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Fractional Calculus Results for Mathieu Series and Generalized Lommel Wright Function

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Abstract

The purpose of this paper is to apply generalized fractional integral and differential operators given by Marichev-Saigo-Maeda to the product of a generalized Mathieu series and a generalized Lommel-Wright function. The results are expressed in terms of generalized Wright function. A number of known results and some new results can be easily found as special cases of our main results.

Keywords: Generalized fractional integral operators; fractional derivative operators; generalized Mathieu series; generalized Lommel-Wright function; Fox-Wright function.

1. Introduction & Preliminaries

Fractional calculus (FC) is an emerging field of mathematics, which has wide applications in all related fields of science and engineering such as electromagnetism, control engineering and signal processing. It has become evident that fractional differential equations can accurately describe more and more processes in the physical and engineering world. The field of fractional calculus has made extraordinary advances, addressing both modelling and control, but new applications and theoretical developments are still needed for explaining and controlling chaotic systems characterized by bifurcations, criticality and symmetry. The fractional integral formulas involving various special functions have gained importance due to the usefulness of these results in the evaluation of generalized integrals and generalized derivatives and the solution of differential and integral equation. For a remarkable number of integral formulas involving a variety of special function one can refer the works of [2, 7–9, 14, 19, 24, 29]. In the present work, we aim at finding generalized integral and differential formulas for the product of generalized Lommel-Wright function and Mahieu series, which are expressed in terms of the generalized (Wright's) hypergeometric functions.

We now recall the generalized fractional integrals and derivatives involving Appell function or Horn function $F_3(.)$. These fractional calculus operators are introduced by Marichev [15] and later extended and studied by Saigo and Maeda [25] and are defined as follows:

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Let $\eta, \eta', \varsigma, \varsigma', \gamma \in \mathbb{C}$ and $x > 0$ and $\mathrm{R}(\gamma) > 0$, then left and right sided fractional integral operators are respectively defined as

$$\left(I_{0+}^{\eta,\eta',\varsigma,\varsigma',\gamma} f\right)(x) = \frac{x^{-\eta}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\eta'} F_3\left(\eta, \eta', \varsigma, \varsigma'; \gamma; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)\, dt \tag{1}$$

and

$$\left(I_{0-}^{\eta,\eta',\varsigma,\varsigma',\gamma} f\right)(x) = \frac{x^{-\eta'}}{\Gamma(\gamma)} \int_x^\infty (t-x)^{\gamma-1} t^{-\eta} F_3\left(\eta, \eta', \varsigma, \varsigma'; \gamma; 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t)\, dt \tag{2}$$

where $F_3(.)$ is the Appell series defined by

$$F_3(\eta, \eta', \varsigma, \varsigma'; \gamma; u, v) = \sum_{m,n=0}^{\infty} \frac{(\eta)_m (\eta')_n (\varsigma)_m (\varsigma')_n}{(\gamma)_{m+n}} \frac{u^m}{m!} \frac{v^n}{n!}, \quad (\max\{|u|, |v| < 1\})$$

Let $\eta, \eta', \varsigma, \varsigma', \gamma \in \mathbb{C}$ and $x > 0$ and $\mathrm{R}(\gamma) > 0$, then corresponding left and right sided generalized fractional differential operators are respectively defined as

$$\left(D_{0+}^{\eta,\eta',\varsigma,\varsigma',\gamma} f\right)(x) = \left(\frac{d}{dx}\right)^{[\mathrm{R}(\gamma)]+1} \left(I_{0+}^{-\eta',-\eta,-\varsigma'+[\mathrm{R}(\gamma)]+1,-\varsigma,-\gamma+[\mathrm{R}(\gamma)]+1} f\right)(x) \tag{3}$$

$$\left(D_{0-}^{\eta,\eta',\varsigma,\varsigma',\gamma} f\right)(x) = \left(-\frac{d}{dx}\right)^{[\mathrm{R}(\gamma)]+1} \left(I_{0-}^{-\eta',-\eta,-\varsigma',-\varsigma+[\mathrm{R}(\gamma)]+1,-\gamma+[\mathrm{R}(\gamma)]+1} f\right)(x) \tag{4}$$

Many interesting applications of fractional integral and differential operators in applicable mathematical analysis can be notably found in [11, 18, 28]. Further, the image formulas for a power function, under operators (1) and (2) are given by [see [25], p. 394, Equations (4.18) and (4.19)]

$$\left[I_{0+}^{\eta,\eta',\varsigma,\varsigma',\gamma} t^{\rho-1}\right](x) = \Gamma\left[\begin{array}{c} \rho, \rho+\gamma-\eta-\eta'-\varsigma, \rho-\eta'+\varsigma' \\ \rho+\varsigma', \rho+\gamma-\eta-\eta', \rho+\gamma-\eta'-\varsigma \end{array}\right] x^{\rho-\eta-\eta'+\gamma-1} \tag{5}$$

where $\mathrm{R}(\gamma) > 0$ and $\mathrm{R}(\rho) > \max\{0, \mathrm{R}(\eta+\eta'+\varsigma-\gamma), \mathrm{R}(\eta'-\varsigma')\}$. Also,

$$\left[I_{0-}^{\eta,\eta',\varsigma,\varsigma',\gamma} t^{\rho-1}\right](x) = \Gamma\left[\begin{array}{c} 1+\eta+\eta'-\gamma-\rho, 1+\eta+\varsigma'-\gamma-\rho, 1-\varsigma-\rho \\ 1-\rho, 1+\eta+\eta'+\varsigma'-\gamma-\rho, 1+\eta-\varsigma-\rho \end{array}\right] x^{\rho+\gamma-\eta-\eta'-1} \tag{6}$$

where $\mathrm{R}(\gamma) > 0$ and $\mathrm{R}(\rho) < 1 + \min\{\mathrm{R}(-\varsigma), \mathrm{R}(\eta+\varsigma'-\gamma), \mathrm{R}(\eta+\eta'-\gamma)\}$ and $\Gamma\left[\begin{array}{c} a, b, c \\ d, e, f \end{array}\right] = \frac{\Gamma a \Gamma b \Gamma c}{\Gamma d \Gamma e \Gamma f}$.

The following infinite series

$$S(l) = \sum_{n=1}^{\infty} \frac{2n}{(n^2+l^2)^2} \quad (l \in \mathbb{R}^+) \tag{7}$$

was investigated by Mathieu [17] in his book of elasticity of solid bodies. Closed form integral representation for $S(l)$ is given by

$$S(l) = \frac{1}{l} \int_0^\infty \frac{x \sin(lx)}{e^x - 1} dx \tag{8}$$

Several interesting problems and solutions dealing with integral representations and bound for the following fractional power generalization of the Mathieu series

$$S_{\sigma}(l) = \sum_{n\geq 1}^{\infty} \frac{2n}{(n^2+l^2)^{\sigma+1}} \quad \left(l, \sigma \in \mathbb{R}^{+}\right) \tag{9}$$

have been widely considered by many authors, see for reference [4, 5, 10, 20, 21, 23, 26, 31]. Various Applications of the Mathieu series and its generalizations can be found in classical, analytical number theory, special functions, mathematical harmonic and numerical analysis physics, probability, quantum field theory, quantum physics, etc. in the book by Tomovski [32]. For our present work, we consider the following family of generalized Mathieu series defined by Tomovski and Mehrez [30] as

$$S_{\sigma,\tau}^{(\theta,\phi)}(l,d;x) = S_{\sigma,\tau}^{(\theta,\phi)}\left(l, \{d_n\}_{n=1}^{\infty}; x\right) = \sum_{n=1}^{\infty} \frac{2d_n^{\phi}(\tau)_n}{\left(d_n^{\theta}+l^2\right)^{\sigma}} \frac{x^n}{n!} \quad \left(l, d, \theta, \phi, \sigma \in \mathbb{R}^{+}; |x| \leq 1\right) \tag{10}$$

In particular, the case $d_n = n$, $\theta = 2$, $\phi, \tau = 1$ and σ with $\sigma + 1$ equation (10) reduces to Mathieu series defined by Tomovski and Pogany [31] of the form

$$S_{\sigma,1}^{(2,1)}(l,n;x) = S_{\sigma}(l;x) = \sum_{n=1}^{\infty} \frac{2nx^n}{(n^2+l^2)^{\sigma+1}} \quad \left(l, \sigma \in \mathbb{R}^{+}; |x| \leq 1\right) \tag{11}$$

For our present investigation, we need to recall the generalized Lommel-Wright function defined by de' Oteiza et al. [22] represented as

$$\begin{aligned} J_{\omega,v}^{\mu,k}(z) &= \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(v+r+1)^k \Gamma(\omega+r\mu+v+1)} \left(\tfrac{z}{2}\right)^{\omega+2v+2r} \\ &= \left(\tfrac{z}{2}\right)^{\omega+2v} {}_1\Psi_{k+1}\left[(1,1);\underbrace{(v+1,1)}_{k-times},(\omega+v+1,\mu);\tfrac{-z^2}{4}\right] \\ & z \in \mathbb{C}\backslash(-\infty,0],\ \mu>0,\ k\in\mathbb{N},\ \omega,v\in\mathbb{C} \end{aligned} \tag{12}$$

where ${}_p\Psi_q(z)$ is the generalized Wright hypergeometric function also called Fox-Wright function defined by Wright (1935) and is given by the following series

$${}_p\Psi_q(z) = {}_p\Psi_q\left[\begin{array}{c} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{array}\middle| z\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)} \frac{z^n}{n!} \tag{13}$$

where, $z, a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}-\{0\}$ $(i = 1,2,...,p; j = 1,2,...q)$ and $\sum_{i=1}^{p}\alpha_i - \sum_{j=1}^{q}\beta_j \leq 1$. Also we have the following relations of generalized Lommel-Wright function with generalized Bessel function and Struve function:

$$J_{\omega,v}^{\mu,1}(z) = J_{\omega,v}^{\mu}(z) \quad \text{(see e.g., [12, p. 353])} \tag{14}$$

$$J^{1,1}_{\omega,1/2}(z) = H_\omega(z) \quad \text{(see e.g., [16, p. 28, Equation (1.170)])} \tag{15}$$

$$J^{1,1}_{\omega,0}(z) = J_\omega(z) \quad \text{(see e.g., [16, p. 27, Equation (1.161)])} \tag{16}$$

A great deal of research work has been carried out to investigate various generalizations and particular cases of Lommel-Wright function. For details one can refer the works of [3,13,27]. Also, Agarwal et al. [1] and Haq et al. [6] have developed integral formulas involving Lommel-Wright functions. In the present paper, we obtain some fractional integrals and derivatives for the product of generalized Mathieu series and generalized Lommel-Wright function.

2. Main Results

In this section, we establish image formulas for the product of generalized Lommel-Wright function $J^{\mu,k}_{\omega,v}(\cdot)$ and generalized Mathieu series $S^{(\theta,\phi)}_{\sigma,\tau}(l,d;x)$ involving Marichev-Saigo-Maeda fractional integral and differential operators.

2.1 Image formulas for fractional integral operators

Theorem 2.1. *Let* $\eta,\eta',\varsigma,\varsigma',\gamma,v \in \mathbb{C}$, $k \in \mathbb{N}$, $\mu > 0$ *and* $x > 0$ *be such that* $\mathrm{R}(\gamma) > 0$ *and* $\mathrm{R}(\omega) > -1$; $\mathrm{R}(\rho + \omega\xi + 2v\xi) > \max\{0, \mathrm{R}(\eta+\eta'+\varsigma-\gamma), \mathrm{R}(\eta'-\varsigma')\}$, *then the following Left sided generalized fractional integration formula holds true*

$$\left[I^{\eta,\eta',\varsigma,\varsigma',\gamma}_{0+} t^{\rho-1} S^{(\theta,\phi)}_{\sigma,\tau}\left(l,d;t^\lambda\right) J^{\mu,k}_{\omega,v}\left(\delta t^\xi\right)\right](x) = x^{A-\eta-\eta'+\gamma-1}\left(\frac{\delta}{2}\right)^{\omega+2v} S^{(\theta,\phi)}_{\sigma,\tau}\left(l,d;x^\lambda\right)$$
$$\times\, {}_4\Psi_{4+k}\left[\begin{array}{c|c} (1,1)\,(A+\lambda n,2\xi)\,(A+\gamma-\eta-\eta'-\varsigma+\lambda n,2\xi) & \\ (A-\eta'+\varsigma'+\lambda n,2\xi) & -\frac{\left(\delta x^\xi\right)^2}{4} \\ (A+\gamma-\eta-\eta'+\lambda n,2\xi)\,(A+\gamma-\eta'-\varsigma+\lambda n,2\xi) & \\ (A+\varsigma'+\lambda n,2\xi)\,(\omega+v+1,\mu)\,\underbrace{(v+1,1)}_{k-times} & \end{array}\right] \tag{17}$$

where, $A = \rho + \omega\xi + 2v\xi$

Proof. On using definitions (10) and (12), writing the functions in series form, and then interchanging the order of integration and summation which is possible under given conditions, the LHS of equation (17) becomes

$$\left[I^{\eta,\eta',\varsigma,\varsigma',\gamma}_{0+} t^{\rho-1} S^{(\theta,\phi)}_{\sigma,\tau}\left(l,d;t^\lambda\right) J^{\mu,k}_{\omega,v}\left(\delta t^\xi\right)\right](x) = \left[\sum_{n=1}^{\infty} \frac{2d_n^\phi(\tau)_n}{(d_n^\theta + l^2)^\sigma}\frac{1}{n!}\right.$$
$$\left.\times \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(v+r+1)^k\Gamma(\omega+r\mu+v+1)}\left(\frac{\delta}{2}\right)^{\omega+2v+2r} I^{\eta,\eta',\varsigma,\varsigma',\gamma}_{0+} t^{\rho+\omega\xi+2v\xi+2r\xi+\lambda n-1}\right](x)$$

Further using the image formula (5), we get

$$\left[I_{0+}^{\eta,\eta',\varsigma,\varsigma',\gamma} t^{\rho-1} S_{\sigma,\tau}^{(\theta,\phi)}\left(l,d;t^{\lambda}\right) J_{\omega,v}^{\mu,k}\left(\delta t^{\xi}\right)\right](x) = x^{A-\eta-\eta'+\gamma-1}\left(\frac{\delta}{2}\right)^{\omega+2v}\sum_{n=1}^{\infty}\frac{2d_n^{\phi}(\tau)_n}{(d_n^{\theta}+l^2)^{\sigma}}\frac{(x^{\lambda})^n}{n!}$$
$$\times\sum_{r=0}^{\infty}\frac{\Gamma(r+1)\Gamma(A+\lambda n+2r\xi)}{\Gamma(A+\gamma-\eta-\eta'+\lambda n+2r\xi)\Gamma(A+\gamma-\eta'-\varsigma+\lambda n+2r\xi)}$$
$$\times\frac{\Gamma(A+\gamma-\eta-\eta'-\varsigma+\lambda n+2r\xi)\Gamma(A-\eta'+\varsigma'+\lambda n+2r\xi)}{\Gamma(A+\varsigma'+\lambda n+2r\xi)\Gamma(\omega+v+1+\mu r)\Gamma(v+r+1)^k}\left(\frac{-\delta^2x^{2\xi}}{4}\right)^r\frac{1}{r!}$$

Rewriting the RHS of above equation, in view of the definition(13), we arrive at result (17). □

Corollary 2.2. *Let $\eta,\varsigma,\gamma,v\in\mathbb{C}$, $k\in\mathbb{N}$, $\mu>0$ and $x>0$ be such that $\mathrm{R}(\omega)>-1$ and $\mathrm{R}(\rho+\omega\xi+2v\xi)>\max\{0,\mathrm{R}(\varsigma-\gamma)\}$, then following formula holds true*

$$\left[I_{0+}^{\eta,\varsigma,\gamma} t^{\rho-1} S_{\sigma,\tau}^{(\theta,\phi)}\left(l,d;t^{\lambda}\right) J_{\omega,v}^{\mu,k}\left(\delta t^{\xi}\right)\right](x) = x^{A-\varsigma-1}\left(\tfrac{\delta}{2}\right)^{\omega+2v} S_{\sigma,\tau}^{(\theta,\phi)}\left(l,d;x^{\lambda}\right)$$
$$\times{}_3\Psi_{3+k}\left[\left.\begin{array}{c}(1,1)\,(A+\lambda n,2\xi)\,(A-\varsigma+\gamma+\lambda n,2\xi)\\(A-\varsigma+\lambda n,2\xi)\,(A+\eta+\gamma+\lambda n,2\xi)\,(\omega+v+1,\mu)\,\underbrace{(v+1,1)}_{k-times}\end{array}\right|\frac{-\left(\delta x^{\xi}\right)^2}{4}\right] \tag{18}$$

where, $A=\rho+\omega\xi+2v\xi$

Theorem 2.3. *Let $\eta,\eta',\varsigma,\varsigma',\gamma,v\in\mathbb{C}$, $k\in\mathbb{N}$, $\mu>0$ and $x>0$ be such that $\mathrm{R}(\gamma)>0$ and $\mathrm{R}(\omega)>-1$; and $\mathrm{R}(-\rho-\omega\xi-2v\xi)<1+\min\{\mathrm{R}(-\varsigma),\mathrm{R}(\eta+\varsigma'-\gamma),\mathrm{R}(\eta+\eta'-\gamma)\}$, then generalized fractional integration $I_{0-}^{\eta,\eta',\varsigma,\varsigma',\gamma}$ of product of $J_{\omega,v}^{\mu,k}(\cdot)$and $S_{\sigma,\tau}^{(\theta,\phi)}(l,d;x)$ is given by*

$$\left[I_{0-}^{\eta,\eta',\varsigma,\varsigma',\gamma} t^{-\rho-1} S_{\sigma,\tau}^{(\theta,\phi)}\left(l,d;t^{\lambda}\right) J_{\omega,v}^{\mu,k}\left(\delta t^{-\xi}\right)\right](x) = x^{-A'-\eta-\eta'+\gamma}\left(\frac{\delta}{2}\right)^{\omega+2v} S_{\sigma,\tau}^{(\theta,\phi)}\left(l,d;x^{\lambda}\right)$$
$$\times{}_4\Psi_{4+k}\left[\left.\begin{array}{c}(1,1)\,(A'+\eta+\eta'-\gamma-\lambda n,2\xi)\,(A'+\eta+\varsigma'-\gamma-\lambda n,2\xi)\\(A'-\varsigma-\lambda n,2\xi)\\(A'-\lambda n,2\xi)\,(A'+\eta+\eta'+\varsigma'-\gamma-\lambda n,2\xi)\,(A'+\eta-\varsigma-\lambda n,2\xi)\\(\omega+v+1,\mu)\,\underbrace{(v+1,1)}_{k-times}\end{array}\right|\frac{-\left(\delta x^{-\xi}\right)^2}{4}\right] \tag{19}$$

where, $A'=1+\rho+\omega\xi+2v\xi$

Proof. On using equation (10) and (12) in the LHS of equation (19) and then interchanging order of integration and summation, we obtain

$$\left[I_{0-}^{\eta,\eta',\varsigma,\varsigma',\gamma} t^{-\rho-1} S_{\sigma,\tau}^{(\theta,\phi)}\left(l,d;t^{\lambda}\right) J_{\omega,v}^{\mu,k}\left(\delta t^{-\xi}\right)\right](x) = \left[\sum_{n=1}^{\infty}\frac{2d_n^{\phi}(\tau)_n}{(d_n^{\theta}+l^2)^{\sigma}}\frac{1}{n!}\right.$$
$$\left.\times\sum_{r=0}^{\infty}\frac{(-1)^r}{\Gamma(v+r+1)^k\Gamma(\omega+r\mu+v+1)}\left(\frac{\delta}{2}\right)^{\omega+2v+2r} I_{0-}^{\eta,\eta',\varsigma,\varsigma',\gamma} t^{-\rho-1-\omega\xi-2v\xi-2r\xi+\lambda n}\right](x)$$

which on using image formula (6), becomes

$$\left[I_{0-}^{\eta,\eta',\varsigma,\varsigma',\gamma}t^{-\rho-1}S_{\sigma,\tau}^{(\theta,\phi)}\left(l,d;t^{\lambda}\right)J_{\omega,v}^{\mu,k}\left(\delta t^{-\xi}\right)\right](x)=x^{-A'-\eta-\eta'+\gamma}$$
$$\left(\frac{\delta}{2}\right)^{\omega+2v}\sum_{n=1}^{\infty}\frac{2d_n^{\phi}\left(\tau\right)_n}{\left(d_n^{\theta}+l^2\right)^{\sigma}}\frac{\left(x^{\lambda}\right)^n}{n!}\sum_{r=0}^{\infty}\frac{\Gamma\left(r+1\right)\Gamma\left(A'+\eta+\eta'-\gamma-\lambda n+2r\xi\right)}{\Gamma\left(A'+\eta+\eta'+\varsigma'-\gamma-\lambda n+2r\xi\right)\Gamma\left(A'+\eta-\varsigma-\lambda n+2r\xi\right)}$$
$$\times\frac{\Gamma\left(A'+\eta+\varsigma'-\gamma-\lambda n+2r\xi\right)\Gamma\left(A'-\varsigma-\lambda n+2r\xi\right)}{\Gamma\left(A'-\lambda n+2r\xi\right)\Gamma\left(\omega+v+1+\mu r\right)\Gamma\left(v+r+1\right)^k}\left(\frac{-\delta^2x^{-2\xi}}{4}\right)^r\frac{1}{r!}$$

Interpreting the RHS of above equation, in view of the definition (13), we arrive at result (19). □

Corollary 2.4. *Let $\eta,\varsigma,\gamma,v\in\mathbb{C},\quad k\in\mathbb{N},\quad\mu>0$ and $x>0$ be such that $\mathrm{R}\left(\gamma\right)>0$ and $\mathrm{R}\left(\omega\right)>-1$; and $\mathrm{R}\left(-\rho-\omega\xi-2v\xi\right)<1+\min\left\{\mathrm{R}\left(\gamma\right),\mathrm{R}\left(\varsigma\right)\right\}$, then following formula holds true*

$$\left[I_{0-}^{\eta,\varsigma,\gamma}t^{-\rho-1}S_{\sigma,\tau}^{(\theta,\phi)}\left(l,d;t^{\lambda}\right)J_{\omega,v}^{\mu,k}\left(\delta t^{-\xi}\right)\right](x)=x^{-A'-\varsigma}\left(\tfrac{\delta}{2}\right)^{\omega+2v}S_{\sigma,\tau}^{(\theta,\phi)}\left(l,d;x^{\lambda}\right)$$
$$\times{}_3\Psi_{3+k}\left[\begin{array}{c}(1,1)\left(A'+\varsigma-\lambda n,2\xi\right)\left(A'+\gamma-\lambda n,2\xi\right)\\\left(A'-\lambda n,2\xi\right)\left(A'+\eta+\varsigma+\gamma-\lambda n,2\xi\right)\left(\omega+v+1,\mu\right)\underbrace{\left(v+1,1\right)}_{k-times}\end{array}\middle|\frac{-\left(\delta x^{-\xi}\right)^2}{4}\right]\tag{20}$$

where, $A'=1+\rho+\omega\xi+2v\xi$

2.2 Image formulas for fractional differential operators

Now, we give image formulas for the product of generalized Lommel-Wright function and generalized Mathieu series involving left and right sided operators of Marichev-Saego-Maeda fractional differential operators by the following theorems:

Theorem 2.5. *The generalized fractional differentiation $D_{0+}^{\eta,\eta',\varsigma,\varsigma',\gamma}$ of the product of generalized Lommel-Wright function $J_{\omega,v}^{\mu,k}\left(\cdot\right)$ and generalized Mathieu series $S_{\sigma,\tau}^{(\theta,\phi)}\left(l,d;x\right)$ is given by*

$$\left[D_{0+}^{\eta,\eta',\varsigma,\varsigma',\gamma}t^{\rho-1}S_{\sigma,\tau}^{(\theta,\phi)}\left(l,d;t^{\lambda}\right)J_{\omega,v}^{\mu,k}\left(\delta t^{\xi}\right)\right](x)=x^{A+\eta+\eta'-\gamma-1}\left(\frac{\delta}{2}\right)^{\omega+2v}S_{\sigma,\tau}^{(\theta,\phi)}\left(l,d;x^{\lambda}\right)$$
$$\times{}_4\Psi_{4+k}\left[\begin{array}{c}(1,1)\left(A+\lambda n,2\xi\right)\left(A-\gamma+\eta+\eta'+\varsigma'+\lambda n,2\xi\right)\\\left(A+\eta-\varsigma+\lambda n,2\xi\right)\\\left(A-\gamma+\eta+\eta'+\lambda n,2\xi\right)\left(A-\gamma+\eta+\varsigma'+\lambda n,2\xi\right)\\\left(A-\varsigma+\lambda n,2\xi\right)\left(\omega+v+1,\mu\right)\underbrace{\left(v+1,1\right)}_{k-times}\end{array}\middle|\frac{-\left(\delta x^{\xi}\right)^2}{4}\right]\tag{21}$$

where, $\eta,\eta',\varsigma,\varsigma',\gamma,v\in\mathbb{C},k\in\mathbb{N},\mu>0$ and $x>0$ be such that $\mathrm{R}\left(\gamma\right)>0$ and $\mathrm{R}\left(\omega\right)>-1$; $\mathrm{R}\left(\rho+\omega\xi+2v\xi\right)>\max\left\{0,\mathrm{R}\left(\gamma-\eta-\eta'-\varsigma'\right),\mathrm{R}\left(\varsigma-\eta\right)\right\}$ and $A=\rho+\omega\xi+2v\xi$.

Proof. On using definitions (10) and (12), writing the functions in series form, and then interchanging the order of integration and summation which is possible under given conditions, the LHS of equation

(21) becomes

$$\left[D_{0+}^{\eta,\eta',\varsigma,\varsigma',\gamma} t^{\rho-1} S_{\sigma,\tau}^{(\theta,\phi)}\left(l,d;t^{\lambda}\right) J_{\omega,v}^{\mu,k}\left(\delta t^{\xi}\right)\right](x) = \left[\sum_{n=1}^{\infty} \frac{2d_n^{\phi}(\tau)_n}{(d_n^{\theta}+l^2)^{\sigma}} \frac{1}{n!}\right.$$
$$\left.\times \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(v+r+1)^k \Gamma(\omega+r\mu+v+1)} \left(\frac{\delta}{2}\right)^{\omega+2v+2r} I_{0+}^{-\eta',-\eta,-\varsigma',-\varsigma,-\gamma} t^{\rho+\omega\xi+2v\xi+2r\xi+\lambda n-1}\right](x)$$

Further using the image formula (5), we get

$$\left[D_{0+}^{\eta,\eta',\varsigma,\varsigma',\gamma} t^{\rho-1} S_{\sigma,\tau}^{(\theta,\phi)}\left(l,d;t^{\lambda}\right) J_{\omega,v}^{\mu,k}\left(\delta t^{\xi}\right)\right](x) = x^{A+\eta+\eta'-\gamma-1}$$
$$\left(\frac{\delta}{2}\right)^{\omega+2v} \sum_{n=1}^{\infty} \frac{2d_n^{\phi}(\tau)_n}{(d_n^{\theta}+l^2)^{\sigma}} \frac{(x^{\lambda})^n}{n!} \sum_{r=0}^{\infty} \frac{\Gamma(r+1)\Gamma(A+\lambda n+2r\xi)}{\Gamma(A-\gamma+\eta+\eta'+\lambda n+2r\xi)\Gamma(A-\gamma+\eta+\varsigma'+\lambda n+2r\xi)}$$
$$\times \frac{\Gamma(A-\gamma+\eta+\eta'+\varsigma'+\lambda n+2r\xi)\Gamma(A+\eta-\varsigma+\lambda n+2r\xi)}{\Gamma(A-\varsigma+\lambda n+2r\xi)\Gamma(\omega+v+1+\mu r)\Gamma(v+r+1)^k} \left(\frac{-\delta^2 x^{2\xi}}{4}\right)^r \frac{1}{r!}$$

Rewriting the RHS of above equation, in view of the definition(13), we arrive at result (21). □

Corollary 2.6. *Let $\eta,\varsigma,\gamma,v \in \mathbb{C}$, $k \in \mathbb{N}$, $\mu > 0$ and $x > 0$ be such that $\mathrm{R}(\gamma) > 0$ and $\mathrm{R}(\omega) > -1$;$\mathrm{R}(\rho+\omega\xi+2v\xi) > \max\{0, \mathrm{R}(-\varsigma), \mathrm{R}(-\eta-\varsigma-\gamma)\}$ then following formula holds true:*

$$\left[D_{0+}^{\eta,\varsigma,\gamma} t^{\rho-1} S_{\sigma,\tau}^{(\theta,\phi)}\left(l,d;t^{\lambda}\right) J_{\omega,v}^{\mu,k}\left(\delta t^{\xi}\right)\right](x) = x^{A+\varsigma-1}\left(\tfrac{\delta}{2}\right)^{\omega+2v} S_{\sigma,\tau}^{(\theta,\phi)}\left(l,d;x^{\lambda}\right)$$
$$\times {}_3\Psi_{3+k}\left[\begin{array}{c|c} (1,1)(A+\lambda n,2\xi)(A+\eta+\varsigma+\gamma+\lambda n,2\xi) & \\ (A+\varsigma+\lambda n,2\xi)(A+\gamma+\lambda n,2\xi)(\omega+v+1,\mu)\underbrace{(v+1,1)}_{k-times} & \frac{-\left(\delta x^{\xi}\right)^2}{4}\end{array}\right] \tag{22}$$

and $A = \rho+\omega\xi+2v\xi$.

Theorem 2.7. *The generalized right-sided fractional differentiation $D_{0-}^{\eta,\eta',\varsigma,\varsigma',\gamma}$ of the product of generalized Lommel-Wright function $J_{\omega,v}^{\mu,k}(\cdot)$ and generalized Mathieu series$S_{\sigma,\tau}^{(\theta,\phi)}(l,d;x)$ is given by*

$$\left[D_{0-}^{\eta,\eta',\varsigma,\varsigma',\gamma} t^{-\rho-1} S_{\sigma,\tau}^{(\theta,\phi)}\left(l,d;t^{\lambda}\right) J_{\omega,v}^{\mu,k}\left(\delta t^{-\xi}\right)\right](x) = x^{-A'+\eta+\eta'-\gamma}\left(\frac{\delta}{2}\right)^{\omega+2v} S_{\sigma,\tau}^{(\theta,\phi)}\left(l,d;x^{\lambda}\right)$$
$$\times {}_4\Psi_{4+k}\left[\begin{array}{c|c} (1,1)(A'-\eta-\eta'+\gamma-\lambda n,2\xi)(A'-\eta'-\varsigma+\gamma-\lambda n,2\xi) & \\ (A'+\varsigma'-\lambda n,2\xi) & \frac{-\left(\delta x^{-\xi}\right)^2}{4} \\ (A'-\lambda n,2\xi)(A'-\eta-\eta'-\varsigma+\gamma-\lambda n,2\xi) & \\ (A'-\eta'+\varsigma'-\lambda n,2\xi)(\omega+v+1,\mu)\underbrace{(v+1,1)}_{k-times} & \end{array}\right] \tag{23}$$

where $\eta,\eta',\varsigma,\varsigma',\gamma,v \in \mathbb{C}$, $k \in \mathbb{N}$, $\mu > 0$ and $x > 0$ be such that $\mathrm{R}(\gamma) > 0$ and $\mathrm{R}(\omega) > -1$; and $\mathrm{R}(-\rho-\omega\xi-2v\xi) < 1+\min\{\mathrm{R}(\varsigma'), \mathrm{R}(\gamma-\eta'-\varsigma), \mathrm{R}(\gamma-\eta-\eta')\}$, where, $A' = 1+\rho+\omega\xi+2v\xi$.

Proof. On using equation (10) and (12) in the LHS of equation (23) and then interchanging order of integration and summation, we obtain

$$\left[D_{0-}^{\eta,\eta',\varsigma,\varsigma',\gamma} t^{-\rho-1} S_{\sigma,\tau}^{(\theta,\phi)}\left(l,d;t^{\lambda}\right) J_{\omega,v}^{\mu,k}\left(\delta t^{-\xi}\right)\right](x) = \left[\sum_{n=1}^{\infty} \frac{2d_n^{\phi}(\tau)_n}{\left(d_n^{\theta}+l^2\right)^{\sigma}} \frac{1}{n!}\right.$$
$$\left.\times \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(v+r+1)^k \Gamma(\omega+r\mu+v+1)} \left(\frac{\delta}{2}\right)^{\omega+2v+2r} I_{0-}^{-\eta',-\eta,-\varsigma',-\varsigma,-\gamma} t^{-\rho-1-\omega\xi-2v\xi-2r\xi+\lambda n}\right](x)$$

which on using image formula (6), becomes

$$\left[D_{0-}^{\eta,\eta',\varsigma,\varsigma',\gamma} t^{-\rho-1} S_{\sigma,\tau}^{(\theta,\phi)}\left(l,d;t^{\lambda}\right) J_{\omega,v}^{\mu,k}\left(\delta t^{-\xi}\right)\right](x) = x^{-A'+\eta+\eta'-\gamma}$$
$$\left(\frac{\delta}{2}\right)^{\omega+2v} \sum_{n=1}^{\infty} \frac{2d_n^{\phi}(\tau)_n}{\left(d_n^{\theta}+l^2\right)^{\sigma}} \frac{\left(x^{\lambda}\right)^n}{n!} \sum_{r=0}^{\infty} \frac{\Gamma(r+1)\Gamma(A'-\eta-\eta'+\gamma-\lambda n+2r\xi)}{\Gamma(A'-\eta-\eta'-\varsigma+\gamma-\lambda n+2r\xi)\Gamma(A'-\eta'+\varsigma'-\lambda n+2r\xi)}$$
$$\times \frac{\Gamma(A'-\eta'-\varsigma+\gamma-\lambda n+2r\xi)\Gamma(A'+\varsigma'-\lambda n+2r\xi)}{\Gamma(A'-\lambda n+2r\xi)\Gamma(\omega+v+1+\mu r)\Gamma(v+r+1)^k} \left(\frac{-\delta^2 x^{-2\xi}}{4}\right)^r \frac{1}{r!}$$

Interpreting the RHS of above equation, in view of the definition (13), we arrive at result (23). □

Corollary 2.8. *Let $\eta,\varsigma,\gamma,v \in \mathbb{C}$, $k \in \mathbb{N}$, $\mu > 0$ and $x > 0$, then following generalized fractional differentiation formula holds true:*

$$\left[D_{0-}^{\eta,\varsigma,\gamma} t^{-\rho-1} S_{\sigma,\tau}^{(\theta,\phi)}\left(l,d;t^{\lambda}\right) J_{\omega,v}^{\mu,k}\left(\delta t^{-\xi}\right)\right](x) = x^{-A'+\varsigma-\gamma}\left(\tfrac{\delta}{2}\right)^{\omega+2v} S_{\sigma,\tau}^{(\theta,\phi)}\left(l,d;x^{\lambda}\right)$$
$$\times {}_3\Psi_{3+k}\left[\begin{array}{c|c} (1,1)\,(A'-\varsigma-\lambda n,2\xi)\,(A'+\eta+\gamma-\lambda n,2\xi) & \\ (A'-\lambda n,2\xi)\,(A'-\varsigma+\gamma-\lambda n,2\xi)\,(\omega+v+1,\mu)\,\underbrace{(v+1,1)}_{k-times} & \frac{-\left(\delta x^{-\xi}\right)^2}{4}\end{array}\right] \tag{24}$$

where, $\mathrm{R}(\gamma) > 0$ *and* $\mathrm{R}(\omega) > -1$; *and* $\mathrm{R}(-\rho-\omega\xi-2v\xi) < 1+\min\{0,\mathrm{R}(\eta+\gamma),\mathrm{R}(-\varsigma)\}$, *and* $A' = 1+\rho+\omega\xi+2v\xi$.

3. Further Remarks and Observations

We conclude our present study by remarking that it is not difficult to obtain several analogues and variations of the derived formulas exhibited here by (17), (18), (19), (20), (21), (22), (23) and (24), involving the generalized Lommel–Wright function $J_{\omega,v}^{\mu,k}(\cdot)$ itself and its other variants. For suitable choices of the parameters μ, ω and v, each of our integral and derivative formulas (17), (18), (19), (20), (21), (22), (23) and (24), (with $k=1$) give some known as well as new results for the generalized Bessel function $J_{\omega,v}^{\mu}(z)$, the Struve function $H_{\omega}(z)$ and the classical Bessel function $J_{\omega}(z)$, which are related to the generalized Lommel–Wright function $J_{\omega,v}^{\mu,k}(z)$ by means of (14), (15) and (16).

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