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Approximation of Fixed Points for (α, β)-generalized Hybrid Mapping via New Three Step Iteration Process

Samir Dashputre[1], Rakesh Tiwari[2], Jaynendra Shrivas[2,*]

[1]*Department of Mathematics, Shahid Durwasha Nishad Govt. College, Arjunda, Balod, Chhattisgarh, India*
[2]*Department of Mathematics, Govt. V.Y.T. PG Autonomous College, Durg, Chhattisgarh, India*

Abstract

In this paper, we establish some strong and Δ-convergence theorems of a newly defined three-step iteration process for (α, β)- generalized hybrid mapping in the setting of CAT(0) spaces. Our results generalize, unify and extend many known results from the existing literature.

Keywords: CAT(0) space; (α, β)-generalized hybrid mapping; fixed point; strong and Δ-convergence theorems.
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1. Introduction

The concept of fixed points theory and its application has proven to be a vital tool in the study of nonlinear functional analysis, as well as a very useful tool in establishing existence and uniqueness theorems for nonlinear ordinary, partial and random differential and integral equations in various abstract spaces. We recall the following:
Let K be a nonempty subset of a Banach space X and $G\colon K \to K$ a self-mapping. A point $x \in X$ is said to be a fixed point of G if $Gx = x$. After existence of fixed point for a mapping, it is natural to find the value of that fixed point. For this purpose, different iteration processes, i.e., those of Mann [20], Ishikawa [12], Agarwal [1], SP [23], Noor [21], Normal-S [11], Thakur New [29] and M-iteration [30], etc. have been introduced. The approximating of a fixed point using an effective iterative technique is an important field of research. For example, Khatoon and Uddin [13] investigated Abbas iterative scheme for G-nonexpansive operators, while Wairojjana [31] demonstrated the strong convergence of a specific scheme for variational inequalities problems. Remember that let (X, d) be a metric space and K be a nonempty subset of X. A mapping $G\colon K \to K$ is said to be nonexpansive if:

$$d(Gx, Gy) \leq d(x, y),$$

*Corresponding author (jayshrivas95@gmail.com)

for all $x, y \in K$.The class of nonexpansive mapping was studied by many authors; see, for example, [6–10, 14, 15]. A point $x \in K$ is called a fixed point of G if $Gx = x$ and $F(G)$ denotes the set of fixed points of the mapping G. A sequence $\{x_n\}$ in K is called the approximate fixed point sequence for G [24] if

$$\lim_{n\to\infty} d(x_n, Gx_n) = 0.$$

Definition 1.1 ([7]). *A mapping $G\colon K \to K$ is said to be quasi-nonexpansive if*

$$d(Gx, p) \leq d(x, p),$$

for all $x \in K$ and $p \in F(G)$ when $F(G) \neq \emptyset$.

In 2008 and 2010, Takahashi and Kohsaka [17] and Takahashi et al. [28] introduced the following two nonlinear mappings in a Hilbert space H as follows:
A mapping $G\colon K \to H$ is said to be:

(1) nonspreading if [18]:

$$2d^2(Gx, Gy) \leq d^2(Gx, y) + d^2(Gy, x) \quad \forall x, y \in K;$$

(2) hybrid if [26]:

$$3d^2(Gx, Gy) \leq d^2(x, y) + d^2(Gx, y) + d^2(Gy, x) \quad \forall x, y \in K;$$

Also, they proved some fixed point theorems for these mappings in Hilbert spaces (for more details, see [10, 17, 18, 27]). It is easy to see that both nonspreading and hybrid mappings with $F(G) \neq \emptyset$ are quasi-nonexpansive. In 2010, Kocourek et al. [16] introduced the following mapping:
A mapping $G\colon K \to K$ is said to be (α, β)-generalized hybrid if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha d^2(Gx, Gy) + (1-\alpha) d^2(x, Gy) \leq \beta d^2(Gx, y) + (1-\beta) d^2(x, y), \tag{1}$$

for any $x, y \in K$. Also, they and others proved some fixed point theorems for this mapping in Hilbert spaces (see [16] and [32]). We have the following properties of this mappings:

(1) An (α, β)-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$;

(2) An (α, β)-generalized hybrid mapping is nonspreading for $\alpha = 2$ and $\beta = 1$;

(3) An (α, β)-generalized hybrid mapping is hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$;

(4) Every (α, β)-generalized hybrid mapping with $F(G) \neq \emptyset$ is quasi-nonexpansive.

On the other hand, in 1953, Mann [20] introduced the following iteration for approximating a fixed point of a nonexpansive mapping G in a Hilbert space H, which is defined by

$$\begin{aligned} &x_1 \in K \\ &x_{n+1} = \alpha_n x_n + (1-\alpha_n) G x_n, \end{aligned} \tag{2}$$

for all $n \geq 1$, where $\{\alpha_n\}$ is a appropriate sequence in (0, 1). In 1974, Ishikawa [12] introduced the following iteration for approximating a fixed point of a nonexpansive mapping G in a Hilbert space H, which is defined by

$$\begin{aligned} &x_1 \in K \\ &x_{n+1} = (1-\alpha_n) G x_n + \alpha_n y_n \\ &y_n = (1-\beta_n) x_n + \beta_n G x_n, \end{aligned} \tag{3}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in (0, 1) with some conditions. This Ishikawa iteration reduces to the Mann iteration when $\beta_n = 0$ for all $n \geq 1$. In 2007, Agarwal et al. [1] introduced the following S-iteration for a nearly asymptotically nonexpansive mapping G in a Banach space E, which is defined by

$$\begin{aligned} &x_1 \in K \\ &x_{n+1} = (1-\alpha_n) G x_n + \alpha_n G y_n \\ &y_n = (1-\beta_n) x_n + \beta_n G x_n, \end{aligned} \tag{4}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in (0, 1) with some conditions. Note that this iteration is independent and converges faster than both of the Ishikawa and Mann iterations. Later, in 2008, Dhompongsa and Panyanak [7] studied the Δ-convergence of the following iteration for a nonexpansive mapping in a complete CAT(0) space, which is defined by

$$\begin{aligned} &x_1 \in K \\ &x_{n+1} = (1-\alpha_n) x_n \oplus \alpha_n G y_n \\ &y_n = (1-\beta_n) x_n \oplus \beta_n G x_n, \end{aligned} \tag{5}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in (0, 1) with some conditions. Recently, in 2017, N. Pakkaranang et al. [22] motivated by the S-iteration introduced by Agarwal et al. [1], he modify the S-iteration for an (α, β)-generalized hybrid mapping in a complete CAT(0) space, which is defined by

$$\begin{aligned} &x_1 \in K \\ &x_{n+1} = (1-\alpha_n) G x_n \oplus \alpha_n G y_n \\ &y_n = (1-\beta_n) x_n \oplus \beta_n G x_n, \end{aligned} \tag{6}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in (0, 1) with some conditions.

Question: Is it possible to develop an iteration process whose rate of convergence is even faster than the iteration processes defined above?

To answer this, we introduce the new iteration process as follows:

Let K be a convex subset of a CAT(0) space and $G\colon K \to K$ be any nonlinear mapping. For each $x_1 \in K$, the sequence $\{x_n\}$ in K is defined by

$$\begin{aligned} & x_1 \in K \\ & z_n = (1-\beta_n)x_n \oplus \beta_n G x_n \\ & y_n = G x_n \\ & x_{n+1} = G((1-\alpha_n)G z_n \oplus \alpha_n G y_n), \end{aligned} \tag{7}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in (0, 1). Motivated by what has been said above, in this research work, we establish some strong and Δ-convergence theorems of the new three steps iterative algorithm for (α, β)-generalized hybrid mapping in the setting of CAT(0). Our results are improved and generalized form of the earlier results.

2. Preliminaries

This section contains some well-known concepts and results that will be referenced throughout the paper.

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a mapping K from a closed interval $[0, r] \subset R$ to X such that $c(0) = x$, $c(r) = y$, $d(c(t), c(s)) = |t - s|$ for all $s, t \in [0, r]$. In particular, K is an isometry and $d(x, y) = r$. The image of K is call a geodesic segment (or metric segment) joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. We denote the point $w \in [x, y]$ such that $d(x, w) = \alpha d(x, y)$ by $w = (1-\alpha)x \oplus \alpha y$, where $\alpha \in [0, 1]$.

The space (X, d) is called a geodesic space if any two points of X are joined by a geodesic and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $D \subseteq X$ is said to be convex if D includes geodesic segment joining every two points of itself. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consist of three points (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for geodesic triangle (or $\Delta(x_1, x_2, x_3)$) in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) = \Delta(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in the Euclidean plane $\mathbb{R}^2$ such that

$$d_{\mathbb{R}^2}(\bar{x_i}, \bar{x_j}) = d(x_i, x_j)$$

for $i, j \in \{1, 2, 3\}$. A geodesic metric space is said to be a CAT(0) space if all geodesic triangle of appropriate size satisfy the following CAT(0) comparison axiom. Let Δ be a geodesic triangle in C and let $\bar{\Delta} \subset \mathbb{R}^2$ be comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) inequality if for all

$x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$,

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y}).$$

If x, y_1, y_2 are points of a CAT(0) space and y_0 is the midpoint of the segment $[y_1, y_2]$ which we will denote by $(y_1 \oplus y_2)/2$, then the CAT(0) inequality implies

$$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2).$$

this inequality is the (CN) inequality of Bruhat and Tits [4]. In fact, a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality. It is well known that all complete, simply combined Riemannian manifold having non-positive section curvature is a CAT(0) space. For other examples, Euclidean buildings ([3]), Pre-Hilbert spaces, $\mathbb{R}$-trees ([2]), the complex Hilbert ball with a hyperbolic metric [8] is a CAT(0) space. Further, Complete CAT(0) spaces are called Hadamard spaces. Now, we give some elementary properties about CAT(0) spaces as follows:

Lemma 2.1 ([7]). *Let X be a CAT(0) space. Then, for any $x, y, z \in X$ and $\lambda \in [0, 1]$,*

$$d((1-\lambda)x \oplus \lambda y, z) \leq (1-\lambda)d(x, z) + \lambda d(y, z).$$

Lemma 2.2 ([7]). *Let X be a CAT(0) space. Then, for any $x, y, z \in X$ and $\lambda \in [0, 1]$,*

$$d^2((1-\lambda)x \oplus \lambda y, z) \leq (1-\lambda)d^2(x, z) + \lambda d^2(y, z) - \lambda(1-\lambda)d^2(x, y).$$

In 2008, Kirk and Panyanak [15] specialized Lim's concept of the Δ-convergence in a general metric space [18] to CAT(0) spaces and proved some weak convergence theorems in Banach spaces by using the Δ-convergence. Now, we introduce some basic properties and the concept of the Δ-convergence. Let $\{x_n\}$ be a bounded sequence in a CAT(0) space (X, d). For any $x \in X$, we put

$$r(x, \{x_n\}) = \lim_{n\to\infty} sup\ d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ is given by

$$r(\{x_n\}) = inf\{r(x, x_n) \colon x \in X\}.$$

The asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined as:

$$A(\{x_n\}) = \{x \in X \colon r(x, x_n) = r(\{x_n\})\}.$$

$A(\{x_n\})$ consists of exactly one point in CAT(0) spaces see ([7], Proposition 7). In 2008, Kirk and

Panyanak [15] gave the following definition of Δ-convergence.

Definition 2.3 ([15]). *A sequence $\{x_n\}$ in a given complete CAT(0) space X is said to Δ-converges to $x \in X$ if x is the unique asymptotic center for every subsequence $\{a_n\}$ of $\{x_n\}$. In this case we write $\Delta - lim_n x_n = x$ and read as x is the $\Delta - limit$ of $\{x_n\}$.*

Note that it is impossible to formulate the concept of the demi-closedness in CAT(0) spaces as in linear spaces, but, if K is a closed convex subset of a CAT(0) space X then, for any mapping $G\colon K \to K$, $I - G$ is said to be demi-closed at zero if a sequence $\{x_n\}$ is a bounded in K such that $\{x_n\}$ is Δ-convergent to x and $d(x_n, Gx_n) \to 0$, then $x \in F(G)$.

Lemma 2.4 ([15]). *Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence.*

Lemma 2.5 ([5]). *If K is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in K, then the asymptotic center of $\{x_n\}$ is in K.*

Definition 2.6 ([25]). *A mapping $G\colon K \to K$ is said to satisfy Condition (I) if there exists a nondecreasing function $f\colon [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that*

$$d(x, Gx) \geq f(d(x, F(G))), \tag{8}$$

for all $x \in K$.

Definition 2.7 ([24]). *Let (X, d) be a metric space and K be a nonempty subset of X. Then a mapping $G\colon K \to K$ is said to be semi-compact if, for any sequence $\{x_n\}$ in K with $\lim_{n\to\infty} d(x_n, Gx_n) = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$.*

In 2011, Lin et al. [19] proved some existence theorems for (α, β)-generalized hybrid mappings in complete CAT(0) spaces as follows:

Lemma 2.8 ([19]). *Let (X, d) be a CAT(0) space and K be a nonempty closed convex subset of X. Let $G\colon K \to K$ be an (α, β)-generalized hybrid mapping such that $\alpha \geq 1$ and $\beta \geq 0$. Then G has a fixed point in K.*

Lemma 2.9 ([19]). *Let K be a nonempty subset of a given CAT(0) space. If $G\colon K \to K$ has the (α, β)-generalized hybrid mapping. Then the set $F(G)$ always closed.*

In 2014, Wangkeeree and Preechasilp [32] obtained the demi-closedness principle of (α, β)-generalized hybrid mappings in complete CAT(0) spaces as follows:

Lemma 2.10 ([22]). *Let (X, d) be a CAT(0) space and K be a nonempty closed convex subset of X. Let $G\colon K \to K$ be an (α, β)-generalized hybrid mapping such that $\alpha \geq 1$ and $\beta \geq 0$. If $\{x_n\}$ be a bounded sequence in K such that $\lim_{n\to\infty} d(x_n, Gx_n) = 0$ and $\{x_n\}$ is Δ-convergent to p, then $G(p) = p$.*

3. Main Results

In this section, we discuss the concept of (α, β)-generalized hybrid mappings and some of their basic properties.

Theorem 3.1. *Let (X, d) be a complete CAT(0) space and K be a nonempty closed convex subset of X. Let $G\colon K \to K$ be an (α, β)-generalized hybrid mapping such that $\alpha \geq 1$ and $\beta \geq 0$. Let $\{x_n\}$ be the sequence in K defined by (7), where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in (0, 1) such that*

$$\lim_{n\to\infty}(1-\alpha_n)\beta_n(1-\beta_n) \neq 0.$$

Then $\lim_{n\to\infty} d(x_n, p)$ exists and $\{x_n\}$ is an approximate fixed point sequence for G.

Proof. By Definition 1.1, $F(G) \neq \emptyset$ and G is quasi-nonexpansive mapping. Thus, for all $p \in F(G)$, we have

$$\begin{aligned} d(z_n, p) &= d((1-\beta_n)x_n \oplus \beta_n Gx_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(Gx_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(x_n, p) \\ &\leq d(x_n, p). \end{aligned} \tag{9}$$

Using (9) and Definition 1.1, we have

$$\begin{aligned} d(y_n, p) &= d(Gx_n, p) \\ &\leq d(x_n, p). \end{aligned} \tag{10}$$

Using (10) and Definition 1.1, we have

$$\begin{aligned} d(x_{n+1}, p) &= d(G((1-\alpha_n)Gz_n \oplus \alpha_n Gy_n), p) \\ &\leq d((1-\alpha_n)Gz_n \oplus \alpha_n Gy_n, p) \\ &\leq (1-\alpha_n)d(Gz_n, p) + \alpha_n d(Gy_n, p) \\ &\leq (1-\alpha_n)d(z_n, p) + \alpha_n d(y_n, p) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(x_n, p) \\ &\leq d(x_n, p). \end{aligned} \tag{11}$$

Hence $\{x_n\}$ is bounded so $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(G)$. Let $\lim_{n\to\infty} d(x_n, p) = r$. From (11), we have

$$\lim_{n\to\infty} d(x_{n+1}, p) = r. \tag{12}$$

On the other hand, we have

$$\begin{aligned} d^2(z_n,p) &= d^2((1-\beta_n)x_n \oplus \beta_n G x_n, p) \\ &\leq (1-\beta_n)d^2(x_n,p) + \beta_n d^2(Gx_n,p) - \beta_n(1-\beta_n)d^2(x_n,Gx_n) \\ &\leq (1-\beta_n)d^2(x_n,p) + \beta_n d^2(x_n,p) - \beta_n(1-\beta_n)d^2(x_n,Gx_n) \\ &\leq d^2(x_n,p) - \beta_n(1-\beta_n)d^2(x_n,Gx_n). \end{aligned} \tag{13}$$

and

$$\begin{aligned} d^2(y_n,p) &= d^2(Gx_n,p) \\ &\leq d^2(x_n,p). \end{aligned} \tag{14}$$

From (13) and (14), we obtain

$$\begin{aligned} d^2(x_{n+1},p) &= d^2(G((1-\alpha_n)Gz_n \oplus \alpha_n G y_n), p) \\ &\leq d^2((1-\alpha_n)Gz_n \oplus \alpha_n G y_n, p) \\ &\leq (1-\alpha_n)d^2(Gz_n,p) + \alpha_n d^2(Gy_n,p) - \alpha_n(1-\alpha_n)d^2(Gz_n,Gy_n) \\ &\leq (1-\alpha_n)d^2(z_n,p) + \alpha_n d^2(y_n,p) - \alpha_n(1-\alpha_n)d^2(Gz_n,Gy_n) \\ &\leq (1-\alpha_n)d^2(z_n,p) + \alpha_n d^2(y_n,p) \\ &\leq (1-\alpha_n)[d^2(x_n,p) - \beta_n(1-\beta_n)d^2(x_n,Gx_n)] + \alpha_n d^2(x_n,p) \\ &\leq d^2(x_n,p) - (1-\alpha_n)\beta_n(1-\beta_n)d^2(x_n,Gx_n). \end{aligned} \tag{15}$$

which implies that

$$(1-\alpha_n)\beta_n(1-\beta_n)d^2(x_n,Gx_n) \leq d^2(x_n,p) - d^2(x_{n+1},p).$$

Therefore, since $\lim_{n\to\infty}(1-\alpha_n)\beta_n(1-\beta_n) \neq 0$ and $\lim_{n\to\infty} d(x_n,p) = r$, we obtain

$$\lim_{n\to\infty} d(x_n,Gx_n) = 0.$$

This completes the proof. □

Theorem 3.2. *Let (X,d) be a complete CAT(0) space and K be a nonempty closed convex subset of X. Let $G\colon K \to K$ be an (α,β)-generalized hybrid mapping such that $\alpha \geq 1$, $\beta \geq 0$ and $F(G) \neq \emptyset$. Let $\{x_n\}$ be the sequence in K defined by (7), where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in (0, 1) such that*

$$\lim_{n\to\infty}(1-\alpha_n)\beta_n(1-\beta_n) \neq 0.$$

Then $\{x_n\}$ is Δ-converges to a fixed point of G.

Proof. By Theorem 3.1, the sequence $\{x_n\}$ is bounded. Hence one can take $A(\{x_n\}) = \{c\}$ for some

$c \in K$. We are going to prove $A(\{x_n\}) = \{c\}$ for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Suppose $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $A(\{x_n\}) = \{c\}$ Since $\{x_{n_k}\}$ is bounded, one can find a subsequence $\{x_{n_j}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_j}\}$ Δ-converges to p for some $p \in K$. By Theorem 3.1, Lemma 2.10 one has $p \in F(G)$ and hence $\lim_{n\to\infty} sup\ d(x_n, p)$ exists. If $p \neq x$, then the singletoness of the cardinality of the asymptotic centers allows us the following

$$\begin{aligned}\lim_{n\to\infty} sup\ d(x_n,p) &= \lim_{j\to\infty} sup\ d(x_{n_j},p) < \lim_{j\to\infty} sup\ d(x_{n_j},x)\\ &\leq \lim_{k\to\infty} sup\ d(x_{n_k},x) < \lim_{k\to\infty} sup\ d(x_{n_k},p)\\ &= \lim_{n\to\infty} sup\ d(x_n,p),\end{aligned} \tag{16}$$

which is contradiction. Therefore, $x = p \in F(G)$. Suppose that $x \neq c$. Then

$$\begin{aligned}\lim_{n\to\infty} sup\ d(x_n,x) &= \lim_{k\to\infty} sup\ d(x_{n_k},x) \leq \lim_{k\to\infty} sup\ d(x_{n_k},c)\\ &\leq \lim_{n\to\infty} sup\ d(x_m,c) < \lim_{n\to\infty} sup\ d(x_m,x)\\ &= \lim_{n\to\infty} sup\ d(x_n,x).\end{aligned} \tag{17}$$

$\{x_n\}$ Δ-converges to an element $c \in F(G)$. □

The following result establishes strong convergence for operators with condition (α,β)-generalized hybrid mapping in CAT(0) spaces under new iterations.

Theorem 3.3. *Let (X, d) be a complete CAT(0) space and K be a nonempty closed convex subset of X. Let $G\colon K \to K$ be an (α,β)-generalized hybrid mapping such that $\alpha \geq 1$, $\beta \geq 0$ and $F(G) \neq \emptyset$. Let $\{x_n\}$ be the sequence in K defined by (7), where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in (0, 1) such that*

$$\lim_{n\to\infty}(1-\alpha_n)\beta_n(1-\beta_n) \neq 0.$$

Then $\{x_n\}$ converges strongly to a fixed point of G if and only if $d(x_n, F(G)) = 0$.

Proof. The necessity is obvious. Next, we prove sufficiency. By Theorem 3.1, $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F(G)$. Thus, $\lim_{n\to\infty} d(x_n, F(G)$ exists. Hence

$$\lim_{n\to\infty} d(x_n, F(G)) = 0. \tag{18}$$

Now, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{x_j\}$ in $F(G)$ such that $d(x_{n_j}, x_j) \leq \frac{1}{2^j}$ for all $j \in \mathbb{N}$. From the proof of Theorem 3.1, we have

$$\begin{aligned}d(x_{n_{j+1}}, x_j) &\leq d(x_{n_j}, x_j)\\ &\leq \frac{1}{2^j}.\end{aligned} \tag{19}$$

The next step is to demonstrate that the sequence $\{x_j\}$ is a Cauchy sequence in $F(G)$. For this, we consider the following

$$\begin{aligned} d(x_{n_{j+1}}, x_j) &\leq d(x_{j+1}, x_{n_{j+1}}) + d(x_{n_{j+1}}, x_j). \\ &\leq \frac{1}{2^{j+1}} + \frac{1}{2^j}. \\ &\leq \frac{1}{2^{j-1}} \to 0 \; as \; j \to \infty. \end{aligned} \tag{20}$$

The above limit showes that the sequence x_j is a Cauchy sequence in the set $F(G)$. By Lemma 2.9, the set $F(G)$ is closed. Hence $x_j \to x$ for some $x \in F(G)$. By theorem 3.1, $\lim_{n\to\infty} d(x_n, x)$ exists. So the proof is completed. □

In this section, we now establish the final result, which is related to condition (I).

Theorem 3.4. *Let (X, d) be a complete CAT(0) space and K be a nonempty closed convex subset of X. Let $G\colon K \to K$ be an (α,β)-generalized hybrid mapping such that $\alpha \geq 1$, $\beta \geq 0$ and $F(G) \neq \emptyset$. Let $\{x_n\}$ be the sequence in K defined by (7), where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in (0, 1) such that*

$$\lim_{n\to\infty} (1-\alpha_n)\beta_n(1-\beta_n) \neq 0.$$

Ii the mapping G satisfy condition (I), then $\{x_n\}$ is converges strongly to a fixed point of G.

Proof. By Theorem 3.1, $\lim_{n\to\infty} d(x_n, F(G))$ exists and also $\lim_{n\to\infty} d(x_n, Gx_n) = 0$. It follows from condition (I) that

$$\lim_{n\to\infty} f(d(x_n, F(G))) \leq \lim_{n\to\infty} d(x_n, Gx_n) = 0,$$

which shows $\lim_{n\to\infty} f(d(x_n, F(G))) = 0$. Thus, by Definition 2.6, we obtain

$$\lim_{n\to\infty} d(x_n, F(G)) = 0.$$

Thus the conclusion follows from Theorem 3.3. This complete the proof. □

4. Numerical Example

Let X be an $\mathbb{R}$-tree with the radial meter d_r. We put

$$K = \left\{(t,0)\colon t \in [0,2] \cup [4, 5\frac{1}{2}]\right\} \cup \left\{(0,t)\colon t \in [0,2] \cup [4, 5\frac{1}{2}]\right\} \subset \mathbb{R}^2$$

and define $G\colon K \to K$ by

$$G(t,0) = \begin{cases} (0,0) & if \quad t \in [0,2] \\ \left(0, \frac{(t-4)^2}{2}\right) & if \quad t \in [4, 5\frac{1}{2}] \end{cases} \quad and \; G(0,t) = \begin{cases} (0,0) & if \quad t \in [0,2] \\ \left(0, \frac{(t-4)^2}{2}\right) & if \quad t \in [4, 5\frac{1}{2}]. \end{cases}$$

Clearly $F(G) = \{(0,0)\}$. Hence, G satisfy (1,1)-generalized hybrid mapping but G is not nonexpansive mapping.

5. Conclusion

In this paper, we have presented a new three-step iteration procedure for (α, β)-generalized hybrid mapping in CAT(0) spaces. Our result generalizes results of Pakkaranang et al. [22] and Wangkeeree et al. [32].

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