords: nabla derivative; solution-tube; forward and backward jump operator; left-scattered; right-dense.

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1. Introduction

In this paper, we study the existence of solution for nabla-differentiable systems of the second order:

$$\begin{cases} u^{\nabla\nabla}(t) &= f(t, u(\rho(t)), u^{\nabla}(t)) \quad \nabla a.e \quad t \in \mathbb{T}_{0,\kappa^2} \\ u(\rho(a)) &= u(\sigma(b)) \\ u^{\nabla}(a) &= u^{\nabla}(\sigma(b)) \end{cases} \tag{1}$$

where the function $f : \mathbb{T}_{0,\kappa^2} \times \mathbb{R}^n \to \mathbb{R}^n$ is $\nabla-$ Caratheodory. Here $\mathbb{T}$ is a compact time scale where $a = \min \mathbb{T}, b = \max \mathbb{T}$ and $\mathbb{T}_{0,\kappa^2}$ will be defined later.

We use the tube solution method for differential systems (1), cited in the works of H. Gilbert and M. Frigon [7]. This notion makes it possible to obtain existence results for systems of second-order differential equations of the type (1). It is a generalization of the method of under and over solutions in a system of differential equations. The main objective is to show the existence of solutions for ∇-differentiable systems (1). This article is organized as follows: first, a review of basic definitions and theorem relating to $\nabla-$differentiation and $\nabla-$integration in time scale and some secondary results. Then, we introduce the notion of tube solution for the differential system (1) and prove our main result.

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2. Preliminaries

Let $\mathbb{T}$ a time scale. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ (resp the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$) by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ (respectively by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$). In this definition, we put $\inf \emptyset = \sup \mathbb{T}$(i.e $\sigma(b) = b$ if $\mathbb{T}$ has a maximum b) and $\sup \emptyset = \inf \mathbb{T}$ (i.e $\rho(a) = a$ if $\mathbb{T}$ has a minimum a), where $\emptyset$ denotes the empty set.

If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$ we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. Also if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. Thus, a point $t \in \mathbb{T}$ is called dense, if it's right-dense and left-dense both.

Note: $L_{\mathbb{T}} = \{t \in \mathbb{T} : \rho(t) < t\}$. The backwards graininess $\nu : \mathbb{T}_\kappa \to [0, \infty[$ is defined by $\nu(t) = t - \rho(t)$. Denote

$$\mathbb{T}_\kappa = \begin{cases} \mathbb{T} \setminus \{m\} = \mathbb{T}_0 & \text{if } m \text{ is right-scattered} \\ \mathbb{T}_\kappa = \mathbb{T} & \text{if } m \text{ is right-dense} \end{cases}$$

Since $\mathbb{T}_\kappa$ is a time scale , denote $\mathbb{T}_\kappa = (\mathbb{T}_\kappa)_\kappa$ and

$$\mathbb{T}_{0,\kappa^2} = \begin{cases} \mathbb{T}_{\kappa^2} \setminus \{m\} & \text{if } m \in \mathbb{T}_{\kappa^2} \\ \mathbb{T}_{\kappa^2} & \text{otherwise} \end{cases}$$

Definition 2.1. *For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$, define nabla derivative of f at t, denoted $f^\nabla(t)$, to be the number (provided it exists) with the property that gives any $\varepsilon > 0$, there is a neighborhood U of t such that*

$$| f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s) | \leqslant \varepsilon \mid \rho(t) - s \mid$$

for all $s \in U$.

If f is $\nabla-$differentiable at t for all $t \in \mathbb{T}_\kappa$, then $f : \mathbb{T} \to \mathbb{R}^n$ is called $\nabla-$derivative of f in $\mathbb{T}_\kappa$.

If f is $\nabla-$differentiable and if f^∇ is $\nabla-$differentiable in $t \in T_\kappa$, on denote $f^{\nabla\nabla}(t) = (f^\nabla)^\nabla(t)$ the second $\nabla-$derivative of f at t.

Proposition 2.2. *We suppose that $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$. then we have:*

i) If f is $\nabla-$differentiable at t, then f is continuous at t;

ii) If f is continuous at a left-scattered t, then f is $\nabla-$differentiable at t with

$$f^\nabla(t) = \frac{f(t) - f(\rho(t))}{\nu(t)};$$

iii) If t is left-dense, then $f$is $\nabla-$differentiable at t iff the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^{\nabla}(t) = \lim_{s\to t} \frac{f(t)-f(s)}{t-s};$$

iv) If f is $\nabla-$differentiable at t, then $f^{\rho}(t) = f(t) - \nu(t) f^{\nabla}(t)$, where $f^{\rho}(t) = f(\rho)(t)$.

Proposition 2.3. *If $f : \mathbb{T} \to \mathbb{R}$ and $g : \mathbb{T} \to \mathbb{R}$ is $\nabla-$differentiables at $t \in \mathbb{T}_{\kappa}$. Then*

i) $f + g$ is $\nabla-$differentiable at t with $(f+g)^{\nabla}(t) = f^{\nabla}(t) + g^{\nabla}(t)$.

ii) fg is $\nabla-$differentiable at t and $(fg)^{\nabla}(t) = f^{\nabla}(t)g(t) + f^{\rho}(t)g^{\nabla}(t) = f(t)g^{\nabla}(t) + f^{\nabla}(t)g^{\rho}(t)$.

iii) If $m = 1$ and $g(t)g^{\rho}(t) \neq 0$, then $\frac{f}{g}$ is $\nabla-$differentiable at t and $\left(\frac{f}{g}\right)^{\nabla}(t) = \frac{f^{\nabla}(t)g(t) - f(t)g^{\nabla}(t)}{g(t)g^{\rho}(t)}$.

Theorem 2.4. *Let W a open of $\mathbb{R}^n$ and $t \in \mathbb{T}$ a point left-dense. If $g : \mathbb{T} \to \mathbb{R}^n$ is $\nabla-$differentiable at t and $f : W \to \mathbb{R}$ is differentiable at $g(t) \in W$, then $f \circ g$ is $\nabla-$differentiable at t with $(f \circ g)^{\nabla}(t) = \langle f'(g(t)), g^{\nabla}(t)\rangle$.*

Example 2.5. *We suppose the $x : \mathbb{T} \to \mathbb{R}^n$ is $\nabla-$differentiable at $t \in \mathbb{T}$. We know that $\|.\| : \mathbb{R}^n \setminus \{0\} \to [0, +\infty)$ is differentiable if $t = \rho(t)$. We prove that*

$$\|x(t)\|^{\nabla} = \frac{\langle x(t), x^{\nabla}(t)\rangle}{\|x(t)\|}.$$

We denote $C(\mathbb{T}, \mathbb{R}^n)$ the space of continuous maps on $\mathbb{T}$ and $C^1(\mathbb{T}, \mathbb{R}^n)$ the space of continuous maps on $\mathbb{T}$ with continuous $\nabla-$derivative on $\mathbb{T}_{\kappa}$. With the norm $\|u\|_0 = \max\{\|u(t)\|, t \in \mathbb{T}\}$ (respectively $\|u\|_1 = \max\{\|u(t)\|_0; \|u^{\nabla}(t)\| : t \in \mathbb{T}_{\kappa}\}$), $C(\mathbb{T}, \mathbb{R}^n)$ (respectively $C^1(\mathbb{T}, \mathbb{R}^n)$) is a Banach space.

Definition 2.6. *The function $f : \mathbb{T} \to \mathbb{R}^n$ is left-dense continuous or $ld-$continuous provided it is continuous at every point left-dense on $\mathbb{T}$ and its left limits exist (finite) at points right-denses on $\mathbb{T}$. If $\mathbb{T} = \mathbb{R}^n$, then f is $ld-$continuous iff f is continuous. The set of functions ld-continuous $f : \mathbb{T} \to \mathbb{R}^n$ is denoted $C_{ld}(\mathbb{T}, \mathbb{R}^n)$. The set of the functions $f : \mathbb{T} \to \mathbb{R}^n$ $\nabla-$differentiable and the $\nabla-$derivative ld-continue is denoted $C^1_{ld}(\mathbb{T}, \mathbb{R}^n)$. If f is ld-continuous, then there exists a function F such as $F^{\nabla} = f$. In the case,*

$$\int_a^b f(t)\nabla t = F(b) - F(a).$$

Theorem 2.7. *We have the following inequalities:*

$$\mid \int_a^b f(t)g(t)\nabla t \mid \leq \int_a^b \mid f(t)g(t) \mid \nabla t \leq (\max_{\sigma(a)\leq t\leq b} \mid f(t) \mid) \int_a^b \mid g(t) \mid \nabla t.$$

The notions of ∇-measure and of ∇-integral for the functions $f : \mathbb{T} \to \mathbb{R}$ are similar to those in the case of $\Delta-$mesurability and of $\Delta-$integrality define in the chapter 5 of [1] or in the chapter dans le chapter 2 in [8].

Theorem 2.8. *[1] For each $t_0 \in \mathbb{T} \setminus \{\min \mathbb{T}\}$, the single-point set $\{t_0\}$ is ∇-measurable, and its $\nabla-$measure is given by*

$$\mu_\nabla(\{t_0\}) = t_0 - \rho(t_0).$$

If $a, b \in \mathbb{T}$ and $a \leq b$, then

$$\mu_\nabla(]a,b]) = b - a; \mu_\nabla(]a,b[) = \rho(b) - a$$

If $a, b \in \mathbb{T} \setminus \min\{\mathbb{T}\}$ and $a \leq b$ then

$$\mu_\nabla([a,b[) = \rho(b) - \rho(a); \mu_\nabla([a,b]) = b - \rho(a)$$

Definition 2.9. *Let $E \subset \mathbb{T}$, $\nabla-$measurable set and $f : \mathbb{T} \to \mathbb{R}$, $\nabla-$measurable function. We will say that $f \in L^1_\nabla(E)$ if*

$$\int_E | f(s) | \nabla s < \infty.$$

We say that the function $f : \mathbb{T} \to \mathbb{R}^n \nabla-$measurable is in the set $L^1_\nabla(E, \mathbb{R}^n)$ provided

$$\int_E | f(s) | \nabla s < \infty$$

The set $L^1_\nabla(\mathbb{T}_0, \mathbb{R}^n)$ is a Banach space endowed with the norm

$$|| f(s) ||_{L^1_\nabla} = \int_{\mathbb{T}_0} || f(s) || \nabla s$$

Proposition 2.10. *Let $f \in L^1_\nabla(E, \mathbb{R}^n)$. Then*

$$\| \int_E f(s) \nabla s \| \leq \int_E \|f(s)\| \nabla s.$$

Theorem 2.11 (Lebesgue dominated convergence theorem)**.** *Let $\{f_k\}_{k \in \mathbb{N}}$ be a sequence of functions in $L^1_\nabla(\mathbb{T}_0, \mathbb{R}^n)$. Assume that there exists a function $f : \mathbb{T}_0 \to \mathbb{R}$ such as $f_k(t) \to f(t)$, $\nabla - p.p \quad t \in \mathbb{T}_0$ and there exists a function $g \in L^1_\nabla(\mathbb{T}_0)$ such as $|| f_k(t) || \leq g(t)$, $\nabla - p.p \quad t \in \mathbb{T}_0$ and for every $k \in \mathbb{N}$, then $f_k \to f$ dans $L^1_\nabla(\mathbb{T}_0, \mathbb{R}^n)$.*

Definition 2.12. *We said that $f : \mathbb{T} \to \mathbb{R}^n$ is absolute continuous function on $\mathbb{T}$ if for every $\varepsilon > 0$, the exists a $\delta > 0$ such as if $\{[a_k, b_k[\}_{k=1}^n$ with $a_k, b_k \in \mathbb{T}$ is a finite pairwise disjoint family of subintervals satisfying $\sum_{k=1}^n (b_k - a_k) < \delta$, then $\sum_{k=1}^n || f(b_k) - f(a_k) || < \varepsilon$.*

Proposition 2.13. *A function $f : \mathbb{T} \to \mathbb{R}$ is absolutely continuous on $\mathbb{T}$ if and only if f is $\nabla-$differentiable $\nabla-$almost everywhere on $\mathbb{T}_0$, $f^\nabla \in L^1_\nabla(\mathbb{T}_0)$ and*

$$\int_{[a,t) \cap \mathbb{T}} f^\nabla(s) \nabla s = f(t) - f(a), \quad for \quad every \quad t \in \mathbb{T}.$$

We will define the Caratheodory function for arbitrary time scales.

Definition 2.14. *A function $f : \mathbb{T}_0 \times \mathbb{R}^{2n} \to \mathbb{R}^n$ is a $\nabla-$Caratheodory function if the following conditions is satisfied:*

(i) $f(.,u,w) : \mathbb{T}_0 \to \mathbb{R}^n$ *is $\nabla-$measure for all* $(u,w) \in \mathbb{R}^{2n}$;

(ii) $f(t,.,.) : \mathbb{R}^{2n} \to \mathbb{R}^n$ *is continuous for* $\nabla - a.e \quad t \in \mathbb{T}_0$;

(iii) For each compact set $K \subset \mathbb{R}^{2n}$, *there exists a function* $h_K \in L^1_\nabla(\mathbb{T}_0, [0,\infty[)$ *such as* $\|f(t,u,w)\| \leq h_K(t)$ $\nabla - a.e \quad t \in \mathbb{T}_0$ *and for* $(u,w) \in K$.

We will now define the notion of Sobolev space with the functions defines on $\mathbb{T}$ is compacts, where $a = \min \mathbb{T} < \max \mathbb{T} = b$.

Definition 2.15. *We will say that a function $u : \mathbb{T} \to \mathbb{R}$ belongs to the set $W^{1,1}_\nabla(\mathbb{T})$ if and only if $u \in L^1_\nabla(\mathbb{T}_0)$ and that there exists a function $g : \mathbb{T}_\kappa \to \mathbb{R}$ such as $g \in L^1_\nabla(\mathbb{T}_0)$ and*

$$\int_{\mathbb{T}_0} u(s)\phi^\nabla(s)\nabla s = -\int_{\mathbb{T}_0} g(s)\phi(\rho(s))\nabla s \quad \forall \quad \phi \in C^1_{0,ld}(\mathbb{T})$$

where

$$C^1_{0,ld}(\mathbb{T}) := \{f : \mathbb{T} \to \mathbb{R} : f \in C^1_{ld}(\mathbb{T}), f(a) = 0 = f(b)\}.$$

We will say that a function $f : \mathbb{T} \to \mathbb{R}^n$ is on the $W^{1,1}_\nabla(\mathbb{T}, \mathbb{R}^n)$ if each of its components f_i are on $W^{1,1}_\nabla(\mathbb{T})$.

Definition 2.16. *We define the space $W^{2,1}_\nabla(\mathbb{T}, \mathbb{R}^n)$ by*

$$W^{2,1}_\nabla(\mathbb{T}, \mathbb{R}^n) = \{u \in W^{1,1}_\nabla(\mathbb{T}, \mathbb{R}^n : u^\nabla \in W^{1,1}_\nabla(\mathbb{T}^\kappa, \mathbb{R}^n)\}.$$

Theorem 2.17. *The sets $W^{1,1}_\nabla(\mathbb{T})$ and $W^{2,1}_\nabla(\mathbb{T})$ are the banach spaces with the norm*

$$\begin{aligned} || u ||_{W^{1,1}_\nabla(\mathbb{T})} &= || u ||_{L^1_\nabla(\mathbb{T})} + || u^\nabla ||_{L^1_\nabla(\mathbb{T})} \\ || u ||_{W^{2,1}_\nabla(\mathbb{T})} &= || u ||_{L^1_\nabla(\mathbb{T})} + || u^\nabla ||_{L^1_\nabla(\mathbb{T})} + || u^{\nabla\nabla} ||_{L^1_\nabla(\mathbb{T})} . \end{aligned}$$

Lemma 2.18. *Let the function $u : \mathbb{T} \to \mathbb{R}^n$ $\nabla-$differentiable.*

(1) On $\{t \in \mathbb{T}_{\kappa^2} : \|u(\rho(t))\| \geqslant 0 \quad \textit{and} \quad u^{\nabla\nabla}(t) \quad \textit{exist}\}$,

$$\|u(t)\|^{\nabla\nabla} \geqslant \frac{\langle u(\rho(t), u^{\nabla\nabla}(t)\rangle}{\|u(\rho(t))\|}$$

(2) On $\{t \in \mathbb{T}_{\kappa^2} \backslash L_\mathbb{T} : \|u(\rho(t))\| \geqslant 0 \quad \textit{and} \quad u^{\nabla\nabla}(t) \quad \textit{exist}\}$,

$$\|u(t)\|^{\nabla\nabla} = \frac{\langle u(t), u^{\nabla\nabla}(t)\rangle + \|u^\nabla(t)\|^2}{\|u(t)\|} - \frac{\langle u(t), u^\nabla(t)\rangle^2}{\|u(t)\|^3}.$$

Proof. Denote $A = \{t \in \mathbb{T}_{\kappa^2} : |u(\rho(t)\| > 0 \quad \text{and} \quad u^{\nabla\nabla}(t) \quad exist\}$. By proposition 2.3, on the set $A \setminus L_T$, we have:

$$\|u(t)\|^{\nabla} = \frac{\langle u(t), u^{\nabla}(t)}{\|u(t)\|}$$

$$\|u(t)\|^{\nabla\nabla} = (\frac{\langle u(t), u^{\nabla}(t)\rangle}{\|u(t)\|})^{\nabla}$$

$$= \frac{\langle u(t), u^{\nabla}(t)\rangle^{\nabla}\|u(t)\| - \langle u(t), u^{\nabla}(t)\rangle\|u(t)\|^{\nabla}}{\|u(t)\|^2}$$

$$\|u(t)\|^{\nabla\nabla} = \frac{\langle u(t), u^{\nabla\nabla}(t)\rangle + \|u(t)\|^2}{\|u(t)\|} - \frac{\langle u(t), u^{\nabla}(t)\rangle^2}{\|u(t)\|^3}$$

Moreover, we have

$$\langle u(t), u^{\nabla}(t)\rangle^2 \leq \|u(t)\|^2\|u^{\nabla}(t)\|^2 \Rightarrow \frac{\langle u(t), u^{\nabla}(t)\rangle^2}{\|u(t)\|^3} \leq \frac{\|u^{\nabla}(t)\|^2}{\|u(t)\|}$$

thus

$$\|u(t)\|^{\nabla\nabla} \geqslant \frac{\langle u(t), u^{\nabla\nabla}(t)\rangle}{\|u(t)\|} = \frac{\langle u(\rho(t), u^{\nabla\nabla}(t)\rangle}{\|u(\rho(t))\|} \qquad on \quad A \setminus L_T$$

If $t \in A$ such as $\rho^2(t) = \rho(t) < t$, then:

$$\begin{aligned}
\|u(t)\|^{\nabla\nabla} &= \frac{\|u(t)\|^{\nabla} - \|u(\rho(t))\|^{\nabla}}{\nu(t)} \\
&= \frac{\|u(t)\| - \|u(\rho(t))\|}{\nu^2(t)} - \frac{\langle u(\rho(t)), u^{\nabla}(\rho(t))\rangle}{\|u(\rho(t))\|\nu(t)} \\
&= \frac{\|u(t)\| - \|u(\rho(t))\|}{\nu^2(t)} - \frac{\langle u(\rho(t)), u(t)\rangle}{\|u(\rho(t))\|\nu^2(t)} + \frac{\|u(\rho(t))\|}{\nu^2(t)} + \frac{\langle u(\rho(t)), u^{\nabla\nabla}(t)\rangle}{\|u(\rho(t))\|} \\
&= \frac{\langle u(\rho(t)), u^{\nabla\nabla}(t)\rangle}{\|u(\rho(t))\|} - \frac{\langle u(\rho(t)), u(t)\rangle}{\|u(\rho(t))\|\nu^2(t)} + \frac{\|u(t)\|}{\nu^2(t)} \\
&\geq \frac{\langle u(\rho(t)), u^{\nabla\nabla}(t)\rangle}{\|u(\rho(t))\|}
\end{aligned}$$

If $t \in A$ such as $\rho^2(t) < \rho(t) < t$, we obtain

$$\begin{aligned}
\|u(t)\|^{\nabla\nabla} &= \frac{\|u(t)\|^{\nabla} - \|u(\rho(t))\|^{\nabla}}{\nu(t)} \\
&= \frac{\|u(t)\| - \|u(\rho(t))\|}{\nu^2(t)} - \frac{\|u(\rho(t))\| - \|u(\rho^2(t))\|}{\nu(\rho(t))\nu(t)}
\end{aligned}$$

we have

$$\frac{\langle u(\rho(t)), u(\rho^2(t))\rangle}{\|u(\rho(t))\|} \leqslant \|u(\rho^2(t))\|$$

$$\begin{aligned}
\|u(\rho(t))\| - \|u(\rho^2(t))\| &\leqslant \frac{\|u(\rho(t))\|^2 - \langle u(\rho(t)), u(\rho^2(t))\rangle}{\|u(\rho(t))\|} \\
\frac{\|u(\rho(t))\| - \|u(\rho^2(t))\|}{\nu(\rho(t))\nu(t)} &\leqslant \frac{\|u(\rho(t))\|^2 - \langle u(\rho(t)), u(\rho^2(t))\rangle}{\nu(\rho(t))\nu(t)\|u(\rho(t))\|}
\end{aligned}$$

$$\leqslant \frac{\langle u(\rho(t)), u(\rho(t)) - u(\rho^2(t))\rangle}{\nu(\rho(t))\nu(t)\|u(\rho(t))\|}$$
$$\leqslant \frac{\langle u(\rho(t)), u^{\nabla}(\rho(t))\rangle}{\nu(t)\|u(\rho(t))\|}$$

thus

$$\|u(t)\|^{\nabla\nabla} \geqslant \frac{\|u(t)\| - \|u(\rho(t))\|}{\nu^2(t)} - \frac{\langle u(\rho(t)), u^{\nabla}(\rho(t))\rangle}{\nu(t)\|u(\rho(t))\|}$$

and we conclude as in previous case. □

Lemma 2.19. *Let $\varepsilon > 0$, the exponential function $e_\varepsilon(., t_0)$ is defined by*

$$e_\varepsilon(t, t_0) = \exp\left(\int_{[t_0,t)\cap\mathbb{T}} \xi_\varepsilon(\mu(s))\nabla s\right),$$

where

$$\xi_\varepsilon(h) = \begin{cases} \varepsilon & \text{if } h = 0 \\ \dfrac{log(1 + h\varepsilon)}{h} & \text{if } h > 0 \end{cases}$$

It is the unique solution to the initial value problem

$$u^{\nabla}(t) = \varepsilon u(t), \qquad u(t_0) = 1$$

Here is a result on times scales, analogous to Gronwall's inequality by:

Theorem 2.20. *Let $\alpha > 0, \varepsilon > 0$ and $y \in C(\mathbb{T}, \mathbb{R})$, if*

$$y(t) = \alpha + \int_{[a,t)\cap\mathbb{T}} \varepsilon y(s)\nabla s$$

then

$$y(t) \leq \alpha e_\varepsilon(t, a) \quad \forall \ t \in \mathbb{T}$$

Lemma 2.21. *Let $f : \mathbb{T} \to \mathbb{R}$ be a function with a local maximum at $t_0 \in]a, b[\cap\mathbb{T}$. If $f^{\nabla\nabla}(\sigma(t_0))$ exists, then $f^{\nabla\nabla}(\sigma(t_0)) \leq 0$ provided t_0 is not the same time right dense and left scattered.*

Lemma 2.22. *Let $f : \mathbb{T} \to \mathbb{R}$ be a function with a local maximum at $t_0 \in \mathbb{T}_\kappa$ left dense. If $f^{\nabla}(t_0) = 0$ and $f^{\nabla\nabla}(t_0)$ exists, then $f^{\nabla\nabla}(t_0) \leq 0$.*

Theorem 2.23. *Let $r \in W^{2,1}_{\nabla}(\mathbb{T})$ a function such as $r^{\nabla\nabla} > 0$ ∇. a.e on $\{t \in \mathbb{T}_{\kappa^2,0} : r(\rho(t)) > 0\}$. If $r(\rho(a)) = r(\sigma(b))$ and $r^{\nabla}(a) \geqslant r^{\nabla}(\sigma(b))$, then $r(t) \leqslant 0$ for every $t \in \mathbb{T}$.*

Proof. Suppose there exist $t_0 \in \mathbb{T}$ such as $r(t_0) = \max_{t\in\mathbb{T}} r(t) > 0$.

1st case : $a < t_0 < \sigma(t_0) < b$, then $r^{\nabla\nabla}(\sigma(t_0))$ exists since

$0 < \mu_{\nabla}(\{\sigma(t_0)\}) = \sigma(t_0) - \rho(\sigma(t_0)) = \sigma(t_0) - t_0$ car $\rho(\sigma(t_0)) = t_0$ and $r \in W^{2,1}_{\nabla}(\mathbb{T})$. By the previous lemma 2.21, $r^{\nabla\nabla}(\sigma(t_0) \leq 0$ which contradicts the fact that $r(\rho(\sigma(t_0))) = r(t_0) > 0$

2nd case : $a < t_0 = \sigma(t_0) < b$ then exist $t_1 > t_0$ such as $r(\sigma(t)) > 0$ for every $t \in (t_0, t_1) \cap \mathbb{T}$. As $r(t_0)$ is a maximum then $r^{\nabla}(t_0) = 0$ and there exist $s \in (t_0, t_1)$ such as $r^{\nabla}(s) \leq 0$ thus

$$0 \geq r^{\nabla}(s) - r^{\nabla}(t_0) = \int_{[t_0,s)\cap\mathbb{T}} r^{\nabla\nabla}(\tau)\nabla\tau > 0,$$

contradiction.

3rd case : $t_0 = a$

Suppose that $\rho(a) < a$ and that $r(a) > r(\rho(a)$ then $r^{\nabla}(a) = \dfrac{r(a) - r(\rho(a))}{a - \rho(a)} > 0$. Then there exists $\delta > 0$ such as for all $t \in [a, a+\delta), r^{\nabla}(t) > 0$, we have, for every $t \in [a, a+\delta)$,

$$r(t) - r(a) = \int_{[a,t)\cap\mathbb{T}} r^{\nabla}(s)\nabla s > 0$$

which contradicts the fact that a is a maximum.

Suppose that $\rho(a) = a$ and $r^{\nabla}(a) > 0$ then there exists $t_1 > a$ such as $r^{\nabla}(t) > 0$ for every $t \in [a, t_1)$. Thus, for all $s \in]a, t_1)$, we have

$$r(s) - r(a) = \int_{[a,s)\cap\mathbb{T}} r^{\nabla}(t)\nabla t > 0$$

impossible because a is a maximum.

If $a = \rho(a)$ and $r^{\nabla}(a) = 0$, there exists a t_2 such as $r(\rho(t)) > 0$ for every $t \in (a, \sigma(t_2))$. So by hypothesis, $\nabla - ppt \in (a, \sigma(t_2)), r^{\nabla\nabla}(t) > 0$ and thus

$$r^{\nabla}(t) - r^{\nabla}(a) = r^{\nabla}(t) = \int_{[a,t)\cap\mathbb{T}} r^{\nabla\nabla}(\omega)\nabla\omega > 0 \tag{2}$$

as $r(a)$ is a maximum, there exists $s \in (a, \sigma(t_2))$ such as $r^{\nabla}(s) \leq 0$. Which contradicts (2)

4th case : $t_0 = b$

If $b = \sigma(b)$, we have $r(\rho(a)) = r(b) = r(\sigma(b))$ if $a > \rho(a)$ then suppose that

$$r(a) < r(\rho(a) \Rightarrow r^{\nabla}(a) = \frac{r(a) - r(\rho(a)}{a - \rho(a)} < 0$$

We have $r^{\nabla}(\sigma(b)) \leq r^{\nabla}(a) < 0$. Which contradicts the fact that b is a maximum. If $a = \rho(a)$, then $r^{\nabla}(\sigma(b)) = r^{\nabla}(a) \leq 0$. Because $t_0 = a$ then $r^{\nabla}(a) = r^{\nabla}(\sigma(b)) = 0$. There exists $t_1 > a$ such as $r(\rho(t)) > 0$. For every $t \in [a, \sigma(t_1))$. Thus for all $s \in (a, t_1)$, we have

$$r^{\nabla}(s) = r^{\nabla}(s) - r^{\nabla}(a) = \int_{[a,s]\cap\mathbb{T}} r^{\nabla\nabla}(t)\nabla t > 0 \tag{3}$$

As $r(a)$ is a maximum, there exist a $s \in (a, t_1)$ such as $r^{\nabla}(s) \leq 0$. which contradicts (3). Thus So in all cases whatever the condition, it is necessary that $r(t) \leq 0$, for every $t \in \mathbb{T}$.

□

Lemma 2.24. *The equation*

$$\begin{cases} u^{\nabla\nabla}(t) - u(\rho(t)) &= 0 \\ u(\rho(a)) &= u(\sigma(b)) \\ u^{\nabla}(a) &= u^{\nabla}(\sigma(b)) \end{cases}$$

admits only one solution which is trivial.

Proof. Suppose that the equation admits u a non-trivial solution. Let $c \in \mathbb{T}$ such as $0 < u(c) = \max_{t\in\mathbb{T}} u(t)$ If $a < c < b$ thus if c is not both scattered on the left and dense on the right then, by the Lemma 2.21, $u^{\nabla\nabla}(\sigma(c)) \leq 0$. So $u^{\nabla\nabla}(\sigma(c)) - u(\rho(\sigma(c)) = u^{\nabla\nabla}(\sigma(c)) - u(c) < 0$. Which contradicts that u is solution of the equation. If $\rho(c) < c = \sigma(c)$ then $\rho(c)$ is not both scattered on the left and dense on the right so we come back to the previous case. Suppose that $c = a$. If $\rho(a) < a$ then we return to the previous case. If $\rho(a) = a$ then $u^a = u^{\nabla}(\sigma(b)) = 0$, by Lemma 2.22, $u^{\nabla\nabla}(a) \leq 0$. So $u^{\nabla\nabla}(a) - u(\rho(a)) < 0$. Contradicts that u is solution of the equation. Suppose that $c = b$. If $b = \rho(b)$ then we come back to the two previous cases. If $b < \rho(b)$ then $\sigma(b)$ is not both scattered on the left and dense on the right then, by the Lemma 2.21, $u^{\nabla\nabla}(\sigma(b)) \leq 0$. So $u^{\nabla\nabla}(\sigma(b)) - u(\rho(\sigma(b)) = u^{\nabla\nabla}(\sigma(b)) - u(b) < 0$. Contradiction. □

Notice (BC) the conditions of boundary value following

$$(BC) : u(\rho(a)) = u(\sigma(b)); u^{\nabla}(a) = u^{\nabla}(\sigma(b))$$

and

$$W^{2,1}_{\nabla,BC} = \{u \in W^{2,1}_{\nabla} : u \in BC\}.$$

Proposition 2.25. *Let $g(t) \in L^1_{\nabla}(\mathbb{T}_{0,\kappa})$ then the tree equations are equivalents*

$$u^{\nabla\nabla}(t) - u(\rho(t)) = g(t) \quad \nabla.p.p \quad t \in \mathbb{T}_{0,\kappa} \tag{4}$$

$$u^{\nabla\nabla}(t) + v(t)u^{\nabla}(t) - u(t) = g(t) \quad \nabla.p.p \quad t \in \mathbb{T}_{0,\kappa} \tag{5}$$

$$u^{\nabla}(t) - u^{\nabla}(a) - \int_{[a,t)\cap\mathbb{T}} u(\rho(s))\nabla(s) = \int_{[a,t[\cap\mathbb{T}} g(s)\nabla(s) \quad \nabla.p.p \quad t \in \mathbb{T}_{0,\kappa} \tag{6}$$

We define the ∇ differential operators L_1 and L_2 associated resp to the problems (4) and (5) define $L_1, L_2 : W^{2,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n) \to L^1_{\nabla}(\mathbb{T}_{0,\kappa}, \mathbb{R}^n)$ by

$$L_1(u)(t) = u^{\nabla\nabla}(t) - u(\rho(t))$$

$$L_2(u)(t) = u^{\nabla\nabla}(t) + v(t)u^{\nabla}(t) - u(t)$$

Definition 2.26. *For two nabla differentiable functions u_1, u_2 we define the nabla Wronskian $W = W(u_1, u_2)$*

by

$$W(t) = det \begin{pmatrix} u_1(t) & u_2(t) \\ u_1^{\nabla}(t) & u_2^{\nabla}(t) \end{pmatrix}$$

We say that two solutions u_1 and u_2 of $L_2 u = 0$ form a fondamental set of solutions for $L_2 u = 0$ provided $W(u_1, u_2)(t) \neq 0$ for all $t \in \mathbb{T}_\kappa$.

Corollary 2.27. *[2] The Wronskian of any two solutions of $L_1 u(t) = 0$ is independant of t.*

Proposition 2.28. *[1] Let $t_0 \in \mathbb{T}_\kappa$. Suppose that u_1 subject to conditions $u(t_0) = 1, u^{\nabla}(t_0) = 0$ and u_2 subject to conditions $u(t_0) = 0, u^{\nabla}(t_0) = 1$ solutions of the homogeneous equation $L_2(u)(t) = 0$, $u(t_0) = u_0$, $u^{\nabla}(t_0) = u_0^{\nabla}$. We have $W(u_1, u_2)(t) = W(u_1, u_2)(t_0) = 1$, then u_1 and u_2 form a fundamental set of solution of this homogeneous equation. Therefore the solution of the initial value problem $L_2(u)(t) = g(t)$, $u(t_0) = u_0$, $u^{\nabla}(t_0) = u_0^{\nabla}$ is given by*

$$u(t) = \int_{[t_0,t)\cap\mathbb{T}} [u_2(t)u_1(\rho(s)) - u_1(t)u_2(\rho(s))]g(s)\nabla.s$$

Proposition 2.29. *[2] If $t_0 \in \mathbb{T}_\kappa$, then the initial value problem $L_2(u)(t) = g(t)$, $u(t_0) = u_0$, $u^{\nabla}(t_0) = u_0^{\nabla}$ has a unique solution and this solution is defined on the whole time scale $\mathbb{T}$.*

3. Existence of Solutions

In this section, we prove the existence of a solution to the problem (1). A solution of the problem is a function $u \in W^{2,1}_{\nabla}(\mathbb{T}; \mathbb{R}^n)$ satisfying (1). Let us introduce the notion of solution-tube for the problem (1) as follows.

Definition 3.1. *Let $(v, M) \in W^{2,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n) \times W^{2,1}_{\nabla}(\mathbb{T}, [0, +\infty[)$. We say that (v, M) is solution-tube for (1) if*

i) $\nabla.a.e$ $t \in \{t \in \mathbb{T}_{0,\kappa^2} : t = \rho(t)\}$, we have

$$\langle u - v(t), f(t, u, w) - v^{\nabla\nabla}(t)\rangle + \|w - v^{\nabla}(t)\|^2 \geq M(t)M^{\nabla\nabla}(t) + (M^{\nabla}(t))^2$$

and for all $(u, w) \in \mathbb{R}^{2n}$ such as $\|u - v(t)\| = M(t)$ and $\langle u - v(t), w - v^{\nabla}(t)\rangle \geq M(t)M^{\nabla}(t)$.

ii) For $t \in \{t \in \mathbb{T}_{0,\kappa^2} : \rho(t) < t\}$, we have $\langle u - v(\rho(t)), f(t, u, w) - v^{\nabla\nabla}(t)\rangle \geq M(\rho(t))M^{\nabla\nabla}(t)$ for every $(u, w) \in \mathbb{R}^{2n}$ such as $\|u - v(\rho(t))\| = M(\rho(t))$.

iii) $v(\rho(a)) = v(\sigma(b)))$ and $M(\rho(a)) = M(\sigma(b))$ and $\| v^{\nabla}(\sigma(b)) - v^{\nabla}(a)\| \leq M^{\nabla}(\sigma(b)) - M^{\nabla}(a)$.

Let $\mathbf{T}(v, M) = \{u \in W^{2,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n) : \|u(t) - v(t)\| \leq M(t) \;\; for \;\; every \;\; t \in \mathbb{T}\}$.

Theorem 3.2. *Let $f : \mathbb{T}_{0,\kappa} \times \mathbb{R}^{2n} \to \mathbb{R}^n$ a function $\nabla-$Carathedory. Suppose that:*
(H_1) : there exists $(v, M) \in W^{2,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n) \times W^{2,1}_{\nabla}(\mathbb{T},]0, \infty[)$ a solution-tube for (1).
(H_2) : There exists the constants $C, D > 0$ such as $\|f(t, u, w)\| \leqslant C + D\|w\| \quad \nabla.p.p \quad t \in \mathbb{T}_{0,\kappa}$ and for every $(u, w) \in \mathbb{R}^{2n}$ such as $\|u - v(t)\| \leqslant M(t)$. Then the problem (1) have a solution $u \in W^{2,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n) \cap T(v, M)$.

Let $K > 0$ a constant which will be defined later. Consider the following modified problem:

$$\begin{cases} u^{\nabla\nabla}(t) - u(\rho(t)) & = & g(t, u(\rho(t)), u^{\nabla}(t)) \quad \nabla p.p \quad t \in \mathbb{T}_{0,\kappa^2} \\ u(\rho(a)) & = & u(\sigma(b)) \quad and \quad u^{\nabla}(a) = u^{\nabla}(\sigma(b)) \end{cases} \tag{7}$$

where

$$g(t,u,w) = \begin{cases} \left(\frac{M(\rho(t))}{\| u - v(\rho(t)) \|} f(t, \bar{u}(\rho(t), \tilde{w}(t)) - \bar{u}(\rho(t))\right) + \\ \left(1 - \frac{M(\rho(t))}{\| u - v(\rho(t)) \|}\right)\left(v^{\nabla\nabla}(t) + \frac{M^{\nabla\nabla}(t)}{\| u - v(\rho(t)) \|}(u - v(\rho(t)))\right) \\ \qquad\qquad\qquad if \quad \|u - v(\rho(t))\| > M(\rho(t)) \\ f(t, \bar{u}(\rho(t), \tilde{w}(t)) - \bar{u} \qquad Otherwise. \end{cases}$$

and

$$\bar{u}(\rho(t)) = \begin{cases} \frac{M(\rho(t))}{\|u - v(\rho(t)))\|}(u((\rho(t)) - v(\rho(t))) + v(\rho(t)), & \text{if } \| u - v(\rho(t))) \| > M(\rho(t)); \\ u(\rho(t)), & \text{otherwise.} \end{cases}$$

and

$$\tilde{w}(t) = \begin{cases} \widehat{w}(t) + \left(M^{\nabla}(t) - \frac{\langle u - v(\rho(t)), \widehat{w}(t) - v^{\nabla}(t)\rangle}{\|u - v(\rho(t))\|}\right)\left(\frac{u - v(\rho(t))}{\|u - v(\rho(t))\|}\right) \\ \qquad if \quad t = \rho(t), \quad \|u - v(\rho(t))\| > M(\rho(t)) \\ \widehat{w}(t) + \left(1 - \frac{K}{\|w - v^{\nabla}(t)\|}\right)\frac{M^{\nabla}(t)}{M(\rho(t)}(u - v(\rho(t))) \\ \qquad if \quad t = \rho(t), \quad \|u - v(\rho(t))\| \le M(\rho(t)) \quad and \quad \|y - v^{\nabla}(t)\| > K \\ \widehat{w}(t) \quad if \quad \rho(t) < t \\ w \qquad otherwise \end{cases}$$

$$\widehat{w}(t) = \begin{cases} \frac{K}{\|w - v^{\nabla}(t)\|}\left(w - v^{\nabla}(t)\right) + v^{\nabla}(t) & \text{if } \|w - v(t)\| > K; \\ w & \text{otherwise} \end{cases}$$

Remark 3.3.

- *If* $\|u - v(\rho(t))\| > M(\rho(t))$*, we have*

$$\|\tilde{u}(\rho(t)) - v(\rho(t))\| = \|\frac{M(\rho(t))}{\|u(\rho(t)) - v(\rho(t))\|}(u(\rho(t)) - v(\rho(t)))\| = M(\rho(t))$$

- *If more* $t = \rho(t)$*, we have*

$$\begin{aligned} \alpha(t) &= \langle \widehat{u}(\rho(t)) - v(\rho(t)), \tilde{u}^{\nabla}(t) - v^{\nabla}(t)\rangle \\ &= \langle \frac{M(\rho(t))}{\|u(\rho(t)) - v(\rho(t))\|}(u(\rho(t)) - v(\rho(t))), \tilde{u}^{\nabla}(t) - v^{\nabla}(t)\rangle \end{aligned}$$

$$\begin{aligned}
&= \frac{M(\rho(t))}{\|u(\rho(t)) - v(\rho(t))\|}\langle u(\rho(t)) - v(\rho(t)), \tilde{u}^{\nabla}(t) - v^{\nabla}(t)\rangle \\
&= \frac{M(\rho(t))}{\|u(\rho(t)) - v(\rho(t))\|}[\langle u(\rho(t)) - v(\rho(t)), \widehat{u}^{\nabla}(t) - v^{\nabla}(t)\rangle \\
&\quad +\langle u(\rho(t)) - v(\rho(t)), M^{\nabla}(t) - \frac{\langle u(\rho(t)) - v(\rho(t)), \widehat{u}^{\nabla}(t) - v^{\nabla}(t)\rangle}{\|u - v(\rho(t))\|}(\frac{u - v(\rho(t))}{\|u - v(\rho(t))\|}\rangle] \\
&= \frac{M(\rho(t))}{\|u(\rho(t)) - v(\rho(t))\|}[\langle u(\rho(t)) - v(\rho(t)), \widehat{u}^{\nabla}(t) - v^{\nabla}(t)\rangle \\
&\quad +(M^{\nabla}(t) - \frac{\langle u(\rho(t)) - v(\rho(t)), \widehat{u}^{\nabla}(t) - v^{\nabla}(t)\rangle}{\|u - v(\rho(t))\|})\|u - v(\rho(t))\|] \\
&= \frac{M(\rho(t))}{\|u(\rho(t)) - v(\rho(t))\|}[M^{\nabla}(t)\|u - v(\rho(t))\|] \\
&= M(\rho(t))M^{\nabla}(t)
\end{aligned}$$

So $\langle \widehat{u}(\rho(t)) - v(\rho(t)), \tilde{u}^{\nabla}(t) - v^{\nabla}(t)\rangle = M(\rho(t))M^{\nabla}(t)$ *and*

$$\|\tilde{u}^{\nabla}(t) - v^{\nabla}(t)\|^2 = \|\widehat{u}^{\nabla}(t) - v^{\nabla}(t)\|^2 + (M^{\nabla}(t))^2 - \frac{\langle u(t) - v(t), \widehat{u}^{\nabla}(t) - v^{\nabla}(t)\rangle^2}{\|u(t) - v(t)\|^2}$$

Note also that $\|\tilde{w}(t)\| \leq 2K + \|v^{\nabla}(t)\| + M^{\nabla}(t)$ *et* $\|\tilde{u}(\rho(t)) - v(\rho(t))\| \leq M(\rho(t))$.

Lemma 3.4. *Every solution* u *of* (7) *is in* $\mathbf{T}(v, M)$

Proof. Consider the set $A = \{t \in \mathbb{T}_{0,\kappa^2} : \|u(\rho(t)) - v(\rho(t))\| > M(\rho(t))\}$. The proof will be done in three parts. First, we will show that for $t \in A$

$$(\|u(t) - v(t)\| - M(t))^{\nabla\nabla} \geq \frac{\langle u(\rho(t)) - v(\rho(t)), u^{\nabla\nabla}(t) - v^{\nabla\nabla}(t)\rangle}{\|u(\rho(t)) - v(\rho(t))\|} - M^{\nabla\nabla}(t)$$

Secondly $(\|u(t) - v(t)\| - M(t))^{\nabla\nabla} > 0 \quad \forall \ t \in A$ and third.

$$\|u(a) - v(a)\| - M(a) = \|u(b) - v(b)\| - M(b)$$

$$\|u^{\nabla}(a) - v^{\nabla}(a)\| - M^{\nabla}(a) \geqslant \|u^{\nabla}(b) - v^{\nabla}(b)\| - M^{\nabla}(b)$$

Step 1: We prove that $t \in A$

$$(\|u(t) - v(t)\| - M(t))^{\nabla\nabla} \geq \frac{\langle u(\rho(t)) - v(\rho(t)), u^{\nabla\nabla}(t) - v^{\nabla\nabla}(t)\rangle}{\|u(\rho(t)) - v(\rho(t))\|} - M^{\nabla\nabla}(t)$$

- If $t \in A$ is left dense i.e.$\rho(t) = t$, then we have

$$|| \ u(t) - v(t) \ ||^{\nabla} = \frac{\langle u(t) - v(t), u^{\nabla}(t) - v^{\nabla}(t)\rangle}{\|u(t) - v(t)\|}$$

And

$$|| \ u(t) - v(t) \ ||^{\nabla\nabla} \quad = \quad (\frac{\langle u(t) - v(t), u^{\nabla}(t) - v^{\nabla}(t)\rangle}{\|u(t) - v(t)\|})^{\nabla}$$

$$\begin{aligned}
&= \frac{\|u(t)-v(t)\|\langle u(t)-v(t),u^{\nabla}(t)-v^{\nabla}(t)\rangle^{\nabla}}{\|u(t)-v(t)\|} \\
&\quad -\frac{\|u(t)-v(t)\|^{\nabla}\langle u(t)-v(t),u^{\nabla}(t)-v^{\nabla}(t)\rangle}{\|u(t)-v(t)\|} \\
&= \frac{\langle u(t)-v(t),u^{\nabla\nabla}(t)-v^{\nabla\nabla}(t)\rangle+\|u^{\nabla}(t)-v^{\nabla}(t)\|^2}{\|u(t)-v(t)\|} \\
&\quad -\frac{\langle u(t)-v(t),u^{\nabla}(t)-v^{\nabla}(t)\rangle^2}{\|u(t)-v(t)\|^3} \\
&= \frac{\langle u(t)-v(t),u^{\nabla\nabla}(t)-v^{\nabla\nabla}(t)\rangle}{\|u(t)-v(t)\|} \\
&\quad +\frac{\|u^{\nabla}(t)-v^{\nabla}(t)\|^2\|u(t)-v(t)\|^2-\langle u(t)-v(t),u^{\nabla}(t)-v^{\nabla}(t)\rangle^2}{\|u(t)-v(t)\|^3} \\
&\geq \frac{\langle u(t)-v(t),u^{\nabla\nabla}(t)-v^{\nabla\nabla}(t)\rangle}{\|u(t)-v(t)\|}
\end{aligned}$$

because $\|u^{\nabla}(t)-v^{\nabla}(t)\|^2\|u(t)-v(t)\|^2 \geq \langle u(t)-v(t),u^{\nabla}(t)-v^{\nabla}(t)\rangle^2$ So

$$(||\ u(t)-v(t)\ ||\ -M(t))^{\nabla\nabla} \geq \frac{\langle u(\rho(t))-v(\rho(t)),u^{\nabla\nabla}(t)-v^{\nabla\nabla}(t)\rangle}{\|u(\rho(t))-v(\rho(t))\|} - M^{\nabla\nabla}(t)$$

- If $t \in A$ is left-scattered ($\rho(t) < t$) and $\rho^2(t) = \rho(t)$, then

$$\|u(t)-v(t)\|^{\nabla} = \frac{\|u(t)-v(t)\|-\|u(\rho(t))-v(\rho(t))\|}{\nu(t)}$$

with $\nu(t) = t-\rho(t)$. We have

$$u^{\nabla\nabla}(t) = \frac{u^{\nabla}(t)-u^{\nabla}(\rho(t))}{\nu(t)} \Rightarrow u^{\nabla}(\rho(t)) = u^{\nabla}(t)-\nu(t)u^{\nabla\nabla}(t)$$

$$\begin{aligned}
\|u(t)-v(t)\|^{\nabla\nabla} &= \frac{\|u(t)-v(t)\|^{\nabla}-\|u(\rho(t))-v(\rho(t))\|^{\nabla}}{\nu(t)} \\
&= \frac{1}{\nu(t)}[\frac{\langle u(t)-v(t),u^{\nabla}(t)-v^{\nabla}(t)\rangle}{\|u(t)-v(t)\|} \\
&\quad -\frac{\langle u(\rho(t))-v(\rho(t)),u^{\nabla}(\rho(t))-v^{\nabla}(\rho(t))\rangle}{\|u(\rho(t))-v(\rho(t))\|}] \\
&= \frac{1}{\nu(t)}[\frac{\langle u(t)-v(t),u^{\nabla}(t)-v^{\nabla}(t)\rangle}{\|u(t)-v(t)\|} \\
&\quad -\frac{\langle u(\rho(t))-v(\rho(t)),u^{\nabla}(t)-v^{\nabla}(t)\rangle}{\|u(\rho(t))-v(\rho(t))\|} \\
&\quad -\nu(t)\frac{\langle u(\rho(t))-v(\rho(t)),u^{\nabla\nabla}(t)-v^{\nabla\nabla}(t)\rangle}{\|u(\rho(t))-v(\rho(t))\|}] \\
&= \frac{\langle u(\rho(t))-v(\rho(t)),u^{\nabla\nabla}(t)-v^{\nabla\nabla}(t)\rangle}{\|u(\rho(t))-v(\rho(t))\|} + \\
&\quad \frac{\langle u(t)-v(t),u^{\nabla}(t)-v^{\nabla}(t)\rangle-\langle u(\rho(t))-v(\rho(t)),u^{\nabla}(t)-v^{\nabla}(t)\rangle}{\nu(t)\|u(t)-v(t)\|}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\langle u(\rho(t)) - v(\rho(t)), u^{\nabla\nabla}(t) - v^{\nabla\nabla}(t)\rangle}{\|u(\rho(t)) - v(\rho(t))\|} + \\
&\quad \frac{\langle (u(t) - v(t)) - (u(\rho(t)) - v(\rho(t))), u^{\nabla}(t) - v^{\nabla}(t)\rangle}{\nu(t)\|u(t) - v(t)\|} \\
&= \frac{\langle u(\rho(t)) - v(\rho(t)), u^{\nabla\nabla}(t) - v^{\nabla\nabla}(t)\rangle}{\|u(\rho(t)) - v(\rho(t))\|} + \\
&\quad \frac{\left\langle \frac{(u(t) - v(t)) - (u(\rho(t)) - v(\rho(t)))}{\nu(t)}, u^{\nabla}(t) - v^{\nabla}(t)\right\rangle}{\|u(t) - v(t)\|} \\
&= \frac{\langle u(\rho(t)) - v(\rho(t)), u^{\nabla\nabla}(t) - v^{\nabla\nabla}(t)\rangle}{\|u(\rho(t)) - v(\rho(t))\|} + \frac{\|u^{\nabla}(t) - v^{\nabla}(t)\|^2}{\|u(t) - v(t)\|}
\end{aligned}$$

thus

$$\|u(t) - v(t)\|^{\nabla\nabla} \geq \frac{\langle u(\rho(t)) - v(\rho(t)), u^{\nabla\nabla}(t) - v^{\nabla\nabla}(t)\rangle}{\|u(\rho(t)) - v(\rho(t))\|}$$

Also

$$(\|u(t) - v(t)\| - M(t))^{\nabla\nabla} \geq \frac{\langle u(\rho(t)) - v(\rho(t)), u^{\nabla\nabla}(t) - v^{\nabla\nabla}(t)\rangle}{\|u(\rho(t)) - v(\rho(t))\|} - M^{\nabla\nabla}(t)$$

- If $t \in A$ is left-scattered $(\rho(t) < t)$ and so $\rho^2(t) < \rho(t)$ then

$$\begin{aligned}
\|u(\rho(t)) - v(\rho(t))\|^{\nabla} &= \frac{\|u(\rho(t)) - v(\rho(t))\| - \|u(\rho^2(t)) - v(\rho^2(t))\|}{\nu(\rho(t))} \\
&= \frac{\|u(\rho(t)) - v(\rho(t))\|^2 - \|u(\rho^2(t)) - v(\rho^2(t))\|\|u(\rho(t)) - v(\rho(t))\|}{\|u(\rho(t)) - v(\rho(t))\|\nu(\rho(t))} \\
&\geq \frac{\langle u(\rho(t)) - v(\rho(t)), (u(\rho(t)) - v(\rho(t))) - (u(\rho^2(t)) - v(\rho^2(t)))\rangle}{\|u(\rho(t)) - v(\rho(t))\|\nu(\rho(t))} \\
&= \frac{\langle u(\rho(t)) - v(\rho(t)), u^{\nabla}(\rho(t)) - v^{\nabla}(\rho(t)))\rangle}{\|u(\rho(t)) - v(\rho(t))\|}
\end{aligned}$$

Thus, it follows that

$$(\|u(t) - v(t)\| - M(t))^{\nabla\nabla} = \frac{\|u(t) - v(t)\|^{\nabla} - \|u(\rho(t)) - v(\rho(t))\|^{\nabla}}{\nu(t)} - M^{\nabla\nabla}(t)$$

As

$$(\|u(t) - v(t)\|)^{\nabla} = \frac{\|u(t) - v(t)\| - \|u(\rho(t)) - v(\rho(t))\|}{\nu(t)}$$

and

$$\|u(\rho(t)) - v(\rho(t))\|^{\nabla} \geq \frac{\langle u(\rho(t)) - v(\rho(t)), u^{\nabla}(\rho(t)) - v^{\nabla}(\rho(t)))\rangle}{\|u(\rho(t)) - v(\rho(t))\|}$$

and

$$w^{\nabla}(\rho(t)) = w^{\nabla}(t) - \nu(t)w^{\nabla\nabla}(t)$$

thus

$$\begin{aligned}
\|u(t)-v(t)\|^{\nabla\nabla} &\geq \frac{1}{\nu(t)}[\frac{\|u(t)-v(t)\|-\|u(\rho(t))-v(\rho(t))\|}{\nu(t)} \\
&\quad -\frac{\langle u(\rho(t))-v(\rho(t)),u^{\nabla}(\rho(t))-v^{\nabla}(\rho(t))\rangle}{\|u(\rho(t))-v(\rho(t))\|}] \\
&\geq \frac{1}{\nu(t)}[\frac{\|u(t)-v(t)\|-\|u(\rho(t))-v(\rho(t))\|}{\nu(t)} \\
&\quad -\frac{\langle u(\rho(t))-v(\rho(t)),u^{\nabla}(t)-v^{\nabla}(t)\rangle}{\|u(\rho(t))-v(\rho(t))\|}+ \\
&\quad \nu(t)\frac{\langle u(\rho(t))-v(\rho(t)),u^{\nabla\nabla}(t)-v^{\nabla\nabla}(t)\rangle}{\|u(\rho(t))-v(\rho(t))\|}] \\
&\geq \frac{\langle u(\rho(t))-v(\rho(t)),u^{\nabla\nabla}(t)-v^{\nabla\nabla}(t)\rangle}{\|u(\rho(t))-v(\rho(t))\|}+ \\
&\quad \frac{1}{\nu(t)}[\frac{\|u(t)-v(t)\|-\|u(\rho(t))-v(\rho(t))\|}{\nu(t)} \\
&\quad -\frac{\langle u(\rho(t))-v(\rho(t)),u^{\nabla}(t)-v^{\nabla}(t)\rangle}{\|u(\rho(t))-v(\rho(t))\|}] \\
&\geq \frac{\langle u(\rho(t))-v(\rho(t)),u^{\nabla\nabla}(t)-v^{\nabla\nabla}(t)\rangle}{\|u(\rho(t))-v(\rho(t))\|}
\end{aligned}$$

As a result

$$(\|u(t)-v(t)\|-M(t))^{\nabla\nabla} \geq \frac{\langle u(\rho(t))-v(\rho(t)),u^{\nabla\nabla}(t)-v^{\nabla\nabla}(t)\rangle}{\|u(\rho(t))-v(\rho(t))\|}-M^{\nabla\nabla}(t)$$

Step 2: We prove now $(\|u(t)-v(t)\|-M(t)))^{\nabla\nabla}>0 \quad \nabla.a.e \quad surA$

⋇ If $\{t\in A:\rho(t)<t\}$

$$(\|u(t)-v(t)\|-M(t))^{\nabla\nabla} \geq \frac{\langle u(\rho(t))-v(\rho(t)),u^{\nabla\nabla}(t)-v^{\nabla\nabla}(t)\rangle}{\|u(\rho(t))-v(\rho(t))\|}-M^{\nabla\nabla}(t)$$

We have

$$\begin{aligned}
\beta(t) &= \langle u(\rho(t))-v(\rho(t)),u^{\nabla\nabla}(t)-v^{\nabla\nabla}(t)\rangle \\
&= \langle u(\rho(t))-v(\rho(t)),g(t,u(\rho(t)),u^{\nabla}(t))+u(\rho(t))-v^{\nabla\nabla}(t)\rangle \\
&= \langle u(\rho(t))-v(\rho(t)),\frac{M(\rho(t))}{\|u(\rho(t))-v(\rho(t))\|}f(t,\bar{u}(\rho(t)),\tilde{u}^{\nabla}(t)) \\
&\quad -\bar{u}(\rho(t))+u(\rho(t))-v^{\nabla\nabla}(t)+\left(1-\frac{M(\rho(t))}{\|u(\rho(t))-v(\rho(t))\|}\right) \\
&\quad \left(v^{\nabla\nabla}(t)+\frac{M^{\nabla\nabla}(t)}{\|u(\rho(t))-v(\rho(t))\|}(u(\rho(t))-v(\rho(t)))\right)\rangle \\
&= \langle u(\rho(t))-v(\rho(t)),\frac{M(\rho(t))}{\|u(\rho(t))-v(\rho(t))\|}(f(t,\bar{u}(\rho(t)),\tilde{u}^{\nabla}(t))-v^{\nabla\nabla}(t)))+u(\rho(t))- \\
&\quad \bar{u}(\rho(t))+\left(1-\frac{M(\rho(t))}{\|u(\rho(t))-v(\rho(t))\|}\right)\left(\frac{M^{\nabla\nabla}(\rho(t))}{\|u(\rho(t))-v(\rho(t))\|}(u(\rho(t))-v(\rho(t)))\right)\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \langle \tilde{u}(\rho(t)) - v(\rho(t)), (f(t, \tilde{u}(\rho(t)), \tilde{u}^{\nabla}(t)) - v^{\nabla\nabla}(t))\rangle \\
&\quad + \langle u(\rho(t)) - v(\rho(t)), u(\rho(t)) - \tilde{u}(\rho(t))\rangle \\
&\quad + \left(1 - \frac{M(\rho(t))}{\|u(\rho(t)) - v(\rho(t))\|}\right) \frac{M^{\nabla\nabla}(t)}{\|u(\rho(t)) - v(\rho(t))\|} \|u(\rho(t)) - v(\rho(t))\|^2 \\
&= \langle \tilde{u}(\rho(t)) - v(\rho(t)), (f(t, \tilde{u}(\rho(t)), \tilde{u}^{\nabla}(t)) - v^{\nabla\nabla}(t))\rangle + \\
&\quad \left(1 - \frac{M(\rho(t))}{\|u(\rho(t)) - v(\rho(t))\|}\right) \|u(\rho(t)) - v(\rho(t))\|^2 \\
&\quad + \left(1 - \frac{M(\rho(t))}{\|u(\rho(t)) - v(\rho(t))\|}\right) M^{\nabla\nabla}(t) \|u(\rho(t)) - v(\rho(t))\| \\
&= \langle \tilde{u}(\rho(t)) - v(\rho(t)), (f(t, \tilde{u}(\rho(t)), \tilde{u}^{\nabla}(t)) - v^{\nabla\nabla}(t))\rangle + \\
&\quad (\|u(\rho(t)) - v(\rho(t))\| - M(\rho(t))) \left[M^{\nabla\nabla}(t) + \|u(\rho(t)) - v(\rho(t))\|\right] \\
&\geqslant M(\rho(t)) M^{\nabla\nabla}(t) + (\|u(\rho(t)) - v(\rho(t))\| - M(\rho(t))) M^{\nabla\nabla}(t) \\
&\quad + [\|u(\rho(t)) - v(\rho(t))\| - M(\rho(t))] \|u(\rho(t)) - v(\rho(t))\| \\
&= \|u(\rho(t)) - v(\rho(t))\| \left[M^{\nabla\nabla}(t) + \|u(\rho(t)) - v(\rho(t))\| - M(\rho(t))\right]
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\frac{\langle u(\rho(t)) - v(\rho(t)), u^{\nabla\nabla}(t) - v^{\nabla\nabla}(t)\rangle}{\|u(\rho(t)) - v(\rho(t))\|} \geqslant M^{\nabla\nabla}(t) + \|u(\rho(t)) - v(\rho(t))\| - M(\rho(t)) \\
&\frac{\langle u(\rho(t)) - v(\rho(t)), u^{\nabla\nabla}(t) - v^{\nabla\nabla}(t)\rangle}{\|u(\rho(t)) - v(\rho(t))\|} - M^{\nabla\nabla}(t) \geqslant \|u(\rho(t)) - v(\rho(t))\| - M(\rho(t)) > 0 \\
&\frac{\langle u(\rho(t)) - v(\rho(t)), u^{\nabla\nabla}(t) - v^{\nabla\nabla}(t)\rangle}{\|u(\rho(t)) - v(\rho(t))\|} - M^{\nabla\nabla}(t) > 0
\end{aligned}
$$

⋇ If $\{t \in A : t = \rho(t)\}$ then

$$
\begin{aligned}
(\|u(t) - v(t)\| - M(t))^{\nabla\nabla} &= \frac{\langle u(t) - v(t), u^{\nabla\nabla}(t) - v^{\nabla\nabla}(t)\rangle + \|u^{\nabla}(t) - v^{\nabla}(t)\|^2}{\|u(t) - v(t)\|} \\
&\quad - \frac{\langle u(t) - v(t), u^{\nabla}(t) - v^{\nabla}(t)\rangle^2}{\|u(t) - v(t)\|^3} - M^{\nabla\nabla}(t)
\end{aligned}
$$

We have

$$
\begin{aligned}
\langle u(t) - v(t), u^{\nabla\nabla}(t) - v^{\nabla\nabla}(t)\rangle &= \langle u(\rho(t)) - v(\rho(t)), g(t, u(\rho(t)), u^{\nabla}(t)) - v^{\nabla\nabla}(t) + u(\rho(t))\rangle \\
&= \langle \overline{u}(\rho(t)) - v(\rho(t)), f(t, \overline{u}(\rho(t)), \tilde{u}^{\nabla}(t)) - v^{\nabla\nabla}(t))\rangle \\
&\quad + \|u(\rho(t)) - v(\rho(t))\| \left[\|u(\rho(t)) - v(\rho(t))\| - M(\rho(t))\right. \\
&\quad \left. + M^{\nabla\nabla}(t) \left(1 - \frac{M(\rho(t))}{\|u(\rho(t)) - v(\rho(t))\|}\right)\right]
\end{aligned}
$$

So thus

$$
(\|u(t) - v(t)\| - M(t))^{\nabla\nabla} = \frac{\langle \overline{u}(t) - v(t), f(t, \overline{u}(t), \tilde{u}^{\nabla}(t)) - v^{\nabla\nabla}(t))\rangle + \|u^{\nabla}(t) - v^{\nabla}(t)\|^2}{\|u(t) - v(t)\|}
$$

$$
\begin{aligned}
&- \frac{\langle u(t)-v(t), u^{\nabla}(t)-v^{\nabla}(t)\rangle^2}{\|u(t)-v(t)\|^3} - M^{\nabla\nabla}(t) \\
&+ \|u(t)-v(t)\| - M(t) + M^{\nabla\nabla}(t)\left(1 - \frac{M(t)}{\|u(t)-v(t)\|}\right) \\
&= \frac{\langle \overline{u}(t)-v(t), f(t,\overline{u}(t),\tilde{u}^{\nabla}(t)) - v^{\nabla\nabla}(t))\rangle + \|\tilde{u}^{\nabla}(t)) - v^{\nabla}(t))\|^2}{\|u(t)-v(t)\|} \\
&+ \frac{\|u^{\nabla}(t)) - v^{\nabla}(t))\|^2 - \|\tilde{u}^{\nabla}(t)) - v^{\nabla}(t))\|^2}{\|u(t)-v(t)\|} \\
&- \frac{\langle u(t)-v(t), u^{\nabla}(t)-v^{\nabla}(t)\rangle^2}{\|u(t)-v(t)\|^3} \\
&+ \|u(t)-v(t)\| - M(t) - M^{\nabla\nabla}(t)\frac{M(t)}{\|u(t)-v(t)\|} \\
&\geqslant \frac{M(t)M^{\nabla\nabla}(t) + (M^{\nabla}(t))^2}{\|u(t)-v(t)\|} + \|u(t)-v(t)\| - M(t) \\
&- M^{\nabla\nabla}(t)\frac{M(t)}{\|u(t)-v(t)\|} \\
&\frac{\|u^{\nabla}(t)) - v^{\nabla}(t))\|^2 - \|\tilde{u}^{\nabla}(t)) - v^{\nabla}(t))\|^2}{\|u(t)-v(t)\|} \\
&- \frac{\langle u(t)-v(t), u^{\nabla}(t)-v^{\nabla}(t)\rangle^2}{\|u(t)-v(t)\|^3} \\
&\geqslant \|u(t)-v(t)\| - M(t) \\
&+ \frac{\|u^{\nabla}(t)) - v^{\nabla}(t))\|^2 - \|\widehat{u}^{\nabla}(t)) - v^{\nabla}(t))\|^2}{\|u(t)-v(t)\|} \\
&+ \frac{\langle u(t)-v(t), \widehat{u}^{\nabla}(t)-v^{\nabla}(t)\rangle^2 - \langle u(t)-v(t), u^{\nabla}(t)-v^{\nabla}(t)\rangle^2}{\|u(t)-v(t)\|^3}
\end{aligned}
$$

◊ If $\|u^{\nabla}(t) - v^{\nabla}(t)\| \leqslant K$ then

$$
(\|u(t)-v(t)\| - M(t))^{\nabla\nabla} \geqslant \|u(t)-v(t)\| - M(t) > 0
$$

◊ If $\|u^{\nabla}(t) - v^{\nabla}(t)\| > K$ then $\|\widehat{u}^{\nabla}(t) - v^{\nabla}(t)\| = K$ and

$$
\begin{aligned}
\langle u(t)-v(t), \widehat{u}^{\nabla}(t)-v^{\nabla}(t)\rangle^2 - \langle u(t)-v(t), u^{\nabla}(t)-v^{\nabla}(t)\rangle^2 &= \frac{K^2}{\|u^{\nabla}(t)-v^{\nabla}(t)\|^2} \\
&\langle u(t)-v(t), u^{\nabla}(t)-v^{\nabla}(t)\rangle^2 - \langle u(t)-v(t), u^{\nabla}(t)-v^{\nabla}(t)\rangle^2 \\
&= \left(\frac{K^2}{\|u^{\nabla}(t)-v^{\nabla}(t)\|^2} - 1\right)\langle u(t)-v(t), u^{\nabla}(t)-v^{\nabla}(t)\rangle^2
\end{aligned}
$$

Thus

$$
\begin{aligned}
&(\|u(t)-v(t)\| - M(t))^{\nabla\nabla} \\
\geqslant\ & \|u(t)-v(t)\| - M(t) + \frac{\|u^{\nabla}(t)) - v^{\nabla}(t))\|^2 - \|\widehat{u}^{\nabla}(t)) - v^{\nabla}(t))\|^2}{\|u(t)-v(t)\|} \\
&+ \frac{\langle u(t)-v(t), \widehat{u}^{\nabla}(t)-v^{\nabla}(t)\rangle^2 - \langle u(t)-v(t), u^{\nabla}(t)-v^{\nabla}(t)\rangle^2}{\|u(t)-v(t)\|^3}
\end{aligned}
$$

$$
\begin{aligned}
&\geqslant \|u(t)-v(t)\| - M(t) + \left(1-\frac{K^2}{\|u^\nabla(t)-v^\nabla(t)\|^2}\right)\frac{\|u^\nabla(t))-v^\nabla(t))\|^2}{\|u(t)-v(t)\|} \\
&\quad -\left(1-\frac{K^2}{\|u^\nabla(t)-v^\nabla(t)\|^2}\right)\frac{\langle u(t)-v(t),u^\nabla(t)-v^\nabla(t)\rangle^2}{\|u(t)-v(t)\|^3} \\
&\geqslant \|u(t)-v(t)\| - M(t) + \\
&\quad \left(1-\frac{K^2}{\|u^\nabla(t)-v^\nabla(t)\|^2}\right)\left[\frac{\|u^\nabla(t))-v^\nabla(t))\|^2}{\|u(t)-v(t)\|}-\frac{\langle u(t)-v(t),u^\nabla(t)-v^\nabla(t)\rangle^2}{\|u(t)-v(t)\|^3}\right] \\
&\geqslant \|u(t)-v(t)\| - M(t) > 0
\end{aligned}
$$

Thus $\nabla.a.e \quad t\in A, (\|u(t)-v(t)\|-M(t))^{\nabla\nabla} > 0$. Denote $r(t)=\|u(t)-v(t)\|-M(t)$, it follows that $r^{\nabla\nabla}(t)>0 \quad \nabla.a.e \quad t\in\{t\in\mathbb{T}_{0,\kappa^2} : r(\rho(t))>0\}$.

Step 3 : Prove that $r(\rho(a))=r(\sigma(b)), \quad r^\nabla(a)\geqslant r^\nabla(\sigma(b))$.

$r(\rho(a))=r(\sigma(b))$ (obvious).

Prove that $r^\nabla(a)\geqslant r^\nabla(\sigma(b))$.

i) If $a=\rho(a)\Rightarrow \|u(a)-v(a)\|^\nabla = \dfrac{\langle u(a)-v(a),u^\nabla(a)-v^\nabla(a)\rangle}{\|u(\rho(a))-v(\rho(a))\|}$.

ii) If $a>\rho(a)\Rightarrow$

$$
\begin{aligned}
\|u(a)-v(a)\|^\nabla &= \frac{\|u(a)-v(a)\|-\|u(\rho(a))-v(\rho(a))\|}{\nu(a)} \\
&= \frac{\|u(a)-v(a)\|\|u(\rho(a))-v(\rho(a))\|-\|u(\rho(a))-v(\rho(a))\|^2}{\nu(a)\|u(\rho(a))-v(\rho(a))\|} \\
&\geqslant \frac{\langle u(a)-v(a),u(\rho(a))-v(\rho(a))\rangle-\|u(\rho(a))-v(\rho(a))\|^2}{\nu(a)\|u(\rho(a))-v(\rho(a))\|} \\
&\geqslant \frac{\langle u^\nabla(a)-v^\nabla(a),u(\rho(a))-v(\rho(a))\rangle}{\|u(\rho(a))-v(\rho(a))\|}.
\end{aligned}
$$

Thus $r^\nabla(a)\geqslant \dfrac{\langle u^\nabla(a)-v^\nabla(a),u(\rho(a))-v(\rho(a))\rangle}{\|u(\rho(a))-v(\rho(a))\|}-M^\nabla(a)$.

Also

iii) $\sigma(b)=\rho(\sigma(b))$ alors $\|u(\sigma(b))-v(\sigma(b))\|^\nabla = \dfrac{\langle u^\nabla(\sigma(b))-v^\nabla(\sigma(b)),u(\sigma(b))-v(\sigma(b))\rangle}{\|u(\sigma(b))-v(\sigma(b))\|}$.

iv) If $\sigma(b)>\rho(\sigma(b))$ then

$$
\begin{aligned}
\|u(\sigma(b))-v(\sigma(b))\|^\nabla &= \frac{\|u(\sigma(b)))-v(\sigma(b))\|-\|u(\rho(\sigma(b)))-v(\rho(\sigma(b)))\|}{\nu(\sigma(b))} \\
&= \frac{\|u(\sigma(b)))-v(\sigma(b))\|^2-\|u(\rho(\sigma(b)))-v(\rho(\sigma(b)))\|\|u(\sigma(b)))-v(\sigma(b))\|}{\nu(\sigma(b))\|u(\sigma(b)))-v(\sigma(b))\|} \\
&\leqslant \frac{\langle u^\nabla(\sigma(b))-v^\nabla(\sigma(b)),u(\sigma(b))-v(\sigma(b))\rangle}{\|u(\sigma(b))-v(\sigma(b))\|}.
\end{aligned}
$$

Thus $r^\nabla(\sigma(b))\leqslant \dfrac{\langle u^\nabla(\sigma(b))-v^\nabla(\sigma(b)),u(\sigma(b))-v(\sigma(b))\rangle}{\|u(\sigma(b))-v(\sigma(b))\|}-M^\nabla(\sigma(b))$. It follows that

$$
r^\nabla(\sigma(b))-r^\nabla(a) \leqslant \frac{\langle u^\nabla(\sigma(b))-v^\nabla(\sigma(b)),u(\sigma(b))-v(\sigma(b))\rangle}{\|u(\sigma(b))-v(\sigma(b))\|}-\frac{\langle u^\nabla(a)-v^\nabla(a),u(\rho(a))-v(\rho(a))\rangle}{\|u(\rho(a))-v(\rho(a))\|}
$$

$$\begin{aligned} & \left(M^{\nabla}(\sigma(b)) - M^{\nabla}(a)\right) \\ \leqslant\ & \frac{\langle \left(u^{\nabla}(\sigma(b)) - v^{\nabla}(\sigma(b))\right) - \left(u^{\nabla}(a) - v^{\nabla}(a)\right), u(\rho(a)) - v(\rho(a))\rangle}{\|u(\rho(a)) - v(\rho(a))\|} \\ & - \left(M^{\nabla}(\sigma(b)) - M^{\nabla}(a)\right) \\ \leqslant\ & \frac{\langle v^{\nabla}(a) - v^{\nabla}(\sigma(b)), u(\rho(a)) - v(\rho(a))\rangle}{\|u(\rho(a)) - v(\rho(a))\|} - \left(M^{\nabla}(\sigma(b)) - M^{\nabla}(a)\right) \\ \leqslant\ & \|v^{\nabla}(a) - v^{\nabla}(\sigma(b))\| - \left(M^{\nabla}(\sigma(b)) - M^{\nabla}(a)\right) \\ r^{\nabla}(\sigma(b)) - r^{\nabla}(a) \leqslant\ & 0. \end{aligned}$$

Thus Therefore for the theorem 2.23 , we have $r(t) \leqslant 0 \Rightarrow \|u(t) - v(t)\| \leq M(t)$ for every $t \in \mathbb{T}$. □

Let the operators:

$$L_1 : W^{2,1}_{\nabla,BC}(\mathbb{T}, \mathbb{R}^n) \to C_0(\mathbb{T}_{0,\kappa}, \mathbb{R}^n) \cap W^{1,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n)$$

define by

$$L_1(u)(t) = u^{\nabla\nabla}(t) - u(\rho(t))$$

and

$$L_3 : C^1(\mathbb{T}, \mathbb{R}^n) \cap W^{2,1}_{\nabla,BC}(\mathbb{T}, \mathbb{R}^n) \to C_0(\mathbb{T}_{0,\kappa}, \mathbb{R}^n) \cap W^{1,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n)$$

define by and

$$L_3(u)(t) = u^{\nabla}(t) - u^{\nabla}(a) - \int_{[a,t)\cap\mathbb{T}} u(\rho(s))\nabla(s)$$

$$N_g : C^1(\mathbb{T}, \mathbb{R}^n) \cap W^{2,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n) \to C_0(\mathbb{T}_{0,\kappa}, \mathbb{R}^n) \cap W^{1,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n)$$

define by

$$N_g(u)(t) = \int_{[a,t)\cap\mathbb{T}} g(s, u(\rho(s)), u^{\nabla}(s))\nabla(s)$$

Remark 3.5. *Since f is $\nabla-$Caratheodory, then there is $h \in L^1_{\nabla}(\mathbb{T}_{0,\kappa}, \mathbb{R}^n)$ such as $\forall\ u, w \in \mathbb{R}^n$, $\|g(t, u, w)\| \leqslant h(t)$ $\nabla-pp$ $t \in \mathbb{T}_{0,\kappa}$.*

Proposition 3.6. *let $f : \mathbb{T}_{0,\kappa} \times \mathbb{R}^n$ a $\nabla-$Caratheodory function. Suppose that (H_1) is satisfied, then the operator N_g defined is continuous and compact.*

Proof. Let $\{u_k\}$ a sequence of $C^1(\mathbb{T}, \mathbb{R}^n)$ converging to $u \in C^1(\mathbb{T}, \mathbb{R}^n)$.
Prove that the function sequence $\{g_k\}_{k\in\mathbb{N}}$ define by $g_k(t) = g(t, u_k(\rho(t)), u_k^{\nabla}(t))$ converges to the function g define by $g(t) = g(t, u(\rho(t)), u^{\nabla}(t))$ $L^1_{\nabla}(\mathbb{T}_{0,\kappa}, \mathbb{R}^n)$.
It is easily shown that $\bar{u}_k(t) \longrightarrow \bar{u}(t)$ et $\widehat{u_k^{\nabla}}(t) \longrightarrow \widehat{u^{\nabla}}(t)$. On $\{t \in \mathbb{T}, \rho(t) < t\}$, we have $g(t, u_k(\rho(t)), u_k^{\nabla}(t)) \longrightarrow g(t, u(\rho(t)), u^{\nabla}(t))$ since f is $\nabla-$Caratheodory. So, for $\nabla-$a.e on $I = \{t \in \mathbb{T}_{0,\kappa} : t = \rho(t), \|u(\rho(t)) - v(\rho(t))\| \neq M(\rho(t))\}$, we have $g_k(t) \longrightarrow g(t)$ as $\tilde{u}_k^{\nabla}(t) \longrightarrow \tilde{u}^{\nabla}(t)$ and f is $\nabla-$Caratheodory.

Notice $S = \{t \in \mathbb{T}_{0,\kappa} : t = \rho(t)$ and $\|u_k(\rho(t)) - v_k(\rho(t))\| = M(\rho(t))\}$. We have $\langle u(\rho(t)) - v(\rho(t)), u^{\nabla}(t) - v^{\nabla}(t)\rangle = M(\rho(t))M^{\nabla}(t) \quad \nabla.a.e \quad t \in S$. So, we have:

$$\begin{aligned}
\tilde{u}_k^{\nabla}(t) &= \hat{u}_k^{\nabla}(t) + [M^{\nabla}(t) - \frac{\langle u_k(\rho(t)) - v(\rho(t), \hat{u}_k^{\nabla}(t) - v^{\nabla}(t))\rangle}{\|u_k(\rho(t)) - v(\rho(t))\|}][\frac{u_k(\rho(t)) - v(\rho(t)}{\|u_k(\rho(t)) - v(\rho(t)\|}] \\
&\to \hat{u}^{\nabla}(t) + [M^{\nabla}(t) - \frac{\langle u(\rho(t)) - v(\rho(t), \hat{u}^{\nabla}(t) - v^{\nabla}(t))\rangle}{\|u(\rho(t)) - v(\rho(t))\|}][\frac{u(\rho(t)) - v(\rho(t)}{\|u(\rho(t)) - v(\rho(t)\|}] \\
&= \begin{cases} \hat{u}^{\nabla}(t) + \frac{M^{\nabla}(t)}{M(\rho(t))}(1 - \frac{K}{\|u^{\nabla}(t) - v^{\nabla}(t)\|})(u(\rho(t)) - v(\rho(t))) & \text{if } \|u^{\nabla}(t) - v^{\nabla}(t)\| > K \\ \hat{u}^{\nabla}(t) & \text{if } \|u^{\nabla}(t) - v^{\nabla}(t)\| \leqslant K \end{cases} \\
&= \tilde{u}^{\nabla}(t)
\end{aligned}$$

It follows that $\nabla-$a.e on S, $\tilde{u}_k^{\nabla}(t) \to \tilde{u}^{\nabla}(t)$ and since f is $\nabla-$Caratheodory, we have $f(t, u_k(\rho(t)), u_k^{\nabla}(t) \to f(t, u(\rho(t)), u^{\nabla}(t)$ and thus $g_k(t) \to g(t)$. By the previous remark, we see there exists a function $h \in L^1_{\nabla}(\mathbb{T}_{0,\kappa}, [0, \infty[)$such as $\|g_k(t)\| \leq h(t)$ ∇.a.e $t \in \mathbb{T}_{\kappa,0}$. Since the assumptions of the dominated convergence theorem are satisfied and thus $g_n \to g$ in $L^1_{\nabla}(\mathbb{T}_{0,\kappa}, \mathbb{R}^n)$ The continuity is verified. Prove now that $N_g(C^1(\mathbb{T}, \mathbb{R}^n))$ is relatively compact in $C_0(\mathbb{T}_\kappa, \mathbb{R}^n)$. Let $\{y_k\}_{k\in\mathbb{N}}$ a sequence of $N_g(C^1(\mathbb{T}, \mathbb{R}^n))$. For all $k \in \mathbb{N}$, the exist $u_k \in C^1(\mathbb{T}, \mathbb{R}^n)$ such as $y_k = N_g(u_k)$. From the above, we can apply $N_g(u_k)(t) = \int_{[a,t)\cap\mathbb{T}} g_k(s)\nabla s$ and so $\{y_k\}_{k\in\mathbb{N}}$ is uniformly bounded and equicontinuous. For the d'Arzela-Ascoli theorem, $\{y_k\}_{k\in\mathbb{N}}$ has a convergent subsequence so $N_g(C^1(\mathbb{T}, \mathbb{R}^n))$ is relatively compact in $C_0(\mathbb{T}_\kappa, \mathbb{R}^n)$. □

Proposition 3.7. *The operator L_1 is linear, continuous and invertible.*

Proof. it's obvious that L_1 is linear and continuous. Let's $g \in L^1_{\nabla}(\mathbb{T}_{\kappa,0}, \mathbb{R}^n)$. By the Proposition 2.27, there exists the unique solution $u \in W^{2,1}_{\nabla,BC}(\mathbb{T}_{\kappa,0}, \mathbb{R}^n)$ such as

$$u^{\nabla\nabla}(t) - u(\rho(t)) = g(t) \qquad \nabla.pp \quad t \in \mathbb{T}_{\kappa,0}$$

By the Lemma 2.24, L_1 is injective. □

Proposition 3.8. *The operator L_3 is linear, continuous and invertible.*

Proof. It's obvious that L_3 is linear and continuous. Let's $g \in C_0(\mathbb{T}_\kappa, \mathbb{R}^n) \cap W^{1,1}_{\nabla}(\mathbb{T}_\kappa, \mathbb{R}^n)$ then $g^{\nabla} L^1_{\nabla}(\mathbb{T}_{\kappa,0}, \mathbb{R}^n)$. By the proposition 2.29,the equation

$$u^{\nabla\nabla}(t) - u(\rho(t)) = g^{\nabla}(t) \qquad \nabla.pp \quad t \in \mathbb{T}_{\kappa,0}$$

have the unique solution $u \in W^{2,1}_{\nabla,BC}(\mathbb{T}_{\kappa,0}, \mathbb{R}^n)$. By integrating the equation on the set $[a, t[\cap\mathbb{T}$, we have

$$u^{\nabla}(t) - u^{\nabla}(a) - \int_{[a,t)\cap\mathbb{T}} u(\rho(s))\nabla s = g(t) - g(a) \qquad \nabla.pp \quad t \in \mathbb{T}_\kappa$$

Since $g \in C_0(\mathbb{T}_\kappa, \mathbb{R}^n)$, the operator is surjective. By the Lemma 2.24, L_3 is injective.So, the invertible of L_3. □

Lemma 3.9. *Assume that (H_1) is satisfied. Let u a solution of the modified problem then $\exists K > 0$ such as*

$$\|u^\nabla(t) - v^\nabla(t)\| \le K \quad \forall t \in \mathbb{T}_\kappa$$

Proof. For the assumption (H_2), Lemma 3.4 and the Proposition 2.13, for all u solution of modified problem (7), we have ∇.a.e $t \in \mathbb{T}_\kappa$,

$$\begin{aligned}
\|u^\nabla(t)\| &\le \|u^\nabla(a)\| + \int_{[a,t)\cap\mathbb{T}} \|u^{\nabla\nabla}(s)\|\nabla s \\
&\le \|u^\nabla(a)\| + \int_{[a,t)\cap\mathbb{T}} \|f(s, u(\rho(s)), \tilde{u}^\nabla(s)\|\nabla s \\
&\le C_0 + \int_{[a,t)\cap\mathbb{T}} (C + D\|\tilde{u}^\nabla(s)\|)\nabla s \\
&\le C_0 + \int_{[a,t)\cap\mathbb{T}} (C + D\|\tilde{u}^\nabla(s)\|)\nabla s \\
&\le C_0 + \int_{[a,t)\cap\mathbb{T}} [C + D(\|\hat{u}^\nabla(s) - v^\nabla(s)\|) + \|v^\nabla(s)\| + |M^\nabla(s)|)]\nabla s \\
&\le C_0 + \int_{[a,t)\cap\mathbb{T}} [C + D(2\|v^\nabla(s)\| + |M^\nabla(s)|)\nabla s] + D\int_{[a,t)\cap\mathbb{T}} \|u^\nabla(s)\|\nabla s \\
&\le C_1 + D\int_{[a,t)\cap\mathbb{T}} \|u^\nabla(s)\|\nabla s
\end{aligned}$$

with $C_0 = 2\|v^\nabla(a)\| + |M^\nabla(a)|$ and $C_1 = C_0 + \int_{[a,t)\cap\mathbb{T}}[C + D(2\|v^\nabla(s)\| + |M^\nabla(s)|)]\nabla s$. By Gronwall's inequality, $\|u^\nabla(t)\| \le C_1 e_D(t,a)$. Fix $K > \|v^\nabla\|_0 + C_1\|e_D(.,a)\|$. So, $\|u^\nabla(t) - v^\nabla(t)\| \le \|u^\nabla(t)\| + \|v^\nabla(t)\| \le \|v^\nabla(t)\| + C_1 e_D(t,a) \le K, \forall t \in \mathbb{T}_\kappa$ □

Proof of Theorem 3.2. From the previous Lemma 3.9, let K defined previously, a solution of the modified problem will be a fixed point of the operator:

$$T = L_3^{-1} \circ N_g : C^1(\mathbb{T}, \mathbb{R}^n) \longrightarrow C^1(\mathbb{T}, \mathbb{R}^n)$$

with L is linear, continuous, invertible and N_g is continuous then T is continuous. By Remark 3.5, there exists $h \in L^1_\nabla(\mathbb{T}_{0,\kappa}, [0,\infty))$ such that for every $y \in T(C^1(\mathbb{T}, \mathbb{R}^n))$, the exists $u \in T(C^1(\mathbb{T}, \mathbb{R}^n))$ such that $y = Tu$ and

$$\|N_g(u)(s)\| \le \int_{[a,t)\cap\mathbb{T}} \|g(s, u(s), u^\nabla(s))\|\nabla s \le \int_{[a,t)\cap\mathbb{T}} h(s)\nabla s \quad \nabla - a.e. \quad s \in \mathbb{T}_{0,\kappa}.$$

Since L_3^{-1} is continuous and affine,they map bounded sets in bounded sets. Thus, there exists a constant k_0 such that

$$\|y\|_1 \le k_0.$$

Moreover, $y \in W^{2,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n)$ and

$$L_3(y)(s) = y^{\nabla}(s) - y^{\nabla}(a) - \int_{[a,t)\cap\mathbb{T}} y(\rho(s))\nabla(s) = N_g(u)(s).$$

So,for every $t < \tau$ in $\mathbb{T}_{\kappa}$,

$$\|y^{\nabla}(t) - y^{\nabla}(\tau)\| \leq \int_{[t,\tau)\cap\mathbb{T}} \|y(\rho(s)) + g(s, u(s), u^{\nabla}(s))\|\nabla s \leq \int_{[t,\tau)\cap\mathbb{T}} (k_0 + h(s))\nabla s.$$

Thus, $T(C^1(\mathbb{T}, \mathbb{R}^n)$ is bounded and equicontinuous in $C^1(\mathbb{T}, \mathbb{R}^n)$. By an analogy of the Arzelà-Ascoli theorem for our context, $T(C^1(\mathbb{T}, \mathbb{R}^n)$ is relatively compact in $T(C^1(\mathbb{T}, \mathbb{R}^n)$. By the Schauder fixed point theorem, T has a fixed point, thus solution of modified problem. So any solution of modified problem $u \in T(v, M)$ and $\|u^{\nabla}(t) - v^{\nabla}(t)\| \leq K \; \forall t \in \mathbb{T}_{\kappa}$. We deduce that u is a solution of problem (1). □

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