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# Stability of Functional Equations in Modular Spaces and Fuzzy Normed Spaces

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**Abstract**

In this paper, we prove the generalized Hyers-Ulam-Stability of quadratic functional equation in Modular Space and Fuzzy Normed Space by direct method.

**Keywords:** quadratic; functional equation; fuzzy normed space; modular space.
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## 1. Introduction

Functional equations play an important role in the study of stability problems in several structures. In 1940, Ulam [17] was the first who suggested the stability problem of functional equations concerning the stability of group homomorphisms and this established the foundation for work on stability problems. In that case if the solution exists, then that equation is stable. Ulam [17] work out such a problem. Using Banach spaces, Hyers [4] solved this stability problem by considering Cauchy's functional equation. Hyers work was expanded upon by Aoki [1] by assuming an unbounded Cauchy difference. In 1982, J. M. Rassias [20] followed the current approach of the Th. M. Rassias theorem in which he replaced the factor product of norms instead of sum of norms. Also, in 1994, Gavruta [3] by replacing a general function $\varepsilon\left(\|x\|^p+\|y\|^p\right)$ by $\varphi\left(x,y\right)$. Last 40 years, the fuzzy hypothesis has become an important tool. A lot of tools of progress has been made in the theory of fuzzy sets to find the analogues of the set theory. In this paper, we research stability of functional equation

$$\begin{aligned} f\left(-2x+y+z\right)+f\left(x-2y+z\right)+f\left(x+y-2z\right) &= 12\left(f\left(x\right)+f\left(y\right)+f\left(z\right)\right) \\ &\quad -3\left(f\left(x+y\right)+f\left(x+z\right)+f\left(y+z\right)\right) \qquad (1) \end{aligned}$$

in modular and fuzzy normed spaces.

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## 2. General Solution of Functional Equation (1)

**Theorem 2.1.** *Let $X$ and $Z$ be linear spaces. If mapping $f : X \to Z$ which satisfies the functional equation* (1), *for all $x, y, z \in X$, then, the mapping $f$ is quadratic.*

*Proof.* Putting $x = y = z = 0$ in equation (1), we obtain $f(0) = 0$. Putting $y = z = 0$ in equation (1), we get

$$f(-2x) = 4f(x), \tag{2}$$

for all $x \in X$. Setting $x = -x$ in (2), we obtain

$$f(2x) = 4f(-x), \tag{3}$$

for all $x \in X$. Replacing $y$ by $x$ and putting $z = 0$ in equation (1), we get

$$\begin{aligned} 2f(-x) + f(2x) &= 24f(x) - 3f(2x) - 6f(x), \\ 2f(-x) + 4f(2x) &= 18f(x), \end{aligned} \tag{4}$$

for all $x \in X$, from (3), we obtain

$$\begin{aligned} \frac{f(2x)}{2} + 4f(2x) &= 18f(x) \\ 9f(2x) &= 36f(x), \\ f(2x) &= 2^2 f(x), \end{aligned} \tag{5}$$

for all $x \in X$. From (3) and (5), we get

$$\begin{aligned} 4f(-x) &= 4f(x), \\ f(-x) &= f(x), \end{aligned}$$

for all $x \in X$. Hence, $f$ is an even mapping. Replacing $x$ by $2x$ in (5), we obtain

$$f\left(2^2 x\right) = 2^4 f(x), \tag{6}$$

for all $x \in X$. Again replacing $x$ by $2x$ in (6), we obtain

$$f\left(2^3 x\right) = 2^6 f(x), \tag{7}$$

for all $x \in X$. Thus, for any non-negative integer $n \geq 1$, we can generalize the result that

$$f(2^n x) = 2^{2n} f(x), \tag{8}$$

for all $x \in X$. Therefore, the mapping $f$ is quadratic. □

## 3. Stability Results of Functional Equation in Modular Spaces

**Definition 3.1** ( [7, 13]). *Let $Z$ be a linear space over $K$ ($\mathbb{R}$ or $\mathbb{C}$). A generalized $\sigma : Z \to [0, \infty)$ is called modular if for any given $x, y \in Z$, the following conditions hold:*

*(i)* $\sigma(x) = 0 \Leftrightarrow x = 0$,

*(ii)* $\sigma(\varepsilon x) = \sigma(x)$ *for any scalar* $\varepsilon$ *with* $|\varepsilon| = 1$,

*(iii)* $\sigma(\varepsilon_1 x + \varepsilon_2 y) \leq \sigma(x) + \sigma(y)$ *for any scalar* $\varepsilon_1, \varepsilon_2 \geq 0$ *with* $\varepsilon_1 + \varepsilon_2 = 1$.
*If the condition (iii) is replaced by*

*(iv)* $\sigma(\varepsilon_1 x + \varepsilon_2 y) \leq \varepsilon_1 \sigma(x) + \varepsilon_2 \sigma(y)$, *for any scalar* $\varepsilon_1, \varepsilon_2 \geq 0$ *with* $\varepsilon_1 + \varepsilon_2 = 1$,

*then $\sigma$ is called a convex modular. Furthermore, the vector space induced by a modular $\sigma$,*

$$Z_\sigma = \{x \in Z : \sigma(ax) \to 0 \ \text{ as } \ a \to 0\}$$

*is a modular space.*

**Definition 3.2** ( [7, 13]). *Let $Z_\sigma$ be a modular space and $\{x_n\}$ is a sequence in $Z_\sigma$. Then*

*(i) A Sequence $\{x_n\}$ is $\sigma$-convergent to a point $x \in Z_\sigma$, if $\sigma(x_n - x) \to 0$ as $n \to \infty$ and we write $x_n \to x$.*

*(ii) A Sequence $\{x_n\}$ is said to be $\sigma$-Cauchy if for any $\varepsilon > 0$, one has $\sigma(x_n - x_m) < \varepsilon$ for sufficiently large $n, m \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers.*

*(iii) $k \subseteq Z_\sigma$ is called a $\sigma$-complete if any $\sigma$-Cauchy sequence is $\sigma$-convergent in $K$.*

**Fatou's Property:** The modular $\sigma$ has the Fatou property if $\sigma(x) \leq \liminf_{n\to\infty} \sigma(x_n)$, whereas the sequence $\{x_n\}$ is $\sigma$-convergent to $x$ in modular space $Z_\sigma$ and conversely.

**Proposition 3.3** ( [5]). *In modular spaces,*

*(i) if $x_n \to x$ and $b$ is a constant vector, then $x_n + b \to x + b$.*

*(ii) if $x_n \to x$ and $y_n \to y$, then $\varepsilon_1 x_n + \varepsilon_2 y_n \to \varepsilon_1 x + \varepsilon_2 y$, where $\varepsilon_1 + \varepsilon_2 \leq 1$ and $\varepsilon_1, \varepsilon_2 \geq 0$.*

**Remark 3.4** ( [16]). *Assume that $\sigma$ satisfies a $\Delta_2$-condition with $\Delta_2$-constant $k > 0$ and is convex. If $k < 2$, then $\sigma(x) \leq k\sigma\left(\frac{x}{2}\right) \leq \frac{k}{2}\sigma(x)$, which indicates $\sigma = 0$. So, we should have the $\Delta_2$-constant $k \geq 2$ if $\sigma$ is convex modular.*

In this section, we prove the stability results of quadratic functional equation in modular spaces by using Direct method, which are improved forms of Wongkum ([18], [19]) and Sadeghi [13]. Consider that $X$ is a linear space and $Z$ is a complete convex modular space. In modular spaces without $\Delta_2$-conditions. We define a mapping $f : X \to Z$ by

$$\begin{aligned} D_{f(x,y,z)} &= f(-2x+y+z) + f(x-2y+z) + f(x+y-2z) - 12(f(x)+f(y)+f(z)) \\ &+ 3(f(x+y)+f(x+z)+f(y+z)), \end{aligned} \tag{*}$$

for all $x, y \in X$.

**Theorem 3.5.** *If there exists a mapping $\Psi : X^3 \to [0, \infty)$ such that*

$$\Psi(x,y,z) = \sum_{n=1}^{\infty} \frac{1}{2^{2n}} \Psi\left(2^{n-1}x, 2^{n-1}y, 2^{n-1}z\right) < \infty, \tag{9}$$

*and an even mapping $f : X \to Z_\sigma$ with $f(0) = 0$ and*

$$\sigma\left(D_{f(x,y,z)}\right) \leq \Psi(x,y,z), \tag{10}$$

*for all $x, y \in X$, then there exists a unique quadratic mapping $Q : X \to Z_\sigma$ satisfying*

$$\sigma(f(x) - Q(x)) \leq \Psi(x,x,0), \tag{11}$$

*for all $x \in X$.*

*Proof.* Replacing $y$ by $x$ and putting $z = 0$ in equation (*), and $\Psi(x,x,0) = \psi(x)$, we get from (10)

$$\sigma(2f(-x) + 4f(2x) - 18f(x)) \leq \Psi(x,x,0) = \psi(x), \tag{12}$$

for all $x \in X$, we get

$$\sigma\left(f(x) - \frac{1}{2^2}f(2x)\right) \leq \frac{1}{2^2}\psi(x), \tag{13}$$

for all $x \in X$. Then, by Principle of Mathematical Induction, we get

$$\sigma\left(f(x) - \frac{1}{2^{2n}}f(2^n x)\right) \leq \sum_{j=1}^{n} \frac{1}{2^{2j}}\psi\left(2^{j-1}x\right), \tag{14}$$

for all $x \in X$ and all natural numbers $n$. Result is true for $n = 1$ arises from (13). We assume that inequality (14) holds for $n \in \mathbb{N}$, then we obtain

$$\begin{aligned} \sigma\left(\frac{f(2^{n+1}x)}{2^{2(n+1)}} - f(x)\right) &= \sigma\left(\frac{1}{2^2}\left(f(2x) - \frac{f(2^n x)}{2^{2n}}\right) + \frac{1}{2^2}\left(2^2 f(x) - f(2x)\right)\right) \\ &\leq \frac{1}{2^2}\sigma\left(\frac{f(2^n x)}{2^{2n}} - f(2x)\right) + \frac{1}{2^2}\sigma\left(f(2x) - 2^2 f(x)\right) \\ &\leq \frac{1}{2^2}\sum_{j=1}^{n}\frac{1}{2^{2j}}\psi\left(2^j x\right) + \frac{1}{2^2}\psi(x) \end{aligned}$$

$$= \sum_{j=1}^{n} \frac{1}{2^{2(j+1)}} \psi\left(2^j x\right) + \frac{1}{2^2} \psi\left(x\right)$$
$$= \sum_{j=1}^{n+1} \frac{1}{2^{2j}} \psi\left(2^{j-1} x\right).$$

Therefore, inequality (14) holds for all $n \in \mathbb{N}$. Let $l$ and $m$ be natural numbers with $m > l$. By inequality (14), we obtain

$$\begin{aligned} \sigma\left(\frac{f\left(2^m x\right)}{2^{2m}} - \frac{f\left(2^l x\right)}{2^{2l}}\right) &= \sigma\left(\frac{1}{2^{2l}}\left(\left(\frac{f\left(2^{m-l} 2^l x\right)}{2^{2(m-l)}}\right) - f\left(2^l x\right)\right)\right) \\ &\leq \frac{1}{2^{2l}} \sum_{j=1}^{m-l} \frac{\psi\left(2^{j-1} 2^l x\right)}{2^{2j}} \\ &= \sum_{j=1}^{m-l} \frac{\psi\left(2^{l+j-1} x\right)}{2^{2(l+j)}} \\ &= \sum_{n=l+1}^{m} \frac{\psi\left(2^{n-1} x\right)}{2^{2n}}. \end{aligned} \tag{15}$$

From (9) and (15), we have the sequence $\left\{\frac{f(2^m x)}{2^{2m}}\right\}$ is a $\sigma$-Cauchy sequence in $Z_\sigma$. The $\sigma$-completeness of $Z_\sigma$ confers its $\sigma$-convergence. Now, we define a mapping $Q : X \to Z_\sigma$ by

$$Q\left(x\right) = \lim_{m \to \infty} \frac{f\left(2^m x\right)}{2^{2m}}, \tag{16}$$

for all $x \in X$. Hereby,

$$\begin{aligned} \sigma\left(\frac{2^2 Q\left(x\right) - Q(2x)}{2^6}\right) &= \sigma\left(\frac{1}{2^6}\left(\frac{f\left(2^{m+1} x\right)}{2^{2m}} - Q(2x)\right) + \frac{1}{2^2}\left(\frac{1}{2^2} Q\left(x\right) - \frac{1}{2^2}\frac{f\left(2^{m+1} x\right)}{2^{2(m+1)}}\right)\right) \\ &\leq \frac{1}{2^6}\sigma\left(Q\left(2x\right) - \frac{f\left(2^{m+1} x\right)}{2^{2m}}\right) + \frac{1}{2^4}\sigma\left(\frac{f\left(2^{m+1} x\right)}{2^{2(m+1)}} - Q(x)\right), \end{aligned} \tag{17}$$

for all $x \in X$. Then, by (16), the right-hand side of (17) tends to 0 as $m \to \infty$. Thus, we get that

$$Q\left(2x\right) = 2^2 Q(x), \tag{18}$$

for all $x \in X$. We observe that for all $m \in \mathbb{N}$, by (18), we have

$$\begin{aligned} \sigma\left(f\left(x\right) - Q(x)\right) &= \sigma\left(\sum_{n=1}^{m} \frac{2^2 f\left(2^{n-1} x\right) - f(2^n x)}{2^{2n}} + \left(\frac{f\left(2^m x\right)}{2^{2m}} - Q(x)\right)\right) \\ &= \sigma\left(\sum_{n=1}^{m} \frac{2^2 f\left(2^{n-1} x\right) - f(2^n x)}{2^{2n}} + \frac{1}{2^2}\left(\frac{f\left(2^{m-1} 2x\right)}{2^{2(m-1)}} - Q(2x)\right)\right), \end{aligned} \tag{19}$$

Because $\sum\limits_{n=1}^{m} \frac{1}{2^{2n}} + \frac{1}{2^2} \leq 1$, from inequality (12) and (19), we get

$$\begin{aligned}\sigma\left(f\left(x\right)-Q(x)\right) &\leq \sum_{n=1}^{m} \frac{1}{2^{2n}}\sigma\left(2f\left(2^{n-1}x\right)-f(2^{n}x)\right)+\frac{1}{2^2}\sigma\left(\frac{f\left(2^{m-1}2x\right)}{2^{2(m-1)}}-Q(2x)\right)\\ &\leq \sum_{n=1}^{m} \psi\left(2^{n-1}x\right)+\frac{1}{2^2}\sigma\left(\frac{f\left(2^{m-1}2x\right)}{2^{2(m-1)}}-Q(2x)\right)\\ &= \sum_{n=1}^{m} \frac{1}{2^{2n}}\Psi\left(2^{n-1}x,2^{n-1}x,0\right)+\frac{1}{2^2}\sigma\left(\frac{f\left(2^{m-1}2x\right)}{2^{2(m-1)}}-Q(2x)\right). \end{aligned} \tag{20}$$

Taking the limit $m \to \infty$ in (20), we get

$$\sigma\left(f\left(x\right)-Q(x)\right) \leq \Psi\left(x,x,0\right),$$

for all $x \in X$. Therefore, we get (11). Now, we want to show that the mapping $Q$ is quadratic. We observe that

$$\begin{aligned}\sigma\left(\frac{1}{2^{2j}}D_{f\left(2^jx,2^jy,2^jz\right)}\right) &\leq \frac{1}{2^{2j}}\sigma\left(D_{f\left(2^jx,2^jy,2^jz\right)}\right)\\ &\leq \frac{1}{2^{2j}}\Psi\left(2^jx,2^jy,2^jz\right) \to 0 \;\; as \;\; j\to\infty, \end{aligned} \tag{21}$$

for all $x,y \in X$. From inequality (21), we have $\sigma(D_{Q(x,y,z)}) \to 0$ as $j \to \infty$. Hence, we get

$$D_{Q(x,y,z)} = 0.$$

Therefore, the mapping $Q$ is quadratic. Next, we show that the mapping $Q$ is unique. We assume that there exists another mapping $R$ which satisfies (11). Then,

$$\begin{aligned}\sigma\left(\frac{Q\left(x\right)-R\left(x\right)}{2^2}\right) &= \sigma\left(\frac{1}{2^2}\left(\frac{Q\left(2^nx\right)}{2^{2n}}-\frac{f\left(2^nx\right)}{2^{2n}}\right)+\frac{1}{2^2}\left(\frac{f\left(2^nx\right)}{2^{2n}}-\frac{R\left(2^nx\right)}{2^{2n}}\right)\right)\\ &\leq \frac{1}{2^2}\sigma\left(\frac{Q\left(2^nx\right)}{2^{2n}}-\frac{f\left(2^nx\right)}{2^{2n}}\right)+\frac{1}{2^2}\sigma\left(\frac{f\left(2^nx\right)}{2^{2n}}-\frac{R\left(2^nx\right)}{2^{2n}}\right)\\ &\leq \frac{1}{2^2 2^{2n}}\left(\sigma\left(Q\left(2^nx\right)-f\left(2^nx\right)\right)+\sigma\left(f\left(2^nx\right)-R\left(2^nx\right)\right)\right)\\ &\leq \frac{1}{2^{2n}}\,\psi\left(2^nx,2^ny,0\right)\\ &\leq \sum_{k=n+1}^{\infty}\Psi\left(2^{k-1}x,2^{k-1}x,0\right) \to 0 \;\; as \;\; n\to\infty. \end{aligned}$$

This implies that $Q = R$. □

**Corollary 3.6.** *If there exists an even mapping $f : X \to Z_\sigma$ with $f\left(0\right) = 0$ and*

$$\sigma\left(D_{f(x,y,z)}\right) \leq \varepsilon, \tag{22}$$

*for all $x, y, z \in X$, then there exists a unique quadratic mapping $Q : X \to Z_\sigma$ satisfying*

$$\sigma\left(f\left(x\right) - Q\left(x\right)\right) \leq \frac{\varepsilon}{2},$$

*for all $x \in X$.*

**Corollary 3.7.** *If there exists an even mapping $f : X \to Z_\sigma$ with $f\left(0\right) = 0$ and*

$$\sigma\left(D_{f(x,y,z)}\right) \leq \theta\left(\|x\|^r + \|y\|^r + \|z\|^r\right),$$

*for all $x, y, z \in X$, and $\theta > 0$ and $0 < r < 1$, then there exists a unique quadratic mapping $Q : X \to Z_\sigma$ satisfying*

$$\sigma\left(f\left(x\right) - Q\left(x\right)\right) \leq \frac{2\theta}{2 - 2^r}\|x\|^r,$$

*for all $x \in X$.*

Next, theorem gives another stability of Theorem 3.5 in modular spaces with the $\Delta_2$-condition.

**Theorem 3.8.** *Suppose that $Z$ is a linear space and $Z_\sigma$ satisfies the $\Delta_2$-condition with the mapping $\Psi : X^3 \to [0, \infty)$ for which there exists a mapping $f : X \to Z_\sigma$ such that*

$$\sigma\left(D_{f(x,y,z)}\right) \leq \Psi\left(x, y, z\right),$$

*and $\lim_{n\to\infty} k^{2n}\Psi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0$ and $\sum_{j=1}^{\infty}\left(\frac{k^2}{2}\right)^j \Psi\left(\frac{x}{2^n}, \frac{x}{2^n}, 0\right) < \infty$, for all $x, y, z \in X$. Then, there exists a unique quadratic mapping $Q : X \to Z_\sigma$ defined by*

$$Q\left(x\right) = \lim_{n\to\infty} 2^{2n} f\left(\frac{x}{2^n}\right),$$

*and*

$$\sigma\left(f\left(x\right) - Q\left(x\right)\right) \leq \frac{\eta}{2k}\sum_{j=1}^{\infty}\left(\frac{k^2}{2}\right)^{2j}\Psi\left(\frac{x}{2^n}, \frac{x}{2^n}, 0\right),$$

*for all $x \in X$.*

*Proof.* $\sigma$ satisfies $\Delta_2$-condition with $\eta$, therefore, the inequality (10) implies that

$$\sigma\left(D_{f(x,y,z)}\right) \leq \eta\Psi\left(x, y, z\right),$$

for all $x, y, z \in X$. Then, we come the required result follows the proof of Theorem 3.5. □

## 4. Stability Results in Fuzzy Normed Spaces

We prove its Ulam's stability in Fuzzy normed spaces by using direct methods. We will use some definitions and notions to study the stability of functional equation (1), in fuzzy normed space.

**Definition 4.1** ( [9]). *Let $E$ be a real vector space.A function $\mathcal{F} : E \times \mathbb{R} \to [0,1]$ is called a fuzzy norm on $E$ if for every $p,q \in E$and $u,v \in \mathbb{R}$,*

$(F_1)$ $\mathcal{F}(p,v) = 0$ *for* $v \leq 0$;

$(F_2)$ $p = 0 \Longleftrightarrow \mathcal{F}(p,v) = 1$ *for all* $v > 0$;

$(F_3)$ $\mathcal{F}(cp,v) = \mathcal{F}\left(p, \frac{v}{|c|}\right)$ *if* $c \neq 0$;

$(F_4)$ $\mathcal{F}(p+q, u+v) \geq \min\{\mathcal{F}(p,\ u), \mathcal{F}(q,\ v)\}$;

$(F_5)$ $\mathcal{F}(p,.)$ *is a non-decreasing function of* $\mathbb{R}$ *and* $\lim_{v\to\infty} \mathcal{F}(p,v) = 1$;

$(F_6)$ *for* $p \neq 0$, $\mathcal{F}(p,.)$ *is continuous on* $\mathbb{R}$.

*The pair $(E,\mathcal{F})$ is called a fuzzy normed vector space.The following fixed-point theorem plays an important role in the study of stability of functional equations* (1).

In this section, we assume that $E$ is a linear space, $(Z,\mathcal{F})$ is a Fuzzy normed space and $(W,G)$ is a fuzzy Banach space.

**Theorem 4.2.** *Let a mapping $X : E^3 \to Z$ satisfy*

$$\mathcal{F}(X(2x,2x,2x),\delta) \geq \mathcal{F}(\rho X(x,x,x),\delta), \tag{23}$$

*for all $x \in E$ and all $\delta > 0$ and*

$$\lim_{l\to\infty} \mathcal{F}\left(X\left(2^l x, 2^l y, 2^l z\right), 2^{2l}\delta\right) = 1, \tag{24}$$

*for all $x,y,z \in E$ and all $\delta > 0$, where 0< $\rho < 4$. Suppose that an even mapping $f : E \to W$ with $f(0) = 0$ satisfies*

$$G(Df(x,y,z),\delta) \geq \mathcal{F}(X(x,y,z),\delta), \tag{25}$$

*for all $x,y,z \in E$ and all $\delta > 0$. Then, the limit*

$$Q_2(x) = G - \lim_{l\to\infty} \frac{f\left(2^l x\right)}{2^{2l}}, \tag{26}$$

*for all $x \in E$, then there exists a unique quadratic mapping $Q_2 : E \to W$ such that*

$$G(f(x) - Q_2(x),\delta) \geq \mathcal{F}\left(X(x,x,x), \delta\left(\frac{(2^2-\rho)9}{4}\right)\right), \tag{27}$$

*for all $x \in E$ and $\delta > 0$.*

*Proof.* Replacing $(x,y,z)$ by $(x,x,x)$ in (25), we have

$$G(9f(2x) - 36f(x),\delta) \geq \mathcal{F}(X(x,x,x),\delta), \tag{28}$$

for all $x \in E$ and $\delta > 0$. From (28), we obtain

$$G\left(\frac{f\left(2x\right)}{2^2} - f\left(x\right), \frac{\delta}{2^2.3^2}\right) \geq \mathcal{F}\left(X\left(x,x,x\right), \delta\right), \tag{29}$$

for all $x \in E$ and $\delta > 0$. Replacing $x$ with $2x$ in (29), we obtain

$$G\left(\frac{f\left(2^2x\right)}{2^4} - \frac{f\left(2x\right)}{2^2}, \frac{\delta}{2^4.3^2}\right) \geq \mathcal{F}\left(X\left(2x,2x,2x\right), \delta\right), \tag{30}$$

Replacing $x$ with $2^lx$ in (29), we obtain

$$G\left(\frac{f\left(2^{l+1}x\right)}{2^{2(l+1)}} - \frac{f\left(2^lx\right)}{2^{2l}}, \frac{\delta}{2^{2(l+1)}3^2}\right) \geq \mathcal{F}\left(X\left(2^lx,2^lx,2^lx\right), \delta\right), \tag{31}$$

using (1) and $\mathcal{F}_3$ in (31), we obtain

$$G\left(\frac{f\left(2^{l+1}x\right)}{2^{2(l+1)}} - \frac{f\left(2^lx\right)}{2^{2l}}, \frac{\delta}{2^{2(l+1)}3^2}\right) \geq \mathcal{F}\left(X\left(x,x,x\right), \frac{\delta}{\rho^l}\right), \tag{32}$$

for all $x \in E$ and $\delta > 0$. Replacing $\delta$ with $\rho^l\delta$ in (32), we obtain

$$G\left(\frac{f\left(2^{l+1}x\right)}{2^{2(l+1)}} - \frac{f\left(2^lx\right)}{2^{2l}}, \frac{\rho^l\delta}{2^{2(l+1)}3^2}\right) \geq \mathcal{F}\left(X\left(x,x,x\right), \delta\right), \tag{33}$$

for all $x \in E$ and $\delta > 0$. Note that

$$\frac{f(2^lx)}{2^{2l}} - f\left(x\right) = \sum_{i=0}^{l-1} \frac{f(2^{i+1}x)}{2^{2(i+1)}} - \frac{f(2^ix)}{2^{2i}}, \tag{34}$$

for all $x \in E$ and $\delta > 0$. From (33) and (34), we get

$$\begin{aligned} G\left(\frac{f\left(2^lx\right)}{2^{2l}} - f\left(x\right), \sum_{i=0}^{l-1} \frac{\rho^i\delta}{2^{2(i+1)3^2}}\right) &\geq \min_{0\leq i\leq l-1} \bigcup \left\{G\left(\frac{f\left(2^{i+1}x\right)}{2^{2(i+1)}} - \frac{f\left(2^ix\right)}{2^{2i}}, \frac{\rho^i\delta}{2^{2(i+1)3^2}}\right)\right\}, \\ &\geq \mathcal{F}(X\left(x,x,\ x\right), \delta), \end{aligned} \tag{35}$$

for all $x \in E$ and $\delta > 0$. Replacing $x$ with $2^mx$ in (35) and with the help of (23) and $\mathcal{F}_3$, we reach at

$$G\left(\frac{f\left(2^{l+m}x\right)}{2^{2l}} - f\left(2^mx\right), \sum_{i=0}^{l-1} \frac{\rho^i\delta}{2^{2(i+1)3^2}}\right) \geq \mathcal{F}\left(X\left(2^mx,2^mx,2^mx\right),\ \delta\right).$$

Hence, we have

$$\begin{aligned} G\left(\frac{f\left(2^{l+m}x\right)}{2^{2(l+m)}} - \frac{\phi\left(2^mx\right)}{2^{2m}}, \frac{1}{2^{2m}}\sum_{i=0}^{l-1} \frac{\rho^i\delta}{2^{2(i+1)3^2}}\right) &\geq \mathcal{F}(\rho^mX\left(x,x,\ x\right), \delta) \\ &\geq \mathcal{F}\left(X\left(x,x,x\right), \frac{\delta}{\rho^m}\right), \end{aligned} \tag{36}$$

replacing $\delta$ by $\rho^m\delta$ in (36), we obtain for all $l, m \geq 0$.

$$G\left(\frac{f\left(2^{l+m}x\right)}{2^{2(l+m)}} - \frac{f(2^m x)}{2^{2m}}, \sum_{i=m}^{l+m-1} \frac{\rho^i \delta}{2^{2(i+1)}3^2}\right) \geq \mathcal{F}(X\left(x, x,\ x\right), \delta),$$

for all $x \in E$ and $\delta > 0$. Replacing $\delta$ with $\frac{\delta}{\sum_{i=m}^{l+m-1} \frac{\rho^i}{2^{2(i+l)}3^2}}$ in above inequality, we get

$$G\left(\frac{f\left(2^{l+m}x\right)}{2^{2(l+m)}} - \frac{f(2^m x)}{2^{2m}}, \delta\right) \geq \mathcal{F}\left(X\left(x, x, x\right), \frac{\delta}{\sum_{i=n}^{l+n-1} \frac{\rho^i}{2^{2(i+l)}3^2}}\right), \tag{37}$$

for all $l, m \geq 0$. As $0 < \rho < 4$ and $\sum_{i=0}^{\infty} \left(\frac{\rho}{2^2}\right)^i \frac{1}{9}$ is convergent.

$(F_5)$ implies that the right-hand side of (37) goes to $l$ as $m \to \infty$. Hence, $\left\{\frac{f(2^m x)}{2^{2m}}\right\}$ is a Cauchy sequence in $(W,\ G)$. As $(W, G)$ is a Fuzzy Banach Space, this sequence converges to same point $Q_2(s) \in W$. Now, we can define a mapping $Q_2 : E \to W$ by

$$Q_2(x) = G - \lim_{l \to \infty} \frac{f\left(2^l x\right)}{2^{2l}}, x \in E.$$

Putting $m = 0$ and taking the limit $l$ tends to $\infty$ in (37), with the help of $\mathcal{F}_6$, we get

$$G\left(f\left(x\right) - Q_2\left(x\right), \delta\right) \geq \mathcal{F}\left(X\left(x, x,\ x\right), \delta\left(\frac{(2^2 - \rho)9}{4}\right)\right),$$

for all $x \in E$ and $\delta > 0$. Now, we show that $Q_2$ is quadratic. Here $f$ and $Q_2$ are even mappings. Replacing $(x, y, z)$ with $\left(2^l x, 2^l y, 2^l z\right)$ in (25), we have

$$\begin{aligned} G\left(\frac{1}{2^{2l}} D_f\left(2^l x, 2^l y, 2^l z\right), \delta\right) &\geq \mathcal{F}\left(\frac{1}{2^{2l}} X\left(2^l x, 2^l y, 2^l z\right), \delta\right) \\ &\geq \mathcal{F}\left(X\left(2^l x, 2^l y, 2^l z\right), 2^{2l}\delta\right) \end{aligned}$$

for all $(x, y, z) \in E$ and all $\delta > 0$. Note $\lim_{n \longrightarrow \infty} \mathcal{F}\left(X\left(2^l x, 2^l y, 2^l z\right), 2^{2l}\delta\right) = 1$. Hence, $Q_2$ satisfies the functional equation (1). Therefore, $Q_2 : E \to W$ is quadratic.

Uniqueness: Now, we show that $Q_2$ is unique. Let $R_2 : E \to W$ be another quadratic mapping satisfying (27). Then

$$\begin{aligned} G(Q_2\left(x\right) - R_2\left(x\right), \delta) &= G\left(\frac{Q_2\left(2^l x\right)}{2^{2l}} - \frac{R_2(2^l x)}{2^{2l}}, \delta\right) \\ &= G\left(\frac{Q_2\left(2^l x\right)}{2^{2l}} - \frac{f(2^l x)}{2^{2l}} + \frac{f(2^l x)}{2^{2l}} - \frac{R_2(2^l x)}{2^{2l}}, \delta\right) \\ &\geq \left\{G\left(\frac{Q_2\left(2^l x\right)}{2^{2l}} - \frac{f\left(2^l x\right)}{2^{2l}}, \frac{\delta}{2}\right), G\left(\frac{f(2^l x)}{2^{2l}} - \frac{R_2\left(2^l x\right)}{2^{2l}}, \frac{\delta}{2}\right)\right\} \end{aligned}$$

$$\geq \mathcal{F}\left(X\left(2^l x, 2^l x, 2^l x\right), \frac{\delta 2^{2l}(2^2-\rho)9}{4}\right)$$
$$\geq \mathcal{F}\left(\rho^l X\left(x,x,x\right), \frac{\delta 2^{2l}(2^2-\rho)9}{4}\right)$$
$$\geq \mathcal{F}\left(X\left(x,x,x\right), \frac{\delta 2^{2l}(2^2-\rho)9}{4\rho^l}\right),$$

$x \in E$, $\delta > 0$. Since $\lim_{l\to\infty} \frac{\delta 2^{2l}(2^2-\rho)9}{4\rho^l} = \infty$, we obtain

$$\lim_{l\to\infty} \mathcal{F}\left(X\left(x,x,x\right), \frac{\delta 2^{2l}(2^2-\rho)9}{4\rho^l}\right) = 1.$$

Thus, $G(Q_2\left(x\right) - R_2\left(x\right), \delta) = 1$. Hence, $Q_2\left(x\right) = R_2\left(x\right)$. □

We have the following result similar to Theorem 4.3, which corresponds to the case $\rho > \frac{1}{4}$.

**Theorem 4.3.** *Let a mapping $X : E^3 \to Z$ satisfy*

$$\mathcal{F}\left(X\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right), \delta\right) \geq \mathcal{F}\left(\frac{1}{\rho} X\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right), \delta\right), \tag{38}$$

*for all $x \in E$ and $\delta > 0$, and*

$$\lim_{l\to\infty} \mathcal{F}\left(X\left(\frac{x}{2^l}, \frac{y}{2^l}, \frac{z}{2^l}\right), \frac{\delta}{2^{2l}}\right) = 1, \tag{39}$$

*for all $x, y, z \in E$ and all $\delta > 0$, where $\rho > \frac{1}{4}$. We assume that an even mapping $f : E \to W$ with $f\left(0\right) = 0$ satisfies*

$$G(Df(x,y,z), \delta) \geq \mathcal{F}(X\left(x,y,z\right), \delta), \tag{40}$$

*for all $x, y, z \in E$ and all $\delta > 0$. Then, the limit*

$$Q_2(x) = G - \lim_{l\to\infty} 2^{2l} f\left(\frac{x}{2^l}\right), \tag{41}$$

*for all $x \in E$ and defines a unique quadratic mapping $Q_2 : E \to W$ such that*

$$G\left(f\left(x\right) - Q_2\left(x\right), \delta\right) \geq \mathcal{F}\left(X\left(x,x,0\right), \delta\left(\frac{9(4\rho-1)}{4\rho}\right)\right), \tag{42}$$

*for all $x \in E$ and $\delta > 0$.*

*Proof.* Replacing $(x,y,z)$ by $\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right)$ in (40), we have

$$G\left(9f\left(x\right) - 36f\left(\frac{x}{2}\right), \delta\right) \geq \mathcal{F}\left(X\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right), \delta\right), \tag{43}$$

for all $x \in E$ and $\delta > 0$.From (43), we obtain

$$G(f(x) - 2^2 f\left(\frac{x}{2}\right), \frac{\delta}{3^2}) \geq \mathcal{F}\left(X\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right), \delta\right), \tag{44}$$

for all $x \in E$ and $\delta > 0$. Replacing $x$ with $2x$ in (44), we obtain

$$G\left(2^2 f\left(\frac{x}{2}\right) - 2^4 f\left(\frac{x}{2^2}\right), \frac{\delta}{2^2 3^2}\right) \geq \mathcal{F}\left(X\left(\frac{x}{2^2}, \frac{x}{2^2}, \frac{x}{2^2}\right), \delta\right), \tag{45}$$

Replacing $x$ with $2^l x$ in (44), we obtain

$$G\left(2^{2l} f\left(\frac{x}{2^l}\right) - 2^{2(l+1)} f\left(\frac{x}{2^{l+1}}\right), \frac{\delta}{2^{2l} 3^2}\right) \geq \mathcal{F}\left(X\left(\frac{x}{2^l}, \frac{x}{2^l}, \frac{x}{2^l}\right), \delta\right), \tag{46}$$

using (1) and $\mathcal{F}_3$ in (46), we obtain

$$G\left(2^{2l} f\left(\frac{x}{2^l}\right) - 2^{2(l+1)} f\left(\frac{x}{2^{l+1}}\right), \frac{\delta}{2^{2l} 3^2}\right) \geq \mathcal{F}\left(X(x, x, x), \rho^l \delta\right), \tag{47}$$

for all $x \in E$ and $\delta > 0$. Replacing $\delta$ with $\frac{\delta}{\rho^l}$ in (47), we obtain

$$G\left(2^{2l} f\left(\frac{x}{2^l}\right) - 2^{2(l+1)} f\left(\frac{x}{2^{l+1}}\right), \frac{\delta}{2^{2l} 3^2 \rho^l}\right) \geq \mathcal{F}(X(x, x, x), \delta), \tag{48}$$

for all $x \in E$ and $\delta > 0$. Note that

$$f(x) - 2^{2l} f\left(\frac{x}{2^l}\right) = \sum_{i=0}^{l-1} 2^{2i} f\left(\frac{x}{2^i}\right) - 2^{2(i+1)} f\left(\frac{x}{2^{i+1}}\right), \tag{49}$$

for all $x \in E$ and $\delta > 0$. From (48) and (49), we get

$$\begin{aligned} G\left(f(x) - 2^{2l} f\left(\frac{x}{2^l}\right), \sum_{i=0}^{l-1} \frac{\delta}{2^{2i} 3^2 \rho^i}\right) &\geq \min_{0 \leq i \leq l-1} \bigcup \left\{G\left(2^{2i} f\left(\frac{x}{2^i}\right) - 2^{2(i+1)} f\left(\frac{x}{2^{i+1}}\right), \frac{\delta}{2^{2l} 3^2 \rho^i}\right)\right\}, \\ &\geq \mathcal{F}(X(x, x, x), \delta), \end{aligned} \tag{50}$$

for all $x \in E$ and $\delta > 0$. Replacing $x$ with $\frac{x}{2^m}$ in (50) and with the help of (38) and using $\mathcal{F}_3$ condition of chapter first, we reach at

$$G\left(2^{2m} f\left(\frac{x}{2^m}\right) - 2^{2(l+m)} f\left(\frac{x}{2^{l+m}}\right), \sum_{i=0}^{l-1} \frac{\delta}{2^{2i} 3^2 \rho^i}\right) \geq \mathcal{F}\left(X\left(\frac{x}{2^m}, \frac{x}{2^m}, \frac{x}{2^m}\right), \delta\right).$$

Hence, we have

$$\begin{aligned} G\left(2^{2m} f\left(\frac{x}{2^m}\right) - 2^{2(l+m)} f\left(\frac{x}{2^{l+m}}\right), \sum_{i=0}^{l-1} \frac{\delta}{2^{2i} 3^2 \rho^i}\right) &\geq \mathcal{F}(\rho^m X(x, x, x), \delta) \\ &\geq \mathcal{F}\left(X(x, x, x), \frac{\delta}{\rho^m}\right), \end{aligned} \tag{51}$$

for all $x \in E$ and $\delta > 0$. Replacing $\delta$ by $\rho^m \delta$ in (51), we obtain for all $l,\ m \geq 0$.

$$G\left(2^{2m} f\left(\frac{x}{2^m}\right) - 2^{2(l+m)} f\left(\frac{x}{2^{l+m}}\right), \sum_{i=m}^{l+m-1} \frac{\delta}{2^{2i}3^2\rho^i}\right) \geq \mathcal{F}(X\left(x,x,x\right),\delta),$$

for all $x \in E$ and $\delta > 0$. Replacing $\delta$ with $\frac{\delta}{\sum_{i=m}^{l+m-1} \frac{1}{2^{2i}3^2\rho^i}}$ in above inequality, we get

$$G\left(2^{2m} f\left(\frac{x}{2^m}\right) - 2^{2(l+m)} f\left(\frac{x}{2^{l+m}}\right),\delta\right) \geq \mathcal{F}\left(X\left(x,x,\ x\right), \frac{\delta}{\sum\limits_{i=m}^{l+m-1} \frac{1}{2^{2i}3^2\rho^i}}\right), \tag{52}$$

for all $l, m \geq 0$. As $\rho > \frac{1}{4}$ and $\sum_{i=0}^{\infty} \frac{1}{2^{2i}3^2\rho^i}$ is convergent.
($F_5$) implies that the right-hand side of (52) goes to $l$ as $m \to \infty$. Hence, $\left\{2^{2m} f\left(\frac{x}{2^m}\right)\right\}$ is a Cauchy sequence in $(W,G)$. As $(W,\ G)$ is a Fuzzy Banach space, this sequence converges to same point $Q_2(s) \in W$. Now, we can define a mapping $Q_2 : E \to W$ by

$$Q_2(x) = G - \lim_{l\to\infty} 2^{2m} f\left(\frac{x}{2^m}\right), x \in E.$$

Putting $m = 0$ and taking the limit $l$ tends to $\infty$ in (52), with the help of $\mathcal{F}_6$ condition, we get

$$G\left(f\left(x\right) - Q_2\left(x\right),\delta\right) \geq \mathcal{F}\left(X\left(x,x,x\right),\delta\left(\frac{9\left(4\rho-1\right)}{4\rho}\right)\right),$$

for all $x \in E$ and $\delta > 0$. Now, we show that $Q_2$ is quadratic. Here $f$ and $Q_2$ are even mappings. Replacing $(x,y,z)$ with $\left(\frac{x}{2^l}, \frac{x}{2^l}, \frac{x}{2^l}\right)$ in (40), we have

$$\begin{aligned} G\left(2^{2l} D_f\left(\frac{x}{2^l}, \frac{x}{2^l}, \frac{x}{2^l}\right),\delta\right) &\geq \mathcal{F}\left(2^{2l} X\left(\frac{x}{2^l}, \frac{x}{2^l}, \frac{x}{2^l}\right),\delta\right) \\ &\geq \mathcal{F}\left(X\left(\frac{x}{2^l}, \frac{x}{2^l}, \frac{x}{2^l}\right), \frac{1}{2^{2l}}\delta\right), \end{aligned}$$

for all $(x,y,z) \in E$ and all $\delta > 0$. Note $\lim\limits_{n\longrightarrow\infty} \mathcal{F}\left(X\left(\frac{x}{2^l}, \frac{x}{2^l}, \frac{x}{2^l}\right), \frac{1}{2^{2l}}\delta\right) = 1$. Hence, $Q_2$ satisfies the functional equation (1). Therefore, $Q_2 : E \to W$ is quadratic.

Uniqueness: Now, we show that $Q_2$ is unique. Let $R_2 : E \to W$ be another quadratic mapping satisfying (42). Then

$$\begin{aligned} G(Q_2\left(x\right) - R_2\left(x\right),\delta) &= G\left(2^{2l} Q_2\left(\frac{x}{2^l}\right) - 2^{2l} R_2\left(\frac{x}{2^l}\right),\delta\right) \\ &= G\left(2^{2l} Q_2\left(\frac{x}{2^l}\right) - 2^{2l} f\left(\frac{x}{2^l}\right) + 2^{2l} f\left(\frac{x}{2^l}\right) - 2^{2l} R_2\left(\frac{x}{2^l}\right),\delta\right) \\ &\geq \left\{G\left(2^{2l} Q_2\left(\frac{x}{2^l}\right) - 2^{2l} f\left(\frac{x}{2^l}\right), \frac{\delta}{2}\right), G\left(2^{2l} f\left(\frac{x}{2^l}\right) - 2^{2l} R_2\left(\frac{x}{2^l}\right), \frac{\delta}{2}\right)\right\} \\ &\geq \mathcal{F}\left(X\left(\frac{x}{2^l}, \frac{x}{2^l}, \frac{x}{2^l}\right),\delta\left(\frac{9\left(4\rho-1\right)}{4\rho}\right)\right) \end{aligned}$$

$$\geq \mathcal{F}\left(\frac{1}{\rho^l} X\left(x,x,x\right), \delta\left(\frac{9\left(4\rho-1\right)}{4\rho 2^{2l}}\right)\right)$$
$$\geq \mathcal{F}\left(X\left(x,x,x\right),\ \delta\left(\frac{9\left(4\rho-1\right)}{4\rho 2^{2l}}\right)\rho^l\right),$$

$x \in E$, $\delta > 0$. Since $\lim_{l\to\infty} \delta\left(\frac{9(4\rho-1)}{4\rho 2^{2l}}\right)\rho^l = \infty$, we obtain

$$\lim_{l\to\infty} \mathcal{F}\left(X\left(x,x,x\right), \delta\left(\frac{9\left(4\rho-1\right)}{4\rho 2^{2l}}\right)\rho^l\right) = 1.$$

Thus, $G(Q_2\left(x\right) - R_2\left(x\right), \delta) = 1$. Hence, $Q_2\left(x\right) = R_2\left(x\right)$. □

**Corollary 4.4.** *Let $\zeta > 0$ be a real constant. If an even mapping $\phi : E \to W$ with $\phi\left(0\right) = 0$ satisfies $G\left(Df\left(x,y,z\right), \delta\right) \geq \mathcal{F}\left(\zeta, \delta\right)$, for all $x, y, z \in E$ and $\delta > 0$, then there exists a unique quadratic mapping $Q_2 : E \to W$ such that $G\left(f\left(x\right) - Q_2\left(x\right), \delta\right) \geq \mathcal{F}\left(\zeta, \delta\left(\frac{9(2^2-1)}{4}\right)\right)$, for all $x \in E$ and $\delta > 0$.*

*Proof.* Let us define $X\left(x,y,z\right) = \zeta$ and $\rho = 2^0$. Then, by Theorem 4.3, we have $G\left(f\left(x\right) - Q_2\left(x\right), \delta\right) \geq \mathcal{F}\left(\zeta, \delta\left(\frac{9(2^2-1)}{4}\right)\right)$, for all $x \in E$ and $\delta > 0$. □

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