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# Extended Methods for Solving Nonconvex Bifunction General Variational Inequalities

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**Abstract**

The nonconvex bifunction extended general variational inequality is another type of variational inequalities that we describe and discuss in this research. We propose and evaluate some iterative solutions for the nonconvex bifunction extended general variational inequalities using the auxiliary principle technique. We show the convergence of these methods either needs only pseudomonotonicity. Our convergence proofs are fairly straightforward. The concepts and methods used in this paper could inspire additional study in this area.

**Keywords:** variational inequalities; auxiliary principle; convergence; Nonconvex function.
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## 1. Introduction

The theory of variational inequalities, which Stampacchia [27] first proposed, can be seen as an essential and substantial extension of the variational principles. It is common knowledge that the variational inequalities serve as the convex function's criterion for optimality. We have another class of variational inequalities, called the bifunction (directional) variational inequalities, for the directional differentiable convex functions. Numerous elements of the bifunction variational inequalities have been studied by Crespi et al. [4–7], Fang et al. [8], Lalitha et al. [10], and Noor et al. [23]. Noor [20] shows that a class of bifunction variational inequalities can serve as a description of the optimality condition for a subclass of directional differentiable nonconvex functions on a nonconvex set.

We introduced and talked about the general nonconvex bifunction variational inequalities on uniformly proximal regular sets as a result of this finding. The prox-regular sets are known to be nonconvex and include convex sets as special cases [3, 27]. Variational inequality on the uniformly prox-regular sets has been studied by Noor [15–20] and Bounkhel et al. [2]. There are many numerical techniques for resolving variational inequalities, such as the projection technique and its variants, the Wiener-Hop equations, the auxiliary principle, and resolvent equations. It is understood that certain

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approaches, such as projection, Wiener-Hop equations, proximal equations, and resolvent equations, cannot be expanded upon or generalised to suggest and examine equivalent iterative solutions to generic nonconvex bifunction variational inequalities.

This aspect promotes the usage of the auxiliary principle technique, which is mostly thanks to Glowinski et al.'s [9] iterative schemes for solving various classes of variational inequalities. We draw attention to the flexibility and lack of operator projection in this technique. This method focuses on locating the auxiliary variational inequality and demonstrating via the fixed-point method that the auxiliary problem's solution is the same as the original problem's solution. It became out that we can create gap (merit) functions by using this technique to identify analogous differentiable optimisation issues. It is well knowledge that this technique can be used to create special cases of a significant number of numerical algorithms.Using this method, we propose and evaluate some explicit predictor-corrector methods for extended general variational inequalities.

## 2. Preliminaries

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$ respectively. Let K be a nonempty and convex set in H. The following well-known notions from nonlinear convex analysis and nonsmooth analysis are owed to Clarke et al. [3] and Poliquin et al. [27], respectively. A novel class of nonconvex sets known as uniformly prox-regular sets has been developed and researched by Poliquin et al. [27] and Clarke et al. [3].

**Definition 2.1.** *The proximal normal cone of K at $u \in H$ is given by*

$$N_K^P(u) := \xi \in H : u \in P_K[u + \alpha\xi],$$

*where $\alpha > 0$ is a constant and*

$$P_K[u] = u^* \in K : d_K(u) = \|u - u^*\|$$

*Here $d_K(.)$ is the usual distance function to the subset K, that is*

$$d_K(u) = \inf_{v \in K} \|v - u\|.$$

The proximal normal cone $N_K^P(u)$ has the following characterization.

**Lemma 2.2.** *Let K be a nonempty, closed and convex subset in H: Then $\zeta \in N_K^P(u)$; if and only if, there exists a constant $\alpha > 0$ such that*

$$\langle \zeta, u - v \rangle \leq \alpha \|v - u\|^2, \quad \forall v \in K$$

**Definition 2.3.** *For a given $r \in (0; 1]$, a subset $K_r$ is said to be normalized uniformly r-prox-regular if and only if every nonzero proximal normal cone to $K_r$ can be realized by an r-ball, that is,$\forall u \in K_r$ and $0 \neq \xi \in N_K^P(u)$*

*one has*

$$\langle(\xi)/\|\xi\|, v-u\rangle \leq (1/2r)\|v-u\|^2, \quad \forall v \in K_r$$

It is clear that the class of normalized uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets. $C_{1.1}$ submanifolds (possibly with boundary) of H, the images under a $C_{1.1}$ diffeomorphism of convex sets and many other nonconvex sets; see Clarke et al. [3] and Poliquin et al. [27]. It is well-known that the union of two disjoint intervals $[a,b]$ and $[c,d]$ is a prox-regular set with $r = \frac{c-b}{2}$. For other examples of prox-regular sets, see Noor [20] . Obviously, for $r = \infty$, the uniformly prox-regularity of $K_r$ is equivalent to the convexity of K. This class of uniformly prox-regular sets have played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions. It is known that if $K_r$ is a uniformly prox-regular set, then the proximal normal cone $N_K^p(u)$ is closed as a set-valued mapping.
For the sake of simplicity, we take $\gamma = \frac{1}{r}$. Then it is clear that for $r = \infty$, we have $\gamma = 0$.
For given bifunction $B(:;:) : H \to H$ and nonlinear operator $g : H \to H$; we consider the problem of finding $u \in H : g(u) \in K_r$ such that

$$B(g(u), g(v) - g(u)) + \gamma\|g(v) - g(u)\|^2 \geq 0 \qquad \forall v \in H : g(v) \in K_r, \tag{1}$$

which is called the nonconvex bifunction general variational inequality.
We now discuss some important special cases nonconvex bifunction general variational inequality.
**Special Cases**
**(I)** We note that, if $K_r \equiv K$; the convex set in H, then problem (1) is equivalent to finding $u \in H : g(u) \in K$ such that

$$B(g(u), g(v) - g(u)) \geq 0 \qquad \forall v \in H : g(v) \in K, \tag{2}$$

Inequality of type (2) is called the bifunction general variational inequality, which appears to be new one.
**(II)** If $B(g(u), g(v) - g(u)) = \langle Tu, g(v) - g(u))\rangle$, where T is a nonlinear operator, then problem (1) is equivalent to finding $u \in H : g(u) \in Kr$ such that

$$\langle Tu, g(v) - g(u))\rangle + \gamma\|g(v) - g(u)\|^2 \geq 0 \qquad \forall v \in H : g(v) \in K_r, \tag{3}$$

which is called the general nonconvex variational inequality, see Noor [15–20].
**(III)** If $B(g(u), g(v) - g(u)) = \langle Tu, g(v) - g(u))\rangle$, where T is a nonlinear operator and $K_r = K$, the convex set, then problem (3) is equivalent to finding $u \in H : g(u) \in K_r$ such that

$$\langle Tu, g(v) - g(u))\rangle \geq 0 \qquad \forall v \in H : g(v) \in K, \tag{4}$$

which is called the general variational inequality, introduced and studied by Noor [11–14]. It has been shown a wide class of nonsymmetric and odd-order obstacle boundary values and initial value problems can be studied in the general framework of general variational inequalities (4). For the applications, numerical methods, sensitivity analysis, dynamical system, merit functions, and other aspects of general variational inequalities, see Al-Said et al. [1], Noor at al. [21–26] and references therein.

**(IV)** If $g \equiv I$, the identity operator, then problem (4) reduces to finding $u \in K_r$ such that

$$\langle Tu, v-u\rangle \geq 0 \qquad \forall v \in K_r, \tag{5}$$

which is called the nonconvex variational inequality, see Noor [16, 20].

**(V)** If $K_r \equiv K$, the convex set, then problem (5) reduces to finding $u \in K$ such that

$$\langle Tu, v-u\rangle \geq 0 \qquad \forall v \in K, \tag{6}$$

which is called the classical variational inequality, introduced and studied by Stampacchia [28].

## 3. Main Result

In this portion, we propose and analyse some iterative methods for solving the general nonconvex bifunction variational inequality (1), which was introduced by Glowinski et al. [9] and developed by Noor [14] and Noor et al [23, 24]. The main advantage of this approach is that it does not utilize the idea of projection. For a given $u \in H : g(u) \in K_r$ satisfying (1), consider the problem of finding $w \in H : g(w) \in K_r$ such that

$$\rho B(g(w), g(v)-g(w)) + \langle w-u, v-w\rangle + \gamma\|g(v)-g(w)\|^2 \geq 0 \qquad \forall v \in H : g(v) \in K_r, \tag{7}$$

where $\rho > 0$ and $\gamma > 0$ is a constant. Inequality of type (7) is called the auxiliary nonconvex bifunction general variational inequality. Note that if w = u, then w is a solution of (1). This simple observation enables us to suggest the following iterative method for solving the general nonconvex bifunction variational inequalities (1).

**Algorithm 3.1.** *For a given $u_0 \in K_r$, compute the approximate solution $u_{n+1}$ by the iterative scheme*

$$\rho B(g(u_{n+1}), g(v)-g(u_{n+1})) + \langle u_{n+1}-u_n, v-u_{n+1}\rangle + \gamma\|g(v)-g(u_{n+1})\|^2 \geq 0 \quad \forall g(v) \in K_r, \tag{8}$$

Algorithm 3.1 is called the proximal point algorithm for solving general nonconvex bifunction variational inequality (1). In particular, if $r = \infty$ and $\gamma = 0$ then the uniformly prox-regular set $K_r$ becomes the standard convex set K, and consequently Algorithm 3.1 reduces to:

**Algorithm 3.2.** *For a given $u_0 \in K$, compute the approximate solution $u_{n+1}$ by the iterative scheme*

$$\rho B(g(u_{n+1}), g(v) - g(u_{n+1})) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \geq 0 \quad \forall g(v) \in K$$

*which is known as the proximal point algorithm for solving bifunction variational inequalities (2) and has been studied extensively, see Noor [11–17,17–20].*

For the convergence analysis of Algorithm 3.1, we recall the following concepts and results.

**Definition 3.3.** *A bifunction $B(:,:) : H \times H \to H$ with respect to the operator $g$ is said to be:*

*(i). monotone, if and only if, $B(g(u), g(v) - g(u)) + B(g(v), g(u) - g(v)) \leq 0, \ \forall \ u, v \in H$;*

*(ii). pseudomonotone, if and only if, $B(g(u), g(v) - g(u)) + \gamma\|g(v) - g(u)\|^2 \geq 0$ implies that $-B(g(v), g(u) - g(v)) - \gamma\|g(v) - g(u)\|^2 \geq 0 \ \forall \ u, v \in H$;*

*(iii). partially relaxed strongly monotone, if and only if, there exists a constant $\alpha > 0$ such that $B(g(z), g(v) - g(u)) + B(g(v), g(u) - g(v)) \leq \alpha\|z - u\|^2, \ \forall \ u, v, z \in H$.*

Note that for $z = u$, partially relaxed strongly monotonicity reduces to monotonicity. It is known that coercivity implies partially relaxed strongly monotonicity, but the converse is not true. It is known that monotonicity implies pseudomonotonicity; but the converse is not true. We also recall the well-known result.

$$2\langle u, v\rangle = \|u + v\|^2 - \|u\|^2 - \|v\|^2 \tag{9}$$

We now consider the convergence criteria of Algorithm 3.1 and this is the main motivation of our next result.

**Theorem 3.4.** *Let the operator $B(:,:) : K_r \times K_r \to H$ be pseudomonotone. If $u_{n+1}$ is the approximate solution obtained from Algorithm 3.1 and $u \in K_r$ is a solution of (1), then*

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2 \tag{10}$$

*Proof.* Let $u \in H : g(u) \in K_r$ be a solution of (1). Then

$$-B(g(v), g(u) - g(v)) - \gamma\|g(v) - g(u)\|^2 \geq 0 \ \forall \ v \in H, g(v) \in K_r \tag{11}$$

since $B(:,:)$ is pseudomonotone. Taking $v = u_{n+1}$ in (11), we have

$$-B(g(u_{n+1}), g(u) - g(u_{n+1})) - \gamma\|g(u) - g(u_{n+1})\|^2 \geq 0 \tag{12}$$

Setting $v = u$ in (2), and using (8), we have

$$\langle u_{n+1} - u_n, u - u_{n+1}\rangle \geq -\rho B(g(u_{n+1}), g(u_{n+1}) - g(u)) - \gamma\|g(u_{n+1}) - g(u)\|^2 \geq 0 \tag{13}$$

Setting $v = u - u_{n+1}$ and $u = u_{n+1} - u_n$ in (3), we obtain

$$2\langle u_{n+1} - u_n, u - u_{n+1}\rangle = \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2 - \|u - u_{n+1}\|^2 \tag{14}$$

From (9) and (14), and using (12) and (13) we get

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2$$

$\square$

**Theorem 3.5.** *Let $H$ be a finite dimension subspace and let $u_{n+1}$ be the approximate solution obtained from Algorithm 3.1. If $u \in K_r$ is a solution of (1), then $\lim_{n\to\infty} u_n = u$.*

*Proof.* Let $u \in H : g(u) \in K_r$ be a solution of (1). Then it follows from (10) that the sequence $u_n$ is bounded and

$$\sum_{n=0}^{\infty} \|u_n - u_{n+1}\|^2 \leq \|u_0 - u\|^2$$

which implies that

$$\lim_{n\to\infty} \|u_n - u_{n+1}\| = 0 \tag{15}$$

Let $\hat{u}$ be a cluster point of the sequence $u_n$ and let the subsequence $u_j$ of the sequence $u_n$ converge to $\hat{u} \in K_r$. replacing $u_n$ by $u_{n_j}$ in (15) and taking the limit $n_j \to \infty$ and using (8), we have

$$B(g(\hat{u}), g(v) - g(\hat{u})) - \gamma\|g(v) - g(\hat{u})\|^2 \geq 0 \qquad \forall v \in H, g(v) \in K_r,$$

which implies that $\hat{u}$ solves the general nonconvex bifunction variational inequality (1) and

$$\|u_n - u_{n+1}\|^2 \leq \|\hat{u} - u_n\|^2$$

Thus it follows from the above inequality that the sequence $u_n$ has exactly one cluster point $\hat{u}$ and $\lim_{n\to\infty} u_n = \hat{u}$, the required result. $\square$

We note that for $r = \infty$, the r-prox-regular set K becomes a convex set and the nonconvex bifunction variational inequality (1) collapses to the bifunction variational inequality (2). Thus our results include the previous known results as special cases. It is well-known that to implement the proximal point methods, one has to calculate the approximate solution implicitly, which is in itself a difficult problem. To overcome this drawback, we suggest another iterative method, the convergence of which requires only partially relaxed strongly monotonicity, which is a weaker condition that cocoercivity. For a given $u \in H : g(u) \in K_r$ satisfying (1), consider the problem of finding $w \in H : g(w) \in K_r$ such that

$$\rho B(g(u), g(v) - g(w)) + \langle w - u, v - w\rangle + \gamma\|g(v) - g(w)\|^2 \geq 0 \quad \forall g(v) \in K_r, \tag{16}$$

which is also called the auxiliary nonconvex bifunction general variational inequality. Note that problems (3) and (16) are quite different. If w = u, then clearly w is a solution of the nonconvex bifunction general variational inequality (1). This fact enables us to suggest and analyze the following iterative method for solving the nonconvex bifunction general variational inequality (1).

**Algorithm 3.6.** *For a given $u_0 \in H$, compute the approximate solution $u_{n+1}$ by the iterative scheme*

$$\rho B(g(u_n), g(v) - g(u_{n+1})) + \langle u_{n+1} - u_n, v - u_{n+1} \rangle + \gamma \|g(v) - g(u_{n+1})\|^2 \geq 0 \;\; \forall \;\; g(v) \in K_r, \tag{17}$$

Note that, for $r = \infty$, the uniformly prox-regular set $K_r$ becomes a convex set K, and Algorithm 3 reduces to:

**Algorithm 3.7.** *For a given $u_0 \in K$, compute the approximate solution $u_{n+1}$ by the iterative scheme*

$$\rho B(g(u_n), g(v) - g(u_{n+1})) + \langle u_{n+1} - u_n, v - u_{n+1} \rangle + \gamma \|g(v) - g(u_{n+1})\|^2 \geq 0 \quad \forall g(v) \in K,$$

**Theorem 3.8.** *Let the operator B(:,:) be partially relaxed strongly monotone with constant $\alpha > 0$. If $u_{n+1}$ is the approximate solution obtained from Algorithm 3.6 and $u \in H : g(u) \in K_r$ is a solution of (1), then*

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - (1 - 2\rho\alpha)\|u_n - u_{n+1}\|^2 \tag{18}$$

*Proof.* Let $u \in H : g(v) \in K_r$ be a solution of(1). Then

$$B(g(u), g(v) - g(u)) + \gamma \|g(v) - g(u)\|^2 \geq 0 \qquad \forall v \in H : g(v) \in K_r, \tag{19}$$

Taking $v = u_{n+1}$ in (19), we have

$$B(g(u), g(u_{n+1}) - g(u)) + \gamma \|g(u_{n+1}) - g(u)\|^2 \geq 0. \tag{20}$$

Letting $v = u$ in (17), we obtain

$$\rho B(g(u_n), g(u) - g(u_{n+1})) + \langle u_{n+1} - u_n, u - u_{n+1} \rangle + \gamma \|g(u) - g(u_{n+1})\|^2 \geq 0,$$

which implies that

$$\begin{aligned} \langle u_{n+1} - u_n, u - u_{n+1} \rangle &\geq -\rho B(g(u_n), g(u) - g(u_{n+1})) - \gamma \|g(u) - g(u_{n+1})\|^2 \\ &\geq -\rho B(g(u_n), g(u) - g(u_{n+1})) + B(g(u), g(u_{n+1}) - g(u)) \\ &\geq -\alpha\rho \|u_n - u_{n+1}\|^2 \end{aligned} \tag{21}$$

since $B(:,:)$ is partially relaxed strongly monotone with constant $\alpha > 0$. Now Combining (20) and (21), we get.

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - (1 - 2\rho\alpha)\|u_n - u_{n+1}\|^2$$

□

## 4. Conclusion

This research introduces and discusses a new class of extended nonconvex bifunction general variational inequalities involving two arbitrary operators. There are some unique cases discussed. Certain iterative solutions to nonconvex bifunction general variational inequalities are proposed using the auxiliary principle method. The proposed approaches convergence analysis is examined in the presence of partially relaxed highly monotonicity and pseudo-monotonicity. Comparing the efficacy of the inertial and proximal approaches with other methods is an outstanding subject; this is another area for future investigation. Additional work is needed to compare these approaches.

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