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Operations on Multisets and Fuzzy Multisets

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Abstract

This paper presents a comprehensive redefinition of Yagar's bag theory [5] by introducing novel notations and adopting the terminology of multisets. The primary objective of this study is to provide a solid foundation for understanding multisets, catering to those interested in delving into the core principles of this concept. We compare two variants of multisets: fuzzy multisets and crisp multisets, and explore their distinct characteristics. Additionally, we investigate various operations on multisets, including those involving fuzzy multisets. Furthermore, we discuss the significance of multiset relations in the context of data set theory.

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1. Introduction

Multisets, also known as bags, are mathematical structures that extend the concept of sets by allowing duplicate elements. They find wide-ranging applications in various domains, including computer science, mathematics, data analysis and artificial intelligence. Yagar's bag theory has been a fundamental reference in understanding and utilizing multisets (see [5]). However, in this paper, we aim to provide a fresh perspective on multisets by introducing new notations and adopting the terminology of "multisets" instead of "bags."

The objective of this study is to offer a comprehensive understanding of multisets from their core principles, making it accessible to those interested in learning about this topic. We focus on comparing two variants of multisets: fuzzy multisets and crisp multisets. Fuzzy multisets incorporate the notion of fuzzy logic, allowing elements to have degrees of membership rather than strict membership values (see [2]). Crisp multisets, on the other hand, adhere to the classical notion of set theory, where elements are either present or absent.

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To establish a solid foundation for understanding multisets, we explore various operations on multisets, including union, intersection, difference, and Cartesian product (see [7]). These operations provide insight into how multisets can be manipulated and analyzed in different contexts. Additionally, we investigate the operations and properties specific to fuzzy multisets, highlighting the unique characteristics they offer (see [1]).

Furthermore, we delve into the applications of multiset relations in the field of data set theory. Multiset relations play a crucial role in analyzing and comparing datasets, allowing for a more nuanced understanding of the relationships between elements. By leveraging multiset relations, researchers and practitioners can gain valuable insights into data patterns, similarities, and differences, leading to more informed decision-making and data-driven solutions [3,4].

In conclusion, this paper provides a comprehensive exploration of multisets, shedding light on their characteristics, operations, and applications. By redefining Yagar's bag theory with new notations and adopting the terminology of multisets, we aim to enhance the understanding and utilization of this mathematical concept. The insights gained from this study pave the way for future research, particularly in the development of graphical representations for multisets and fuzzy multisets, which can provide intuitive and visual interpretations of complex multiset structures.

2. Crisp Multiset

This section focuses on the definition of crisp multisets and their associated operations. We will explore the fundamental characteristics of crisp multisets, highlighting their distinction from traditional sets. Additionally, we will delve into the operations performed on crisp multisets, including union, intersection, and difference. These operations provide essential tools for manipulating and analyzing the contents of crisp multisets, facilitating various computations and comparisons.

Definition 2.1 (Multiset). *Let X is any non-empty set then multiset A drawn from set X can be represented by a function*

$$Count_A \colon X \to \mathbb{N}_0,$$

where $\mathbb{N}_0$ is the set of non-negative integers. In the above for any $x \in X$, $Count_A(x)$ indicates the number of times the element x appears in multiset A.

$$A = \left\{ \frac{Count_A(x)}{x} : x \in X \right\} \tag{1}$$

Example 2.2. *If $X = \{d, g, i, m, n, o, r\}$ then multiset A is*

$$A = \{g, o, o, d, m, o, r, n, i, n, g\}$$

$$A = \left\{\frac{1}{d}, \frac{2}{g}, \frac{1}{i}, \frac{1}{m}, \frac{2}{n}, \frac{3}{o}, \frac{1}{r}\right\}$$

Every set is a multiset, i.e. X be any set then $Count_X \colon X \to \{0,1\}$ is defined as

$$Count_X(x) = \begin{cases} 1, & \text{if } x \in X \\ 0, & \text{if } x \notin X \end{cases}$$

Definition 2.3 (Support). *Let A is a multiset drawn from the set X. The subset B of X with membership function U_B is called the supporting the multiset A, U_B can be specified as:*

$$U_B(x) = \min\{Count_A(x), 1\}$$

i.e. $x \in B$ if $Count_A(x) > 0$ and $x \notin B$ if $Count_A(x) = 0$.

It should be noted that many different multiset may have the same support set.

Example 2.4. *For X in ex. 2.2, $A_1 = \{g, o, o, d\}$ and $A_2 = \{d, o, g\}$ are multisets drawn from X and their support set B is*

$$B = \{d, g, o\} \subset X$$

Definition 2.5 (Submultiset). *Let A and B are two multisets drawn from the set X, we say A is a submultiset of B, denoted as $A \subset B$ if*

$$Count_A(x) \leq Count_B(x) \ \forall \ x \in X$$

It is called proper submultiset if

$$Count_A(x) < Count_B(x) \ for \ some \ x \in X$$

It obviously follows that for any two multisets $A = B \iff A \subset B$ *and* $B \subset A$.

Definition 2.6 (Empty Multiset). *A multiset is called an empty multiset if*

$$Count(x) = 0 \ \forall \ x \in X$$

Support set of empty multiset is null set ($\emptyset$).

Definition 2.7 (Cardinality). *Let A is a multiset drawn from X. The cardinality of A, denoted $card(A)$ is defined as*

$$card(A) = \sum_{x \in X} Count_A(x).$$

Let A is a set and P_A be the set of all multisets which have A as its support set. For any $B \in P_A$

$$card(B) \geq card(A).$$

Definition 2.8 (Peak Value and Peak Element). *Let A is a multiset then $\max_{x \in X} Count_A(x)$ is known as peak value of multiset. Any $x^* \in X$ such that $Count_A(x^*) = \max_{x \in X} Count_A(x)$, is known as peak element of the multiset.*

2.1 Operations on Multisets

Definition 2.9 (Insertion of a Element). *Let $x \in X$ and A is a multiset drawn from set X. The insertion of x into A gives a new multiset B, denoted as $B = x \oplus A$ and for $y \in X$ membership function is defined as*

$$Count_B(y) = \begin{cases} Count_A(y) + 1, & \text{if } y = x \\ Count_A(y), & \text{if } y \neq x \end{cases}$$

Definition 2.10 (Addition of Multiset). *Let A and B are two multisets drawn from set X. The addition of multisets A and B gives a new multiset C, denoted as $C = A \oplus B$ such that for any $x \in X$*

$$Count_C(x) = Count_A(x) + Count_B(x).$$

The fundamental step in creating multisets is the action of insertion or addition.The following characteristics of the multiset addition operation are easily demonstrable.

1. $A \oplus B = B \oplus A$; (commutative)
2. $A \oplus (B \oplus C) = (A \oplus B) \oplus C$; (associative)
3. $A \oplus \emptyset = A$.

We point out in particular that multiset addition is not idempotent and there is no inverse.
It appears that the multiset operation of addition and the set theoretic action of union are closely related.

Theorem 2.11 ([5]). *Let A and B are two multisets and let $C = A \oplus B$. A^*, B^* and C^* are the support sets of the multisets A, B and C respectively, then $C^* = A^* \cup B^*$.*

Definition 2.12 (Removal of an Element). *Let $x \in X$ and A is a multiset drawn from X, the removal of x from A gives a new multiset B, denoted as $B = A \ominus x$ and for any $y \in X$ membership function is defined as*

$$Count_B(y) = \begin{cases} \max\{Count_A(y) - 1, 0\}, & \text{if } y = x \\ Count_A(y), & \text{if } y \neq x \end{cases}$$

Definition 2.13 (Removal of a Multiset). *Let A and B are multisets drawn from set X, the removal of multiset B from multiset A gives a new multiset C, denoted as $C = A \ominus B$ and for any $x \in X$*

$$COunt_C(x) = \max\{Count_A(x) - Count_B(x), 0\}$$

Definition 2.14 (Union). *Let A and B are multisets drawn from set X, the union of A and B is a new multiset C, deonoted as*

$$C = A \Cup B$$

such that for any $x \in X$

$$Count_C(x) = \max\{Count_A(x), Count_B(x)\}.$$

It is simple to demonstrate that the union of the two multisets A and B is defined as the smallest multiset for which both A and B are submultisets.

Definition 2.15 (Intersection). *Let A and B are multisets drawn from set X, the intersection of A and B is a new multiset D, deonoted as*

$$D = A \Cap B$$

such that for any $x \in X$

$$Count_D(x) = \min\{Count_A(x), Count_B(x)\}.$$

We can demonstrate that the largest multiset that is contained in both A and B is the intersection of multisets A and B.

The following properties of union, intersection, and addition of multisets are easily established:

1. Commutativity
   (a) $A \Cap B = B \Cap A$
   (b) $A \Cup B = B \Cup A$;
2. Associativity
   (a) $A \Cup (B \Cup C) = (A \Cup B) \Cup C$
   (b) $A \Cap (B \Cap C) = (A \Cap B) \Cap C$;
3. Idempotency
   (a) $A \Cup A = A$
   (b) $A \Cap A = A$;
4. Distributivity
   (a) $A \Cup (B \Cap C) = (A \Cup B) \Cap (A \Cup C)$
   (b) $A \Cap (B \Cup C) = (A \Cap B) \Cup (A \Cap C)$
   (c) $A \oplus (B \Cup C) = (A \oplus B) \Cup (A \oplus C)$
   (d) $A \oplus (B \Cap C) = (A \oplus B) \Cap (A \oplus C)$;
5. (a) $A \Cap (A \oplus B) = A$

(b) $A \Cup (A \oplus B) = A \oplus B$;

6. $A \oplus B = (A \Cup B) \oplus (A \Cap B)$.

We take note of the following theorem without providing proof.

Theorem 2.16. *Let A, B, C and D are multisets such that $C = A \Cap B$ and $D = A \Cup B$. Let A^*, B^*, C^* and D^* be the support sets of these multisets respectively. Then $C^* = A^* \cap B^*$ and $D^* = A^* \cup B^*$.*

2.2 Set-Multiset Selection

Definition 2.17 (Multiset-to-Set Inclusion)**.** *Let A be a multiset drawn from the set X, and let B be a subset of X. In certain scenarios, there may be a need to create a new multiset comprising elements from multiset A that are also members of set B. For simplicity, we denote this resulting multiset as $D = A \circledast B$. The count function for multiset D can be defined as follows:*

$$Cout_D(x) = \begin{cases} Count_A(x), & \text{if } x \in B \\ 0, & \text{if } x \notin B. \end{cases}$$

Definition 2.18 (Multiset-to-Set Removal)**.** *Let A be a multiset drawn from the set X, and let B be a subset of X. Create a new multiset comprising elements from multiset A that are not members of set B. For simplicity, we denote this resulting multiset as $E = A \circledast \bar{B}$. The count function for multiset E can be defined as follows:*

$$Cout_E(x) = \begin{cases} 0, & \text{if } x \in B \\ Count_A(x), & \text{if } x \notin B \end{cases}$$

Theorem 2.19. *Let A and B are any sets. Consider A as a multiset along with the Count function defined as*

$$Count_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

then the set $D = A \circledast B$ is defined as intersection of A and B, i.e. $D = A \cap B$.

Proof. If $A \cap B = \emptyset$ then for any $x \in A$ implies $x \notin B$ then $Count_D(x) = 0$ for all $x \in A$, thus $D = \emptyset$. When $A \cap B \neq \emptyset$. Let $x \in A \cap B$ is arbitrary, so

$$\begin{aligned} x \in A \cap B &\implies x \in A \text{ and } x \in B \\ &\implies Count_D(x) = Count_A(x) \\ &\implies Count_D(x) = 1 \\ &\implies x \in D \end{aligned}$$

Thus for all $x \in A \cap B$ implies $x \in D$ which gives

$$A \cap B \subset D \tag{2}$$

Now for any $x \in D$

$$Count_D(x) = Count_A(x) = 1 \text{ and } x \in B$$

implies $x \in A \cap B$, which gives

$$D \subset A \cap B \tag{3}$$

by (2) and (3) we can say $D = A \circledast B = A \cap B$. □

We can easily prove the followings:

1. $A \circledast X = A$ where $A \in P(X)$
2. $A \circledast \emptyset = \emptyset$
3. $A \circledast (B_1 \cap B_2) = (A \circledast B_1) \Cap (A \circledast B_2)$
4. $A \circledast (B_1 \cup B_2) = (A \circledast B_1) \Cup (A \circledast B_2)$
5. $A_1 \Cap A_2) \circledast B = (A_1 \circledast B) \Cap (A_2 \circledast B)$
6. $A_1 \Cup A_2) \circledast B = (A_1 \circledast B) \Cup (A_2 \circledast B)$

2.3 Relation on Multiset

Relational databases are one area where the theory of multisets may prove to be valuable.The projection procedure is one that is frequently employed in relational data stores. Assume that relation R has the following attributes: $A_1, A_2, ..., A_n$. Therefore, the components of Rare n-tuples. By limiting each of the n-tuples to be three tuples, only taking into account the top three values in each tuple, and then removing any tuple duplications, a new set, the projection of R onto A_1, A_2 and A_3, for instance, can be created.

Example 2.20. *Think of a relationship with the staff members over the plan.*

Name Age

As a result, the entries in the relation staff members are 2-tuples, with each pair representing a worker's name and age. For instance

Staff Members						
Name	*Harry*	*Hermione*	*Luna*	*Ron*	*Ginny*	*Draco*
Age	*16*	*15*	*13*	*14*	*14*	*13*
Name	*Fred*	*Neville*	*George*	*Albus*	*Severus*	*Minerva*
Age	*17*	*14*	*17*	*75*	*35*	*65*

Let's say we want to know the ages of the staff members. Normally, we would use the age projection to determine

$$Proj_{age} = \{13, 14, 15, 16, 17, 35, 65, 75\}$$

The set $Proj_{age}$ really offers us the set of "all the different ages of the staff members" rather than the "set of ages of all the staff members," which is more accurate. We can state that any staff member's age falls within the specified $Proj_{age}$ It seems that the multiset would provide a more accurate response when a data base is requested for the staff member's age.

$$\{16, 15, 14, 13, 14, 13, 14, 17, 17, 75, 35, 65\}$$

or

$$\left\{\frac{2}{13}, \frac{3}{14}, \frac{1}{15}, \frac{1}{16}, \frac{2}{17}, \frac{1}{35}, \frac{1}{65}, \frac{1}{75}\right\}$$

As a result, we propose that a more practical implementation of database calculus can be achieved by including, in addition to the standard projection operations, a new operation called Multiset Projection. Multiset Projection essentially performs the same function as the standard projection but does not remove redundant elements, leading to multisets.

We will now discuss the selection operation from relational data base theory. Selection involves choosing elements from a relation R that possess a specific property. To define a selection, we specify a subset S of $X \times Y$, and the selection operation is represented by $R \cap S = R_S$. In a special case, the selection is based on the condition that the first member of the tuples $(x, y) \in R$ belongs to a subset B of X. In this scenario, the selection subset S becomes $S = B \times Y$, and we can denote the selection operation as $R \cap S = R \cap (B \times Y) = R_B$.

Definition 2.21 (Multi-projection Relation)**.** *Let R be a relationship in $X \times Y$, Let $B \subset X$ and let*

$$H = Multi - Proj_X R.$$

Let H be a multiset over X that is obtained by projecting R onto X without removing duplicates. Additionally, let M be a multiset in X that is obtained by intersecting the set B with the multiset H.

$$M = H \circledast B.$$

Let us consider the selection operation of relation R using the set B, which results in a set in $X \times Y$ denoted as R_B. This set R_B is defined as follows:

$$R_B = R \cap (B \times Y) = R \cap B$$

It can be readily demonstrated that:

$$M = Multi - Proj_X[R_B].$$

Thus

$$(Multi - Proj_X R) \circledast B = Multi - Proj_X [R \cap B].$$

Example 2.22. *Relation R in ex. 2.20. Assume we want to know how many teenagers are in this database. Let B be the set of teenagers.*

$$B = \{13, 14, 15, 16, 17\}$$

then

$$M = (Multi - Proj_{age} R) \circledast B$$

offers us a multiset containing all of the ages in B. Finally, $card(M)$ represents the number of teenagers in R.

$$M = \{16, 15, 14, 13, 14, 13, 14, 17, 17\}$$

or

$$M = \left\{ \frac{2}{13}, \frac{3}{14}, \frac{1}{15}, \frac{1}{16}, \frac{2}{17} \right\}$$

3. Fuzzy Multisets

The idea of a fuzzy multiset will be introduced in this part, and we'll create a algebra for working with these structures later on. As we recall, L. A. Zadeh introduced the fuzzy set.

Definition 3.1 ([6]). *A fuzzy set $\tilde{A}$ in a Set X is defined as*

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\} \tag{4}$$

Here $\mu_{\tilde{A}} : X \to [0, 1] = I$.

When developing a concept that corresponds to a multiset with fuzzy elements, it is essential to consider that a specific object x may have multiple membership grades within the fuzzy multiset. The subsequent description of a fuzzy multiset acknowledges and accommodates this particular scenario.

Definition 3.2 (fuzzy multiset). *Let's assume X is a set of elements. A fuzzy multiset $\tilde{A}$, drawn from X, can be described by a function $Count.\mu_{\tilde{A}}$, which has the following characterization:*

$$Count.\mu_{\tilde{A}} : X \to Q,$$

where Q is the crisp multiset drawn from the unit interval.

For any element x in the set X, $Count.\mu_{\tilde{A}}(x)$ represents a multiset drawn from the unit interval. Since any multiset, such as $Count.\mu_{\tilde{A}}(x)$, can itself be characterized by a count function over its set (in this case, $I = [0,1]$), we can rewrite the expression using the earlier notation as follows:

$$Count_{Count.\mu_{\tilde{A}}(x)} : I \to \mathbb{N} \cup \{0\}$$

Here, $Count_{Count.\mu_{\tilde{A}}(x)}$ represents the count function associated with the multiset $Count.\mu_{\tilde{A}}(x)$ at element α in the set I, i.e. the number of times x appears with membership α in the fuzzy multiset $\tilde{A}$. It represents the count of $(x, \mu_{\tilde{A}}(x))$ pairs in the fuzzy multiset $\tilde{A}$.

To simplify the notation, we will adopt the following convention:

In the context of characterizing a fuzzy multiset using the $Count.\mu_{\tilde{A}}$ function, we will denote it as $C.\mu_{\tilde{A}}$, where $C.\mu_{\tilde{A}}(x)$ represents the multiset associated with element x. Similarly, for the function $Count_{Count.\mu_{\tilde{A}}(x)}$, we will use the notation $C_{C.\mu_{\tilde{A}}(x)}$, indicating the count of graded x objects in the fuzzy multiset $\tilde{A}$. It is important to highlight that $C_{C.\mu_{\tilde{A}}(x)}(x)$ evaluates to zero when there are no elements matching the grading criteria, implying that non-existent elements are not considered in the count.

A regular or crisp multiset B on X can be transformed into a fuzzy multiset using the following formulation, where $Count(x)$ denotes the standard count of x in B. If $C_{C.\mu_B(x)}(\alpha)$ represents the fuzzy multiset representation of the crisp multiset B, then for every x in X.

$$C_{C.\mu_B(x)}(\alpha) = \begin{cases} Count_B(x), & \text{if } \alpha = 1 \\ 0, & \text{if } \alpha \neq 1 \end{cases}$$

In this scenario, the elements $C.\mu_B(x)$ are treated as multisets, where only elements with a count of $\alpha = 1$ can have a non-zero count. This observation indicates that the concept of a multiset is, to some extent, a generalization of a count. From now on, when we mention a fuzzy multiset, we can also imply that it may be a regular (crisp) multiset.

Definition 3.3 (Fuzzy Submultiset)**.** *Let $\tilde{A}$ and $\tilde{B}$ are two multisets drawn from the set X, we say $\tilde{A}$ is a submultiset of $\tilde{B}$, denoted as $A \subset B$ if any $x \in X$*

$$C.\mu_{\tilde{A}}(x) \leq C.\mu_{\tilde{B}}(x)$$

In order to effectively implement the aforementioned requirement, it is necessary to consider for every $x \in X$ and for every $\alpha \in I$

$$C_{C.\mu_{\tilde{A}}(x)}(\alpha) \leq C_{C.\mu_{\tilde{A}}(x)}(\alpha)$$

It obviously follows that for any two fuzzy submultisets $\tilde{A} = \tilde{B} \iff \tilde{A} \subset \tilde{B}$ and $\tilde{B} \subset \tilde{A}$.
i.e. for all $x \in X$

$$C.\mu_{\tilde{A}}(x) = C.\mu_{\tilde{B}}(x)$$

To operationalize this definition, it is crucial to determine when the fuzzy multiset characterizing x in $\tilde{A}$ is equal *to the fuzzy multiset characterizing x in $\tilde{B}$. For the purpose of equality between fuzzy multisets, it is necessary to compare the characterizing counts of $C.\mu_{\tilde{A}}(x)$ and $C.\mu_{\tilde{B}}(x)$, which are crisp multisets over the unit interval. Therefore, the equality of two fuzzy multisets requires that for all $x \in X$ and $\alpha \in I$*

$$C_{C.\mu_{\tilde{A}}(x)}(\alpha) = C_{C.\mu_{\tilde{B}}(x)}(\alpha)$$

Definition 3.4 (Cardinality of Fuzzy Multiset)**.** *Let $\tilde{A}$ is a fuzzy multiset drawn from set X. The cardinality of $\tilde{A}$ is given by*

$$Card(A) = \sum_{x \in X} Card_{\tilde{A}}(x)$$

Where, $Card_{\tilde{A}}(x)$ is cardinality of x in $\tilde{A}$, defined as

$$Card_{\tilde{A}}(x) = \sum_{\alpha \in I} \alpha * C_{C.\mu_{\tilde{A}}(x)}(\alpha).$$

Therefore, the cardinality of x in $\tilde{A}$ can be calculated by evaluating the product of the membership grade and the count of x with that particular grade, representing the frequency of x's occurrence in $\tilde{A}$.

Definition 3.5 (Absolute Cardinality of Fuzzy Multiset)**.** *Absolute cardinality is stand for unweighted measure of the number of elements in $\tilde{A}$, where $\tilde{A}$ is any fuzzy multiset drawn form any set X. Thus the absolute cardinality of $\tilde{A}$ is defined as*

$$|Card(A)| = \sum_{x \in X} |Card_{\tilde{A}}(x)|$$

where $|Card_{\tilde{A}}(x)|$ is the absolute cardinality of x in $\tilde{A}$, defined as

$$|Card_{\tilde{A}}(x)| = \sum_{\alpha \in I} |C_{C.\mu_{\tilde{A}}(x)}(\alpha)|.$$

At this stage, it is important to reiterate that the multisets, $C.\mu_{\tilde{A}}(x)$, extracted from set I, serve as representations of counts. It is worth noting that there is also a possibility to discuss type II fuzzy multisets. In this scenario, the functions $C.\mu_{\tilde{A}}(x)$ represent fuzzy multisets over I. Essentially, for each $x \in X$ and $\alpha \in I$, $C_{C.\mu_{\tilde{A}}(x)}(\alpha)$ is not a single value from the set of non-negative integers ($\mathbb{N} \cup \{0\}$), but rather a multiset itself. However, we will not explore this topic any further.

Analogous to the crisp case, where crisp sets can be considered as special instances of crisp multisets, we can observe a similar relationship in the fuzzy domain. Fuzzy sets can be seen as specific conditions of fuzzy multisets.

Assume $\tilde{A}$ is a fuzzy subset of X with a membership grade function $\mu_{\tilde{A}}$. We can consider this as a multiset in which $C.\mu_{\tilde{A}}(x)$ is the multiset $\langle \frac{1}{\mu_{\tilde{A}}(x)} \rangle$. That is, for each $x \in X$,

$$C_{C.\mu_{\tilde{A}}(x)}(\alpha) = \begin{cases} 1, & \text{if } \mu_{\tilde{A}} = \alpha \\ 0, & \text{if } \mu_{\tilde{A}} \neq \alpha \end{cases}$$

Definition 3.6 (Fuzzy Support Set). *Assume that $\tilde{A}$ is a fuzzy multiset chosen from set X. The fuzzy supporting set $\tilde{B}$ of $\tilde{A}$ is a fuzzy subset of X and its membership function is defined as*

$$\mu_{\tilde{B}}(x) = \max_{\alpha \in I}\{C_{C.\mu_{\tilde{A}}(x)}(\alpha) \vee \alpha\}.$$

The highest membership grade with a non-zero count in the multiset, $C.\mu_{\tilde{A}}(x)$, is equal to $\mu_{\tilde{B}}(x)$, as we have shown.

3.1 Operations on Fuzzy Multisets

Definition 3.7 (Addition of Fuzzy Multisets). *Assume that $\tilde{A}$ and $\tilde{B}$ are fuzzy multisets chosen from set X. The addition of $\tilde{A}$ and $\tilde{B}$ gives a new fuzzy multiset $\tilde{C}$ on X, denoted by $\tilde{C} = \tilde{A} \oplus \tilde{B}$ and for any $X \in X$*

$$C.\mu_{\tilde{C}}(x) = C.\mu_{\tilde{A}}(x) \oplus C.\mu_{\tilde{B}}(x) \tag{5}$$

The incorporation of multisets into equation 5 can be accomplished using Definition 2.10.
For any $\alpha \in I$ and $x \in X$ It can be demonstrated that for any $\alpha \in I$ and $x \in X$ the operation of addition effectively results in the following outcome:

$$C_{C.\mu_{\tilde{C}}(x)}(\alpha) = C_{C.\mu_{\tilde{A}}(x)}(\alpha) + C_{C.\mu_{\tilde{B}}(x)}(\alpha)$$

Definition 3.8 (Insertion of a Fuzzy Element). *Let (a, y) is a fuzzy element and $\tilde{A}$ fuzzy multiset drawn from set X. Insertion of (a, y) in $\tilde{A}$ gives a new multiset $\tilde{D}$ for which the count function is defined as*

$$C_{C.\mu_{\tilde{D}}(x)}(\alpha) \begin{cases} C_{C.\mu_{\tilde{A}}(y)}(a) + 1, & \text{if } \alpha = a \& x = y \\ C_{C.\mu_{\tilde{A}}(x)}(\alpha), & \text{otherwise} \end{cases}$$

Definition 3.9 (Union of Fuzzy Multisets). *Assume $\tilde{A}$ and $\tilde{B}$ are two fuzzy multisets drawn from the set X. The union of $\tilde{A}$ and $\tilde{B}$ is a new fuzzy multiset $\tilde{D}$, denoted $\tilde{D} = \tilde{A} \Cup \tilde{B}$, such that for each element x in X*

$$C.\mu_{\tilde{D}}(x) = C.\mu_{\tilde{A}}(x) \Cup C.\mu_{\tilde{B}}(x). \tag{6}$$

Here $C.\mu_{\tilde{A}}(x)$ and $C.\mu_{\tilde{B}}(x)$ are crisp multisets on I thus from definition 2.14 for any $x \in X$ and $\alpha \in I$,

$$C_{C.\mu_{\tilde{D}}(x)}(\alpha) = \max\{C_{C.\mu_{\tilde{A}}(x)}(\alpha), C_{C.\mu_{\tilde{B}}(x)}(\alpha)\}.$$

Definition 3.10 (Intersection of Fuzzy Multisets). *Assume $\tilde{A}$ and $\tilde{B}$ are two fuzzy multisets drawn from the set X. The intersection of $\tilde{A}$ and $\tilde{B}$ is a new fuzzy multiset $\tilde{E}$, denoted $\tilde{D} = \tilde{A} \Cap \tilde{B}$, such that for each element x in X*

$$C.\mu_{\tilde{E}}(x) = C.\mu_{\tilde{A}}(x) \Cap C.\mu_{\tilde{B}}(x). \tag{7}$$

Here $C.\mu_{\tilde{A}}(x)$ and $C.\mu_{\tilde{B}}(x)$ are crisp multisets on I thus from definition 2.15 for any $x \in X$ and $\alpha \in I$,

$$C_{C.\mu_{\tilde{E}}(x)}(\alpha) = \min\{C_{C.\mu_{\tilde{A}}(x)}(\alpha), C_{C.\mu_{\tilde{B}}(x)}(\alpha)\}.$$

Definition 3.11 (Removal of Fuzzy Multisets). *Let $\tilde{A}$ and $\tilde{B}$ are two fuzzy multisets drawn from the set X. The removal of multiset $\tilde{B}$ from $\tilde{A}$ gives a new fuzzy multiset $\tilde{D}$, denoted as $\tilde{D} = \tilde{A} \ominus \tilde{B}$ and for any $x \in X$, Count function is defined as*

$$C.\mu_{\tilde{D}}(x) = C.\mu_{\tilde{A}}(x) \ominus C.\mu_{\tilde{B}}(x).$$

Here $C.\mu_{\tilde{A}}(x)$ and $C.\mu_{\tilde{B}}(x)$ are crisp multisets on I thus from definition 2.13, for any $x \in X$ and $\alpha \in I$,

$$C_{C.\mu_{\tilde{D}}(x)}(\alpha) = \max\{C_{C.\mu_{\tilde{A}}(x)}(\alpha) - C_{C.\mu_{\tilde{B}}(x)}(\alpha), 0\}.$$

3.2 Fuzzy Set-Multiset Selection

Consider a scenario where $\tilde{A}$ represents a fuzzy multiset drawn from the set X, and $\tilde{B}$ represents a fuzzy subset of X.

Definition 3.12 (Fuzzy Multiset to Fuzzy Set Inclusion). *Let $\tilde{D} = \tilde{A} \circledast \tilde{B}$, the fuzzy multiset where elements are in it from $\tilde{A}$ and also members of $\tilde{B}$. Since $\tilde{A}$ is fuzzy multiset then for any $x \in X$ we get*

$$C.\mu_{\tilde{A}}(x) = \bigcup_{\alpha \in I} \left\{ \frac{C_{C\mu_{\tilde{A}}(x)}(\alpha)}{\alpha} \right\}$$

Here $C_{C\mu_{\tilde{A}}(x)}(\alpha)$ represents number of times α appeared along with x in fuzzy multiset $\tilde{A}$ then

$$C.\mu_{\tilde{D}}(x) = \bigcup_{\alpha \in I} \left\{ \frac{C_{C\mu_{\tilde{A}}(x)}(\alpha)}{\min\{\alpha, \mu_{\tilde{B}}(x)\}} \right\}$$

Theorem 3.13. *If $\tilde{A}$ and $\tilde{B}$ are fuzzy sets on X then $\tilde{A} \circledast \tilde{B} = \tilde{A} \wedge \tilde{B}$.*

Proof. Let $\tilde{D} = \tilde{A} \circledast \tilde{B}$. Since $\tilde{A}$ is a fuzzy set then $C_{C\mu_{\tilde{A}}(x)}(\alpha) \in \{0, 1\}$. Thus for any $x \in X$ and $\alpha \in I$

$$C_{C\mu_{\tilde{D}}(x)}(\alpha) = \begin{cases} C_{C\mu_{\tilde{A}}(x)}(\alpha), & \text{if } \min\{\alpha, \mu_{\tilde{B}}(x)\} \neq 0 \\ 0, & \text{if } \min\{\alpha, \mu_{\tilde{B}}(x)\} = 0 \end{cases}$$

And $\mu_{\tilde{D}}(x) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$. Hence $\tilde{A} \circledast \tilde{B} = \tilde{A} \wedge \tilde{B}$. □

4. Conclusion

This research article proposes an alternative perspective on Yagar's Bag Theory by adopting the term "multiset" instead of "bag" to align with community preferences. The study introduces revised definitions and different notations to enhance the understanding of multiset structures within Yagar's theory. The upcoming graphical representations will aid in visual comprehension and stimulate further research and discussions in the field. Overall, this work contributes to the refinement and development of Yagar's Theory within the context of multisets.

References

[1] G. Beliakov, A. Pradera and T. Calvo, *Aggregation Functions: A Guide for Practitioners*, Springer Berlin, Heidelberg, 2010.

[2] D. Dubois and H. Prade, *Representation and combination of uncertainty with belief functions and possibility measures*, Computational Intelligence, 4(3)(1988), 244–264.

[3] B. Li, *Fuzzy bags and applications*, Fuzzy Sets and Systems, 34(1)(1990), 61–71.

[4] S. Sebastian, *Multi-fuzzy sets*, International Mathematical Forum, 5(1)(2010), 2471 –2476.

[5] R. R. YAGER, *On the theory of bags*, International Journal of General Systems, 13(1)(1986), 23–37.

[6] L. A. Zadeh, *Fuzzy sets*, Information and Control, 8(3)(1965), 338–353.

[7] X. R. Zhao and B. Q. Hu, *Three-way decisions with decision-theoretic rough sets in multiset-valued information tables*, Information Sciences, 507(2020), 684–699.