Study of Forchheimer, Ohmic, Joule Effects and Variable Viscosity on Magnetohydrodynamic Viscous Flow Aacross the Slendering Stretching Surface

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Abstract: The boundary layer viscous flow across the slendering sheet has caught considerable attention due to its wide applications in microfluidics, duct design, nano-fluids and nano-devices and plastics molding, and many other fields. Noticing this we have examined the two dimensional magnetohydrodynamic boundary layer viscous flow across the slendering sheet under the influence of Forchheimer, Ohmic, Joules effect and as well as the variable viscosity effect on the fluid flow. Similarity transformations are used to simplify the partial differential equations to ordinary differential equations which are further solved analytically using the generalized homotopy method. The computational results without these effects agree excellently with the previous results by [11].

1.   Introduction

The behavior of laminar boundary layer flow under the effect of heat transfer has been a tantalizing concept for several researchers owing to its diverse applications not only in the fields of microfluidics, duct design, nano-fluids and nano-devices and plastics molding but also in hydraulics and in hemodynamics for the study of blood flow. It is widely used in nuclear reactor technology and also in aerodynamics, in the study of aircraft design in order to analyse the prospects of enhancing speed and proficiency of the aircraft.

In this paper we study the laminar boundary layer flow across a slendering stretching sheet when subjected to the Joules effect. The laminar flow across a slendering stretching sheet has got extravagant attention due to its far-reaching applications in acoustical components, in nuclear science,chemical and manufacturing procedures; for example, in polymer extrusion, and machine design. Hayat [6] investigated via boundary layer access, Brownian motion as well as thermophoresis phenomena, the (magnetohydrodynamic) MHD boundary layer flow of Powell- Eyring nano-fluid past a non-linear stretching sheet of variable thickness. An electrically conducting fluid was considered under the characteristics of magnetic field applied transverse to the sheet. Hayat [8] also examined the magnetohydrodynamic (MHD) three-dimensional nonlinear convective flow of Maxwell nanofluid towards a stretching surface. Simultaneous effects of heat transfer of this flow under thermal radiation, heat generation/absorption and prescribed heat flux condition were examined using a nanofluid model that included Brownian

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motion and thermophoresis. Kandasamy [4] probed into the thermal and solutal stratification on heat and mass transfer induced due to a nanofluid over a porous vertical plate. The effect of Brownian motion and thermophoresis particle deposition were included in the corresponding transport equations of the study.

Reddy [5] on the other hand provided a similarity solution for the hydromagnetic motion of a nano-fluid over a slendering stretching sheet, in the presence of thermophoresis and Brownian movement. They modelled the unsteady MHD nano-fluid flow over a slendering stretching surface with slip effects under thermophoresis and brownian motion effects. This notion of diminishing the governing partial differential equations using suitable similarity transformations, into ordinary differential equations facilitates in making our analysis uncomplicated. Srinivas [7] studied under the assumption that the sheet is non-flat, the magneto hydrodynamic boundary layer flow with heat and mass transfer of the Williamson nano-fluid over a stretching sheet with variable thickness and variable thermal conductivity under the radiation effect. The governing partial differential equations here were for this objective broken down using suitable similarity transformations, into nonlinear coupled ordinary differential equations.

Bilal [9] studied the three dimensional MHD upper-convected Maxwell nano-fluid flow with nonlinear radiative heat flux. They examined the effects of nanoparticles and magnetohydrodynamics (MHD) on heat and mass transfer by incorporating a nonlinear radiative heat flux in the formulation of energy equation. Similarity transformations were again employed to chop down the nonlinear partial differential equations of the problem to the ordinary differential equations, which were then solved by the well-known shooting technique through Runge-Kutta integration procedure of order four. Sandeep [10] analyzed the momentum and heat transfer behavior of MHD nano-fluid embedded with conducting dust particles past a stretching surface in the presence of volume fraction of dust particles. The governing equations of the flow and heat transfer were for this purpose transformed into nonlinear ordinary differential equations using similarity transformation and then solved numerically using RungeKutta based shooting technique. Pandey [2] studied the effects of viscous dissipation and suction/injection on MHD flow of a nano-fluid past a wedge with convective surface in the appearance of slip flow and porous medium by again altering the basic non-linear PDEs of flow and energy into a set of non-linear ODEs using auxiliary similarity transformations. The eventualizing system of equations together with the coupled boundary conditions were solved numerically by applying Runge-Kutta-Fehlberg procedure via shooting scheme.

The influence of relevant parameters on non-dimensional velocity and temperature profiles were also investigated. Daniel [3] investigated the unsteady mixed convection flow of electrical conducting nano-fluid and heat transfer due to a permeable linear stretching sheet with the combined effects of an electric field, magnetic field, thermal radiation, viscous dissipation, and chemical reaction. A similarity transformation was applied, for this intention to transform the constitutive equations into a system of nonlinear ordinary differential equations so that the arising system of equations could then be solved numerically using implicit finite difference method. This facilitated the entropy analysis in electrical magnetohydrodynamic flow of nano-fluid with many of the substantial effects mentioned above. Prasad [1] later analysed using a slightly distinct technique the heat and mass transfer analysis for the MHD flow of nano fluid with radiation absorption. They studied the effects of Diffusion, thermo, radiation absorption and chemical reaction on MHD free convective heat and mass transfer flow of a nano-fluid encompassed by a semi-infinite flat plate, using two types of nano-fluids namely Cu-water nanofluid and TiO2-water nano-fluid.

The plate was moved with a constant velocity and temperature and the concentration are assumed to be fluctuating with time harmonically from a constant mean at the plate. The analytical solutions of the boundary layer equations are assumed of oscillatory type and are solved by using the small perturbation technique. Babu [11] have also done comparable studies on the magnetohydrodynamic dissipative flow across the slendering stretching sheet with variable viscosity debased on temperature. We have involved the nonlinear term of the Forchheimer effect in the momentum equation, and the Ohmic and

Joule effects in the energy equation, which admit significant industrial applications and are apt for mathematical modelling. Such an investigation of this system has not been given attention previously as per our knowledge.

1.1. Nomenclature

A	non-dimensional variable viscosity parameter
u, v	velocity components in x, y directions
f	dimensionless velocity of the fluid
N	coefficient related to stretching sheet
n	velocity power index parameter
c	physical parameter related to stretching sheet
$B(x)$	magnetic field parameter
T	temperature of the fluid (K)
k	thermal conductivity $(Wm^{-1}K)$
k_0	chemical reaction parameter
U_w	stretching velocity of the sheet (ms^{-1})
C	concentration of the fluid (kgm^{-2})
C_p	specific heat at constant pressure $(JkgK^{-1})$
T	temperature of the fluid in the free stream (K)
C	concentration of the fluid in the free stream (kgm^2)
Pr	Prandtl number
B_0	magnetic field strength
$B(x)$	dimensional magnetic field parameter
M	magnetic field parameter
Sc	Schmidt number
Kr	dimensionless chemical reaction parameter
h_1	heat transfer coefficient
h_2	concentration transfer coefficient
a_1, b_1	constants
E_c	Eckert number
C_f	skin friction coefficient
Nu_x	local Nusselt number
Sh_x	local Sherwood number

Greek Symbols

ϕ	dimensionless concentration
c	similarity variable
σ	electrical conductivity of the fluid (mXm^{-1})
θ	dimensionless temperature
ρ	density of the fluid (kgm^{-3})
λ	wall thickness parameter
μ	dimensional variable viscosity parameter
μ^*	constant value of the coefficient of viscosity far away from sheet

μ	kinematic viscosity $(m^2 s^{-1})$
β_1	heat transfer Biot number
β_2	mass transfer Biot number

2.   Mathematical Formulation

We examine a steady, laminar, two-dimensional Magnetohydrodynamic flow of an electrically conducting fluid across a slendering stretching sheet. We deem the convective boundary conditions in variable viscosity and viscous dissipation parameters. Here our presumptions are $U_w(x) = b(x+c)^n$ and $y = N(x+c)^{\frac{1-n}{2}}$. We foreknow the magnetic Reynolds number as low as conceivable to disregard the induced magnetic field. Thereby, we can bypass the induced magnetic field in our work. Based on the above suppositions, the governing equations are,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) - \sigma B^2(x)u + \frac{c_b}{\rho_0\sqrt{k}}u^2, \tag{2}$$

$$\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2} - \frac{\mu}{\rho c_p}(T_\infty\left(\frac{\partial u}{\partial y}\right)^2) - \frac{c_b}{\rho_0 c_p}u^2 + \frac{\mu}{\rho_0 c_p k}u^2 \tag{3}$$

$$\left(u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y}\right) = D_m\frac{\partial^2 C}{\partial y^2} - k_0(x)(C - C_\infty), \tag{4}$$

with the boundary conditions,

$$u(x,y) = U_w(x), v(x,y) = 0, -k\frac{\partial T}{\partial y} = h_1(T_w - T), -D_m\frac{\partial C}{\partial y} = h_2(C_w - C) \text{ at } y = 0, u = 0, T = T_\infty, C = C_\infty \text{ at } y = \infty, \tag{5}$$

where

$$\mu(T) = \mu[a_1 + b_1(1-\theta)(T_w - T_\infty)], B(x) = B_0(x+c)^{\frac{1-n}{2}}, T_w = T + T_0(x+c)^{1-n2}, C_w = C_\infty + C_0(x+c)^{\frac{1-n}{2}}\rho \tag{6}$$

We use the following similarity transformations,

$$\zeta = y\sqrt{\left(\frac{n+1}{2}\right)}\left(\frac{b}{-}\right)(x+c)^{\frac{1-n}{2}},$$

$$u = b(x+c)^n\frac{df}{d\zeta},$$

$$v = -\sqrt{\frac{n+1}{2}v(x+c)^{\frac{1-n}{2}}}\left[\zeta\frac{df}{d\zeta}\frac{n-1}{n+1} + f(\zeta)\right], \tag{7}$$

$$T = T_\infty + (T_w - T_\infty)\theta(\zeta), C = (C_\infty + C)\Phi(\zeta)$$

to convert the above PDE to the system of differential equations, i.e. substituting (7) in (1)-(6), we have

$$(a_1 + A(1-\theta))\frac{d^3 f}{(d\zeta^3} + f\frac{d^2 f}{d\zeta^2} - \frac{2M}{n+1}\frac{df}{d\zeta} - A\frac{d\theta}{d\zeta}\frac{d^2 f}{d\zeta^2} + \frac{2}{n+1}\left(\frac{c_b}{\rho\rho_0\sqrt{k}}\left(\frac{U_w(x)}{b}\right)^{\frac{1}{n}} - n\right)\left(\frac{df}{d\zeta}\right)^2 = 0, \tag{8}$$

$$\frac{d^2\theta}{d\zeta^2} + Pr\left\{f\frac{d\theta}{d\zeta} + Ec\left(\left(\frac{d^2\theta}{d\zeta^2}\right)^2 - \left(\frac{1-n}{n+1}\right)\left(\frac{df}{d\zeta}\right)\theta\right)\right\} + \frac{b^2(x+c)^{n+1}}{k}\left[\frac{\rho B_0^2}{\rho_0} - k\right]\left(\frac{df}{d\zeta}\right)^2 = 0, \tag{9}$$

$$\frac{d^2\phi}{d\zeta^2} + Sc\left\{f\frac{d\theta}{d\zeta} - \left(\frac{1-n}{n+1}\right)\frac{df}{d\zeta}\phi - \left(\frac{2}{n+1}\right)Kr\phi\right\} = 0 \tag{10}$$

with corresponding boundary conditions,

$$f(0) = 0, \frac{df}{d\zeta}\mid_{\zeta=0} = 1, \frac{d\theta}{d\zeta}\mid_{\zeta=0} = -\beta_1[1-\theta(0)], \beta_1 = -h_1\frac{y}{\zeta}, \frac{d\phi}{d\zeta}\mid_{\zeta=0} = -\beta_2[1-\phi(0)], \beta_2 = h_2\frac{y}{\zeta}, \frac{df}{d\zeta}\mid_{\zeta=\infty} = 0, \theta(\infty) = 0, \phi(\infty) = 0, \tag{11}$$

where,

$$A = b_1(T_w - T_\infty), Pr = \frac{\mu c_p}{k}, Ec = \frac{b^2(x+c)^{5n-1}}{c_p}, Sc = \frac{}{D_m}, Kr = \frac{(x+c)^{n-1}}{b}.$$

Considering expressions in the form,

$$K_1(x) = \frac{2}{n+1}\left(\frac{c_b}{\rho\rho_0\sqrt{k}}(x+c) - n\right), \quad K_2(x) = \frac{b^2(x+c)^{n+1}}{k}\left[\frac{\rho B_0^2}{\rho_0} - \frac{\mu}{k}\right],$$

$$A_1 = A + a_1, \ M_1 = \frac{2M}{n+1}, \ N_1 = \frac{1-n}{n+1}, \ N_2 = \frac{2}{n+1}, \ M = \frac{\sigma B_0}{\rho b}.$$

Establish the following homotopy equations,

$$(1-p)\left(A_1\frac{d^3f}{d\zeta^3} - M_1\frac{df}{d\zeta}\right) + p\left(A_1 - A\theta\right)\frac{d^3f}{d\zeta^3} + f\frac{d^2f}{d\zeta^2} - M_1\frac{df}{d\zeta} - A\frac{d\theta}{d\zeta}\frac{d^2f}{d\zeta^2} + K_1(x)\left(\frac{df}{d\zeta}\right)^2 = 0, \tag{12}$$

$$(1-p)\left(\frac{d^2\theta}{d\zeta^2}\right) - p\left(\frac{d^2\theta}{d\zeta^2} + Pr\left\{f\frac{d\theta}{d\zeta} + Ec\left(\frac{d^2\theta}{d\zeta^2}\right)^2 - N_1\left(\frac{df}{d\zeta}\right)\theta\right\}\right) + K_2(x)\left(\frac{df}{d\zeta}^2\right) = 0, \tag{13}$$

$$(1-p)\left[\frac{d^2\phi}{d\zeta^2} - ScN_2Kr\phi\right] - p\left[\frac{d^2\phi}{d\zeta^2} + Sc\left\{f\frac{d\theta}{d\zeta} - N_1\frac{df}{d\zeta}\phi - N_2Kr\phi\right\}\right] = 0. \tag{14}$$

According to the generalized homotopy method, assume the solution for (12), (13) and (14) in the form

$$f_p = f_0 + ln[1 + f_1p + f_2p^2 + f_3p^3 + \ldots], \tag{15}$$

$$\theta_p = \theta_0 + ln[1 + \theta_1p + f_2p^2 + \theta_3p^3 + \ldots], \tag{16}$$

$$\phi_p = \phi_0 + ln[1 + \phi_1p + \phi_2p^2 + \phi_3p^3 + \ldots] \tag{17}$$

Taking Taylor's series of (15), (16), (17),

$$f_p = f_0 + f_1p + (f_2 - \frac{f_1^2}{2})p^2 + (f_3 - \frac{f_1f_2}{3} - \frac{f_1(-2f_2 + f_1^2)}{3})p^3 + \ldots \tag{18}$$

$$\theta_p = \theta_0 + \theta_1p + (\theta_2 - \frac{\theta_1^2}{2})p^2 + (\theta_{(3)} - \frac{\theta_1\theta_2}{3} - \frac{\theta_1(-2\theta_2 + \theta_1^2)}{3})p^3 + \ldots \tag{19}$$

$$\phi_p = \phi_0 + \phi_1p + (\phi_2 - \frac{\phi_1^2}{2})p^2 + (\phi_{(3)} - \frac{\phi_1\phi_2}{3} - \frac{\phi_1(-2\phi_2 + \phi_1^2)}{3})p^3 + \ldots \tag{20}$$

Substituting (18 , (19), (20) into (8), (9), (10) and rearranging the terms of order p, we have, concerning f,

$$A_1\frac{d^3f_0}{d\zeta^3} - M_1\frac{df_0}{d\zeta} = 0, \tag{21}$$

$$A\frac{d^3f_0}{d\zeta^3} + \frac{d^3f_0}{d\zeta^3}(A\theta_0 - 2A) - \frac{d^2f_0}{d\zeta}(\frac{Ad\theta_0}{d\zeta} + f_0) - M_1\frac{df_1}{d\zeta} - 2M_1\frac{df_0}{d\zeta} + K_1(x)(\frac{df_0}{d\zeta})^2 = 0, \tag{22}$$

$$A\frac{d^3f_2}{d\zeta^3} - 3A\frac{d^2f_1}{d\zeta^2}\frac{df_1}{d\zeta} - Af_1\frac{d^3f_1}{d\zeta^3} - 2Ad^3f_1d\zeta^3 - M_1\frac{df_2}{d\zeta} + M_1f_1\frac{df_1}{d\zeta} + 2M_1\frac{df_1}{d\zeta} + A\frac{d^2f_1}{d\zeta^3}\theta_2 + A\frac{d^3f_0}{d\zeta}\theta_1$$

$$- f_1\frac{d^2f_0}{d\zeta^2} - - f_0\frac{d^2f_1}{d\zeta^2} + A\frac{d\theta_1}{d\zeta}\frac{d^2f_0}{d\zeta^2} + 2K_1(x)\frac{df_1}{d\zeta}\frac{df_0}{d\zeta} = 0. \tag{23}$$

Concerning θ,

$$\frac{d^2\theta_0}{d\zeta^2} = 0, \tag{24}$$

$$-2\frac{d^2\theta_0}{d\zeta^2} - Prf_0\frac{d\theta_0}{d\zeta} - Pr(\frac{d^2f_0}{d\zeta^2})^2 - PrN_1\theta_0\frac{df_0}{d\zeta} - Kr\frac{df_0}{d\zeta} = 0, \tag{25}$$

$$\frac{d^2\theta_2}{d\zeta^2} - \frac{d^2\theta_1}{d\zeta^2}(\theta_1 + 2) + (\frac{d\theta_1}{d\zeta})^2 - Prf_0\frac{d\theta_1}{d\zeta} - Prf_1\frac{d\theta_0}{d\zeta} - 2PrEc\frac{d^2f_0}{d\zeta^2}\frac{d^2f_1}{d\zeta^2}$$

$$+PrN_1\frac{df_0}{d\zeta}\theta_1 + PrN_1\theta_0\frac{df_1}{d\zeta} - -2K_1(x)\frac{df_1}{d\zeta}\frac{df_0}{d\zeta} = 0. \tag{26}$$

Concerning Φ,

$$\frac{d^2\Phi_0}{d\zeta^2} - ScN_2Kr\Phi_0 = 0, \tag{27}$$

$$\frac{d^2\Phi_1}{d\zeta^2} - ScN_2Kr\Phi_1 - 2\frac{d^2\Phi_0}{d\zeta^2} - ScN_2Kr\Phi_0 + Scf_0\frac{d\Phi_0}{d\zeta} - ScN_1\Phi_0\frac{df_0}{d\zeta} = 0, \tag{28}$$

$$\frac{d^2\Phi_2}{d\zeta^2} - (\frac{d\Phi_1}{d\zeta})^2 - \Phi_1\frac{d^2\Phi_1}{d\zeta^2} - ScN_2Kr\Phi_0 + ScN_2Kr\frac{\Phi_1^2}{2} - 2\frac{d^2\Phi_1}{d\zeta^2} + ScN_2Kr\Phi_1 + f_0\frac{d\Phi_1}{d\zeta}Sc$$

$$+Scf_1\frac{d\Phi_0}{d\zeta} - ScN_1\frac{df_0}{d\zeta}\Phi_1 - ScN_1\Phi_0\frac{df_1}{d\zeta} = 0. \tag{29}$$

The boundary conditions become,

$$f_0(0) = 0, f_1(0) = 0, f_2(0) = 0, \frac{df_0}{d\zeta}(0) = 1, \frac{df_1}{d\zeta}(0) = 0, \frac{df_2}{d\zeta}(0) = 0, \frac{df_1}{d\zeta}(\infty) = 0, \frac{df_2}{d\zeta}(\infty) = 0,$$

$$\theta_1(0) = -\beta_1, \theta_2(0) = 0, \theta_0(\infty) = 0, \theta_1(\infty) = 0, \theta_2(\infty) = 0, \tag{30}$$

$$\Phi_1(0) = -\beta_2 + \beta_2\Phi_0(0), \Phi_2(0) = \frac{\beta_2\Phi_1(0) + \Phi_1^2(0)}{2}, \Phi_0(\infty) = 0, \Phi_1(\infty) = 0, \Phi_2(\infty) = 0,$$

The solutions to (21)-(29) are,

$$f_0 = \left(-1 + \exp\left(-\frac{M_1\zeta}{A}\right)\right),$$

$$f_1 = c_6 + c_8\exp\left(-\sqrt{\frac{M_1}{A_1}}\zeta\right) + L_3\exp\left(-2\sqrt{\frac{M_1}{A_1}}\zeta\right) - L_4x\exp\left(-\sqrt{\frac{M_1}{A_1}}\zeta\right)$$

$$f_2 = c_{12} + (c_14 + T_1)\exp\left(-\sqrt{\frac{M_1}{A_1}}\zeta\right) + T_2\exp\left(-2\sqrt{\frac{M_1}{A_1}}\zeta\right) + T_3\exp\left(-3\sqrt{\frac{M_1}{A_1}}\zeta\right) + T_4\exp\left(-4\sqrt{\frac{M_1}{A_1}}\zeta\right)$$

$$+ T_5\exp\left(-5\sqrt{\frac{M_1}{A_1}}\zeta\right) + T_6x\exp\left(-2\sqrt{\frac{M_1}{A_1}}\zeta\right) + T_7x^2\exp\left(-2\sqrt{\frac{M_1}{A_1}}\zeta\right) + T_0x\exp\left(-3\sqrt{\frac{M_1}{A_1}}\zeta\right)$$

$$+ T_8xexp\left(-\sqrt{\frac{M_1}{A_1}}\zeta\right) + T_9x^2\exp\left(-\sqrt{\frac{M_1}{A_1}}\zeta\right) + T_{10}x^2\exp\left(-3\sqrt{\frac{M_1}{A_1}}\zeta\right) + T_{11}x\exp\left(-4\sqrt{\frac{M_1}{A_1}}\zeta\right),$$

$$\theta_0 = 0,$$

$$\theta_1 = K_3\exp\left(-2\sqrt{\frac{M_1}{A_1}}\zeta\right),$$

$$\theta_2 = \exp\left(-3\sqrt{\frac{M_1}{A_1}}\zeta\right)L_{10} + L_{11}\exp\left(-2\sqrt{\frac{M_1}{A_1}}\zeta\right) + L_10^*\exp\left(2\sqrt{\frac{M_1}{A_1}}\zeta\right) + L_{11}^*x\exp\left(2\sqrt{\frac{M_1}{A_1}}\zeta\right),$$

$$\Phi_0 = c_11\exp\left(-\sqrt{ScN_2Kr}\zeta\right),$$

$$\Phi_1 = \exp\left(-\sqrt{ScN_2Kr}\zeta\right)\left(c_{13} + L_14 + L_15\exp\left(\sqrt{\frac{M_1}{A_1}}\right)\zeta\right),$$

$$\Phi_2 = d_2\exp\left(-\sqrt{ScN_2Kr}\zeta\right) + P_9\exp\left(-2\sqrt{ScN_2Kr}\zeta\right) + P_{10}\exp\left(-\left(2\sqrt{\frac{M_1}{A_1}}\right) + 2\sqrt{ScN_2Kr}\right)\zeta$$

$$+ P_{11}exp\left(-\left(\sqrt{\frac{M_1}{A_1}}\right) + 2\sqrt{ScN_2Kr}\right)\zeta + P_{12}\exp\left(-\left(\sqrt{\frac{M_1}{A_1}}\right) + \sqrt{ScN_2Kr}\right)\zeta$$

$$+ P_{13}x\exp\left(-\left(\sqrt{\frac{M_1}{A_1}}\right) + \sqrt{ScN_2Kr}\right)\zeta + P_{14}\exp\left(-2\sqrt{\frac{M_1}{A}}\zeta\right) + P_{15}\exp\left(-\left(2\sqrt{\frac{M_1}{A}} + \sqrt{ScN_2Kr}\right)\zeta\right)$$

$$+ P_{16}\exp\left(-\left(3\sqrt{\frac{M_1}{A}} + 2\sqrt{ScN_2Kr}\right)\zeta\right).$$

Now the solutions to (8), (9), (10) satisfying (11) is given by

$$f = \lim_{p \to 1} f_p = f_0 + log(1 + f_1 + f_2 + f_3 + f_4 + f_5 + \dots),$$

$$\theta = \lim_{p \to 1} \theta_p = \theta_0 + log(1 + \theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 + \dots),$$

$$\Phi = \lim_{p \to 1} \Phi_p = \Phi_0 + log(1 + \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + \Phi_5 + \dots).$$

So we can write the first and second approximations to f, θ, Φ, respectively as,

$$f = f_0 + log(1 + f_1) = \left(-1 + \exp\left(-\frac{M_1\zeta}{A}\right)\right) + log\left(1 + c_6 + c_8 \exp\left(-\sqrt{\frac{M_1}{A_1}}\zeta\right)\right)$$

$$+ L_3 \exp\left(-2\sqrt{\frac{M_1}{A_1}}\zeta\right) - L_4 x \exp\left(-\sqrt{\frac{M_1}{A_1}}\zeta\right) \tag{31}$$

$$f = f_0 + log\left(1 + f_1 + f_2\right) = \left(-1 + \exp\left(\frac{M_1\zeta}{A}\right)\right) + log\left(1 + c_6 + c_1 2 + c_8 \exp\left(-\sqrt{\frac{M_1}{A_1}}\zeta\right)\right)$$

$$+ (L_3 + T_2) \exp\left(-2\sqrt{\frac{M_1}{A_1}}\zeta\right) - L_4 x \exp\left(-\sqrt{\frac{M_1}{A_1}}\zeta\right) + (c_1 4 + T_1) \exp\left(-\sqrt{\frac{M_1}{A_1}}\zeta\right) + T_5 \exp\left(-5\sqrt{\frac{M_1}{A_1}}\zeta\right)$$

$$+ T_6 x \exp\left(-2\sqrt{\frac{M_1}{A_1}}\zeta\right) + T_7 x^2 \exp\left(-2\sqrt{\frac{M_1}{A_1}}\zeta\right) + T_0 x \exp\left(-3\sqrt{\frac{M_1}{A_1}}\zeta\right) + T_8 x \exp\left(-\sqrt{\frac{M_1}{A_1}}\zeta\right)$$

$$+ T_9 x^2 \exp\left(-\sqrt{\frac{M_1}{A_1}}\zeta\right) + T_{10} x^2 \exp\left(-3\sqrt{\frac{M_1}{A_1}}\zeta\right) + T_{11} x \exp\left(-4\sqrt{\frac{M_1}{A_1}}\zeta\right) \tag{32}$$

$$\theta = \theta_0 + log\left(1 + \theta_1\right) = log\left(-K_3(x) \exp\left(-\sqrt{\frac{M_1}{A_1}}\zeta\right)\right), \tag{33}$$

$$\theta = \theta_0 + log\left(1 + \theta_1 + \theta_2\right) = log\left(-K_3(x) \exp\left(-\sqrt{\frac{M_1}{A_1}}\zeta\right) + \exp\left(-3\sqrt{\frac{M_1}{A_1}}\zeta\right) L_{10}\right)$$

$$+ L_{11} \exp\left(-2\sqrt{\frac{M_1}{A_1}}\zeta\right) + L_{10}^* \exp\left(2\sqrt{\frac{M_1}{A_1}}\zeta\right) + L_{11}^* x \exp\left(2\sqrt{\frac{M_1}{A_1}}\zeta\right)\right), \tag{34}$$

$$\Phi = \Phi_0 + log\left(1 + \Phi_1\right)$$

$$= c_{11} \exp\left(-\sqrt{ScN_2Kr}\zeta\right) + log\left(1 + \exp\left(-\sqrt{ScN_2Kr}\zeta\right)\right)\left(c_{13} + L_{14} + L_{15} \exp\left(\sqrt{\frac{M_1}{A_1}}\zeta\right)\right) \tag{35}$$

$$\Phi = \Phi_0 + log\left(1 + \Phi_1 + \Phi_2\right) = c_{11} \exp\left(-\sqrt{ScN_2Kr}\zeta\right) + log\left(1 + \exp\left(-\sqrt{ScN_2Kr}\zeta\right)\left(c_{13} + L_{14} + L_{15} \exp\left(\sqrt{\frac{M_1}{A_1}}\zeta\right)\right)\right.$$

$$+ d_2 \exp\left(-\sqrt{ScN_2Kr}\zeta\right) + P_9 \exp\left(-2\sqrt{ScN_2Kr}\zeta\right) + P_{10} \exp\left(-\left(2\sqrt{\frac{M_1}{A_1}} + 2\sqrt{ScN_2Kr}\zeta\right)\right)$$

$$+ P_{11} \exp\left(-\left(\sqrt{\frac{M_1}{A_1}} + 2\sqrt{ScN_2Kr}\right)\zeta\right) + P_{12} + P_{14} \exp\left(-\left(\sqrt{\frac{M_1}{A_1}} + \sqrt{ScN_2Kr}\right)\zeta\right)$$

$$+ P_{13} x \exp\left(-\left(\sqrt{\frac{M_1}{A_1}} + \sqrt{ScN_2Kr}\right)\zeta\right) + P_{14} \exp\left(-2\sqrt{\frac{M_1}{A}}\zeta\right)$$

$$+ P_{15} \exp\left(-\left(2\sqrt{\frac{M_1}{A}} + \sqrt{ScN_2Kr}\right)\zeta\right) + P_{16} \exp\left(-\left(3\sqrt{\frac{M_1}{A}} + 2\sqrt{ScN_2Kr}\right)\zeta\right)\right) \tag{36}$$

Where the evaluated constants are as mentioned in the appendix. As the series is convergent, we ignore terms f_3, θ_3, Φ_3 onwards as their effects are negligible.

3.   Results and Discussion

The system of partial differential equations with the boundary conditions, are converted to a system of ordinary differential equations (1)-(3) using similarity transformations. These ODE are solved using the homotopy technique. First we calculate

f^p, θ^p, Φ^p and then taking $p \to 1$ we get the solution to f, θ, Φ. We analyse the profiles of velocity, temperature, concentration through graphs, for impacts of the Joule's effect and the Ohmic effects. We use $Pr = 0.71$, $Sc = 0.01$, $Ec = 0.01$, $Kr = 1$, $A = 1$, $a_1 = 0.5$, $n = 1$, $\beta_1 = 0.5$, $\beta_2 = 0.5$. In absence of the Joule's effect and the Ohmic effects velocity increases with increase in ζ. As we choose slightly bigger values of the Joule's effect and the Ohmic effects, the velocity increases till at some point of ζ where velocity then begins to reduce till it reaches 0, after which it starts increasing again as ζ increases. When the Joule's effect and the Ohmic effects take value of 3, velocity increases at a reasonably slow rate as ζ increases, as compared to that when the Joule's effect and the Ohmic effects are in the vicinity of the interval (0,1). In the absence of the Joule's effect and the Ohmic effects the temperature increases steadily with increases with increase in ζ till eventually it tends to remain constant taking value 0 after some particular ζ. When the Joule's effect and the Ohmic effects take increasing values the temperature increases at a faster rate than that for which the Joule's effect and the Ohmic effects are absent. On the other hand concentration reduces as ζ increases in the absence of the Joule's effect and the Ohmic effects, but as we increase the Joule's effect and the Ohmic effects concentration reduces at a much faster rate as ζ increases.
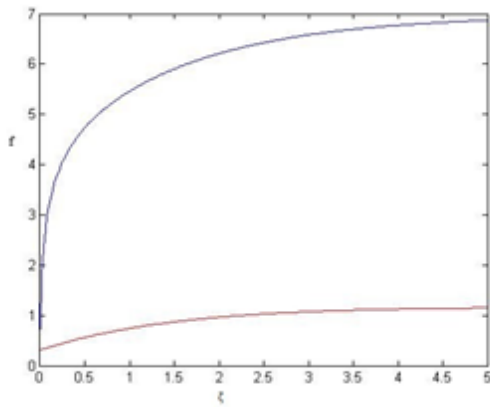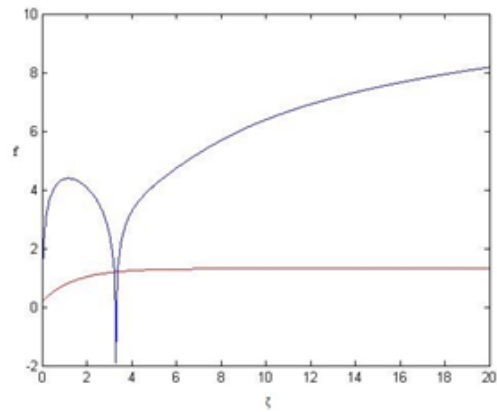

Figure 1: f for $K_1 = 0, K_2 = 0$

Figure 2: f for $K_1 = 0.5, K_2 = 0.5$
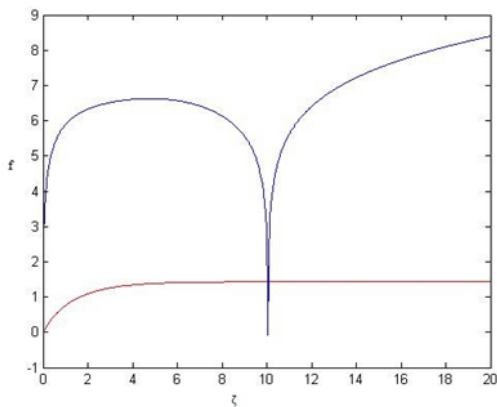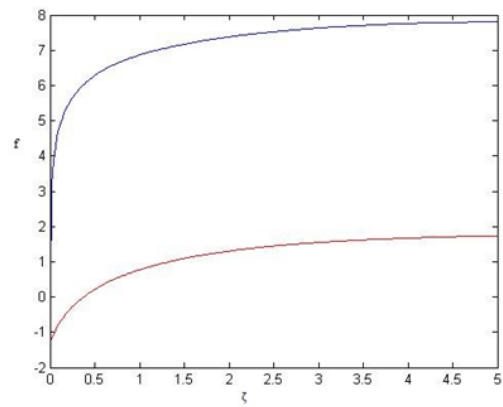

Figure 3: f for $K_1 = 1, K_2 = 1$
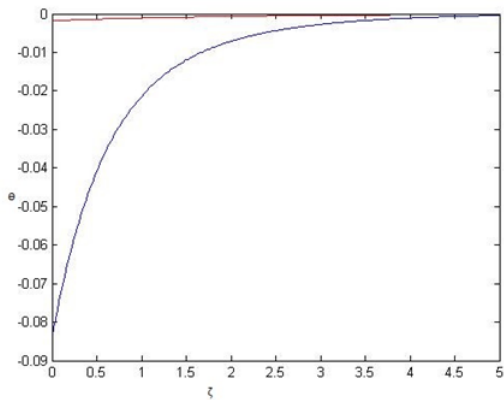

Figure 4: f for $K_1 = 3, K_2 = 3$

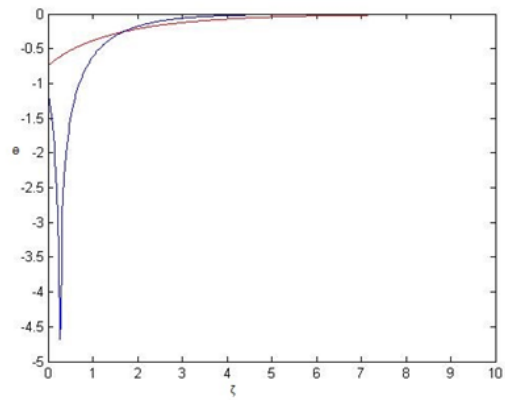Figure 5: θ for $K_1 = 0, K_2 = 0$

Figure 6: θ for $K_1 = 0.5, K_2 = 0.5$
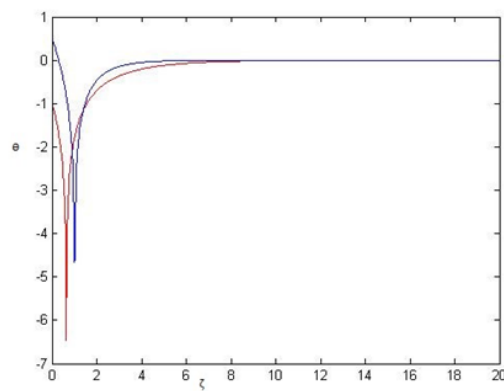

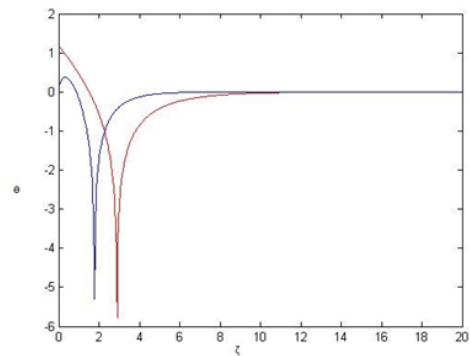Figure 7: θ for $K_1 = 1, K_2 = 1$

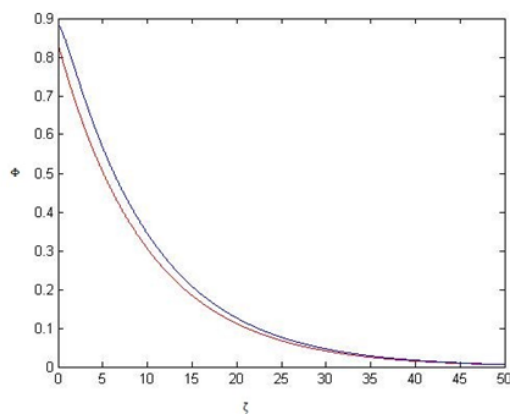Figure 8: θ for $K_1 = 3, K_2 = 3$
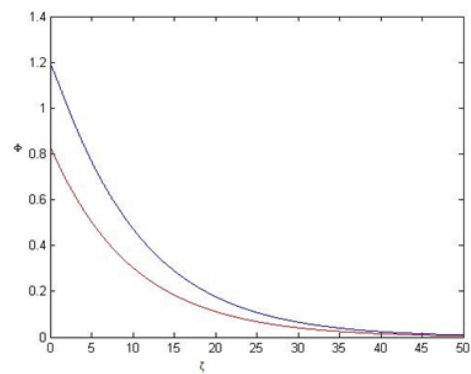

Figure 9: Φ for $K_1 = 0, K_2 = 0$

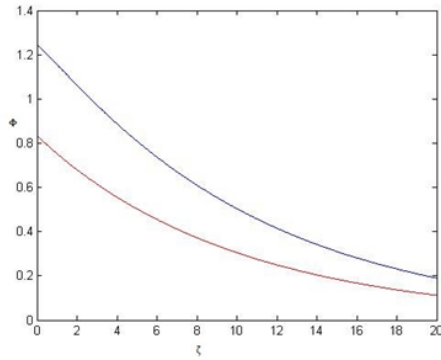Figure 10: Φ for $K_1 = 0.5, K_2 = 0.5$
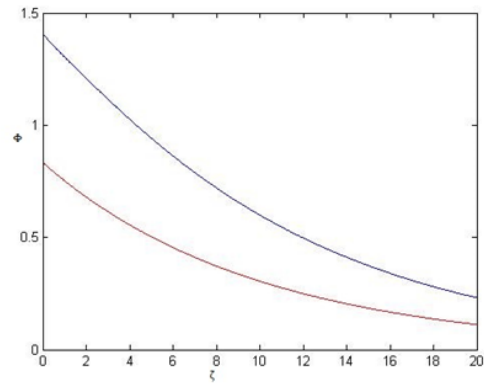
Figure 11: Φ for $K_1 = 1, K_2 = 1$

Figure 12: Φ for $K_1 = 3, K_2 = 3$

Babu [11] have studied the Magnetohydrodynamic dissipative flow across a slendering stretching sheet with temperature dependent variable viscosity in the absence of the Joule and Ohmic effect in [11] where they have compared the different values of $\theta'(0)$ using multiple numerical approaches like the Runge-Kutta method. The comparison of the table obtained by them and by us is drawn below,

1	0.115245	0.1152451342	0.1152451342
2	0.100166	0.1001668710	0.1001668711
3	0.086884	0.0868845644	0.0868845643
4	0.006452	0.0064522314	0.0064522314

Table 1: Comparison of $\theta'(0)$ as per [11]

K_1	K_2	1st approximation of $\theta'(0)$	2nd approximation of $\theta'(0)$
0	0	0.0018	0.1296
0.5	0.5	1.1053	-2.0599
1	1	-3.6667	-1.426
3	3	-1.3077	-1.3346

Table 2: Comparison of $\theta'(0)$ using homotopy method

Whereas our analysis using the homotopy method gives the following values of $\theta'(0)$ for the different values of K_1, K_2. The Joule's and Homic effect cannot be detected in the 1st approximation for $\theta'(0)$ but it changes the profile of θ in the 2nd approximation. Table 2 validates our analytical method using $\theta'(0)$ for different values of K_1 representing the Joule's effect and, K_2 representing the Ohmic effect.

4.   Conclusion

The boundary layer viscous flow across the slendering sheet has caught considerable attention due to its wide applications in microfluidics, duct design, nano-fluids and nano-devices and plastics molding and many other fields. Noticing this we have examined the two - dimensional magnetohydrodynamic boundary layer viscous flow across the slendering sheet under the Joules effect as well as the Homic effect. The major discoveries are :

- M, Ec, Pr, Sc have no effect on temperature.

- The behaviour of velocity and concentration are opposite to each other not only in the absence of the Joule's and Homic effects, but also when the Joule's and Homic effects are prominent

Acknowledgement

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Appendix

The constants in (31) -(36) are established to be,

$$L_1 = \frac{M_1}{A_1}(1-K_1), L_3 = \frac{L_1}{6M_1\sqrt{\frac{M_1}{A_1}}}, L_4 = \frac{1}{2A_1}, C_6 = L_4\sqrt{\frac{A}{M_1}} -2\ L_3, C_8 = -L_4\sqrt{\frac{A}{M_1}} +3\ L_3,$$

$$L_{16} = 3\left(C_8\frac{M_1}{A_1} + L_4\right)\left(C_8\frac{M_1}{A_1} + 2\,L_4\sqrt{\frac{M_1}{A_1}}\right) + 2\,L_4\sqrt{\frac{M_1}{A_1}} + C_8\left(C_8\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}} + 3L_4\frac{M_1}{A_1}\right) - 10C_8L_3\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}}$$

$$- \frac{M_1}{A_1}C_8\left(C_8\sqrt{\frac{M_1}{A_1}} + L_4\right) - C_8\frac{M_1}{A_1} - 2K_1\sqrt{\frac{M_1}{A_1}}\frac{1}{A_1}\left(L_4 + C_8\sqrt{\frac{M_1}{A_1}}\right),$$

$$L_{17} = 3\left(C_8\frac{M_1}{A_1} + L_4\right)L_4\sqrt{\frac{M_1}{A_1}} - L_4\left(C_8\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}} + 3L_4\frac{M_1}{A_1}\right) + C_8L_4\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}} + \frac{M_1}{A_1}L_4\left(C_8\sqrt{\frac{M_1}{A_1}} + L_4\right)$$

$$- C_8L_4\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}} + L_4\frac{M_1}{A_1} + L_4\frac{M_1}{A_1}\frac{1}{A_1} - 2\frac{1}{A_1}K_1L_4\frac{M_1}{A_1},$$

$$L_{18} = 22L_3L_4\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}},$$

$$L_{19} = -13\frac{M_1}{A_1}L_3\left(C_8\sqrt{\frac{M_1}{A_1}} + L_4\right) - 6\,L_3\sqrt{\frac{M_1}{A_1}}\left(C_8\frac{M_1}{A_1} + 2L_4\sqrt{\frac{M_1}{A_1}}\right) + L_3\left(C_8\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}} + 3L_4\frac{M_1}{A_1}\right)$$

$$- C_8\,10\,L_3\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}} - 3\,A\frac{1}{A_1}K_3\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}} - L_3\frac{M_1}{A_1} + 4\,L_3\frac{M_1}{A_1}\frac{1}{A_1} + 4\,L_3K_1\frac{M_1}{A_1}\frac{1}{A_1},$$

$$L_{20} = C_6\left(C_8\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}} + 3\,L_4\sqrt{\frac{M_1}{A_1}}\right) - C_6\sqrt{\frac{M_1}{A_1}}\left(C_8\sqrt{\frac{M_1}{A_1}} + L_4\right) - C_6\sqrt{\frac{M_1}{A_1}} + \frac{1}{A_1}\left(C_8\frac{M_1}{A_1} + 2\,L_4\sqrt{\frac{M_1}{A_1}}\right),$$

$$L_{21} = -68L_3^2\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}}, \quad L_{22} = -5L_4^2\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}}, \quad L_{23} = 2C_6L_4\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}} - \frac{1}{A_1}L_4\frac{M_1}{A_1},$$

$$T_1 = L_{20}\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}} - 0.5\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}}L_{16} + 0.5L_{17}\left(\frac{A_1}{M_1}\right)^2 + L_{22}\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}}\frac{M_1}{A_1},$$

$$T_2 = 0.5\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}}L_{16} + L_{18}0.5\left(\frac{A_1}{M_1}\right)^2,$$

$$T_3 = -\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}}L_{16}\frac{1}{6} - L_{17}\left(\frac{A_1}{M_1}\right)^2\frac{1}{18} - \frac{S_{13}}{27},$$

$$T_4 = \left(\frac{M_1}{A_1}\right)^{\frac{3}{2}}L_{21}\frac{1}{4} - \left(\frac{M_1}{A_1}\right)^{\frac{3}{2}}L_{19}\frac{1}{8} - L_{18}\left(\frac{A_1}{M_1}\right)^2,$$

$$T_5 = -\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}}L_{21}\frac{1}{10},$$

$$T_6 = -\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}}L_{17}\frac{1}{2} + 0.5\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}}L_{22}\left(\frac{A_1}{M_1}\right)^2 - \left(\frac{M_1}{A_1}\right)^{\frac{3}{2}}L_{18}0.5 - L_{23}\left(\frac{A_1}{M_1}\right)^2 + L_{23}\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}}\frac{1}{8},$$

$$T_7 = \left(\frac{M_1}{A_1}\right)^{\frac{3}{2}}L_{22}0.5 - L_{23}\left(\frac{A_1}{M_1}\right)^2\frac{1}{8},$$

$$T_0 = \left(\frac{M_1}{A_1}\right)^{\frac{3}{2}}L_{18}\frac{1}{3} - \left(\frac{M_1}{A_1}\right)^{\frac{3}{2}}L_{22}\frac{1}{6},$$

$$T_8 = -0.5\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}}L_{17} - L_{22}\left(\frac{A_1}{M_1}\right)^2 + L_{23}\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}},$$

$$T_9 = -0.5\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}}L_{22},$$

$$T_{11} = \left(\frac{M_1}{A_1}\right)^{\frac{3}{2}}L_{18}\frac{1}{8},$$

$$T_{12} = L_{20}\frac{M_1}{A_1}0.5, T_{13} = L_{23}\frac{M_1}{A_1}0.5,$$

$$C_{14} = T_1 + 2T_2 + 3T_3 + 4T_4 + 5T_5 - \sqrt{\frac{A_1}{M_1}}\left(T_6 + T_0 + T_8 + T_{11} + T_{12}\right),$$

$$C_{12} = -\left(2\,T_1 + 3\,T_2 + 4\,T_3 + 5\,T_4 + 6\,T_5\right) + \sqrt{\frac{A_1}{M_1}}\left(T_6 + T_0 + T_8 + T_{11} + T_{12}\right),$$

$$L_5 = 2Pr\sqrt{\frac{M_1}{A_1}}\,K_3 - PrEc\left(\frac{M_1}{A_1}\right)^2 8\,L_3 - PrN_1\sqrt{\frac{A_1}{M_1}}K_3 - 4\,K1\frac{M_1}{A_1}L_3,$$

$$L_6 = 4\,(K_3)^2\left(\frac{M_1}{A_1} + \left(\frac{M_1}{A_1}\right)^2\right),$$

$$L_8 = 2PrK_3\, b_2\sqrt{\frac{M_1}{A_1}} - 2PrEc\,\frac{M_1}{A_1}\left(C_8\frac{M_1}{A_1} + 2\,L_4\sqrt{\frac{M_1}{A_1}}\right) - 2K_1\sqrt{\frac{M_1}{A_1}}\left(L_4 + C_8\sqrt{\frac{M_1}{A_1}}\right),$$

$$L_9 = 2PrEc\,L_4\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}} + 2\,K_1L_4\frac{M_1}{A_1}, L_{10} = \left(L_5\frac{A_1}{M_1}\right)\frac{1}{9},$$

$$L_{11} = \left(L_8\frac{A_1}{M_1}\right)\frac{1}{4} - L_9\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}}\frac{1}{4}, L_{10}^* = \left(L6\frac{A_1}{M_1}\right)\frac{1}{16},$$

$$L_{11}^* = L_9\frac{A_1}{M_1}\frac{1}{4}, C_{11} = \frac{b_2}{b_2 + \sqrt{ScN2\ Kr}}, L_{14} = \frac{ScC11\sqrt{ScN_2Kr} - N1\frac{M_1}{A_1}}{\frac{M_1}{A_1} + 2\sqrt{ScN_2Kr}\sqrt{\frac{M_1}{A_1}}},$$

$$L_{15} = C_{11}\frac{Sc}{2},$$

$$C_{13} = -L_{14} + \left(L_{15}\frac{-b_2 + \sqrt{ScN2\ Kr}\sqrt{\frac{M_1}{A_1}}}{b_2 + \sqrt{ScN2\ Kr}}\right),$$

$$P_1 = -ScN_2Kr\frac{(C_{13} + L_{14})^2}{2},$$

$$P_2 = -\frac{ScN_2Kr}{2},$$

$$P_3 = -ScN_2KrL_{15}\,(C_{13} + L_{14})\left(-\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}} - L_{15}\,(C_{13} + L_{14})\,\frac{M_1}{A_1}\right),$$

$$P_4 = ScN_1\sqrt{\frac{M_1}{A_1}}\,(C_{13} + L_{14}) - ScN_1C_{11}\left(L_4 - C_8\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}}\right),$$

$$P_5 = ScN_1C_{11}\left(\frac{M_1}{A_1}\right)^{\frac{3}{2}}L_4, P_6 = ScN_1\sqrt{\frac{M_1}{A_1}}L_{15} - 2\,ScN_1C_{11}\sqrt{\frac{M_1}{A_1}}L_3,$$

$$P_7 = 2\,ScK_3\sqrt{\frac{M_1}{A_1}},$$

$$P_8 = 2\,ScK_3\sqrt{\frac{M_1}{A_1}},$$

$$P_9 = 2\frac{P_1}{3\,ScN_2Kr},$$

$$P_{10} = \left(\frac{1}{2\,\sqrt{ScN_2Kr}}\right)\left(\left(\frac{P_2}{\sqrt{\frac{M_1}{A_1}} + 3\sqrt{ScN_2Kr}}\right) + \left(\frac{P_2}{2\sqrt{\frac{M_1}{A_1}} + \sqrt{ScN_2Kr}}\right) + \left(\frac{P_7}{-2\sqrt{\frac{M_1}{A_1}} + \sqrt{ScN_2Kr}}\right)\right),$$

$$P_{11} = \left(\frac{1}{2\,\sqrt{ScN_2Kr}}\right)\left(\left(\frac{P_3}{\sqrt{\frac{M_1}{A_1}} + 3\sqrt{ScN_2Kr}}\right) + \left(\frac{P_3}{\sqrt{\frac{M_1}{A_1}} + 3\sqrt{ScN_2Kr}}\right)\right),$$

$$P_{12} = \left(\frac{1}{2\,\sqrt{ScN_2Kr}}\right)\left(\left(\frac{P_4}{S5 + 2\sqrt{ScN_2Kr}}\right) + P_4\sqrt{\frac{A_1}{M_1}} - 2\,P_5 + \left(\frac{P_3}{\sqrt{\frac{M_1}{A_1}} + 3\sqrt{ScN_2Kr}}\right)\right),$$

$$P_{13} = \left(-\frac{1}{2\,\sqrt{ScN_2Kr}}\right)P_5\sqrt{\frac{A_1}{M_1}},$$

$$P_{14} = \left(\frac{1}{2\,\sqrt{ScN2\ Kr}}\right)\left(\frac{P_7 + P_6}{2\sqrt{\frac{M_1}{A_1}} + \sqrt{ScN2\ Kr}}\right),$$

$$P_{15} = \frac{-P_6\sqrt{\frac{A_1}{M_1}}}{4\,\sqrt{ScN_2Kr}},$$

$$P_{16} = \left(-\frac{1}{2\,\sqrt{ScN_2Kr}}\right)\left(\frac{P_8}{-3\sqrt{\frac{M_1}{A_1}} + \sqrt{ScN2\ Kr}}\right),$$

$$d_2 = -\left(2\,P_9 + 2\,P_{10} + 2\,P_{11} + P_{12} + P_{15} + 2\,P_{16}\right) + \left(\frac{P_{13}}{\sqrt{ScN2\ Kr}}\right) - \left(\left(\frac{\sqrt{\frac{M_1}{A_1}}}{\sqrt{ScN2\ Kr}}\right)(P_9 + P_{10} + P_{11} + P_{12} + P_{14} + P_{15} + P_{16})\right).$$