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Some More One Point Union Cordial Graphs Related to C_4 and Paths

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Abstract: In this paper we define tail graph and obtain one point union graphs $G^{(k)}$ on it and show them to be cordial. We take all structures on $G = tail(C_4, P_m)$, $m = 2, 3, 4, 5$. We have split P_m to obtain two tails (attached to same fixed point) keeping sum of edges fixed to $m - 1$. We show that all these structures are cordial.

MSC: 05C78.

Keywords: Label, cordial, tail graph, antenna, one point union.

1. Introduction

The graphs we consider are simple, finite, undirected and connected. For terminology and definitions we depend on Graph Theory by Harary [6], A dynamic survey of graph labeling by J.Gallian [8] and Douglas West [9]. I. Cahit introduced the concept of cordial labeling [5]. $f : V(G) \to \{0, 1\}$ be a function. From this label of any edge (uv) is given by $|f(u) - f(v)|$. Further number of vertices labeled with 0 i.e., $v_f(0)$ and the number of vertices labeled with 1 i.e.., $v_f(1)$ differ at most by one. Similarly number of edges labeled with 0 i.e. $e_f(0)$ and number of edges labeled with 1 i.e. $e_f(1)$ differ at most by one. Then the function f is called as cordial labeling. I. Cahit has shown that: every tree is cordial; K_n is cordial if and only if $n \leq 3$; $K_{m,n}$ is cordial for all m and n; the friendship graph $C_3^{(t)}$ (i.e., the one-point union of t copies of C_3) is cordial if and only if t is not congruent to 2 $(mod\ 4)$; all fans are cordial; the wheel W_n is cordial if and only if n is not congruent to 3 $(mod\ 4)$. A lot of work has been done in this type of labeling. One may refer dynamic survey by J. Gallian [8].

Our focus of attention is on one point unions on different graphs. For a given graph there are different one point unions (up to isomorphism) structures possible. It depends on which point on G is used to fuse to obtain one point union. We have shown that for G = bull on C_3, bull on C_4, C_3^+, $C_4^+ - e$ the different path union $P_m(G)$ are cordial [4]. It is called as invariance under cordial labeling. We use the convention that $v_f(0, 1) = (a, b)$ to indicate the number of vertices labeled with 0 are a in number and that number of vertices labeled with 1 are b in number. Further $e_f(0, 1) = (x, y)$ we mean the number of edges labeled with 0 are x and number of edges labeled with 1 are y in number. The graph whose cordial labeling is known is cordial graph. In this paper we define tail graph and obtain one point union graphs on it and show them to be cordial. On tail graph $tail(G, P_m)$ we restrict our attention to at most two tails with due care that sum of edges on all tails will be $m - 1$.

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2. Preliminaries

Definition 2.1 (Tail graph)**.** *Let G be a (p,q) graph. A path P_m with it's one end is fused with some vertex of G is called as antenna of G by some authors. We refer it as P_m tail of G and denote by $tail(G, P_m)$.*

Definition 2.2 (Fusion of vertex [9])**.** *Let G be a (p, q) graph. Let $u \neq v$ be two vertices of G. We replace them with single vertex w and all edges incident with u and that with v are made incident with w. If a loop is formed is deleted. The new graph has $p-1$ vertices and at least $q-1$ edges 3.*

Definition 2.3. *$G^{(K)}$ is One point union of k copies of G is obtained by taking k copies of G and fusing a fixed vertex of each copy with same fixed vertex of other copies to create a single vertex common to all copies. If G is a (p, q) graph then $|V(G^{(k)}| = k(p-1)+1$ and $|E(G)| = k.q$*

Definition 2.4. *$(G, 2-P_m)$ it has two paths P_m attached at the same vertex of G. We discuss the case $m = 2$.*

Definition 2.5. *(G, P_m, P_n) it has a copy of P_m and a copy of P_n attached at the same point of G.*

3. Main Results

Theorem 3.1. *$G^{(k)}$ is cordial where $G = tail(C_4, P_2)$.*

Proof. Define A function $f : V(G) \to \{0,1\}$ as follows. It introduces different types of labeling units as follows. We combine them suitably to obtain a labeled copy of $G^{(k)}$.

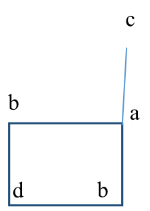

Fig 4.1 Tail(C_4,P_2)

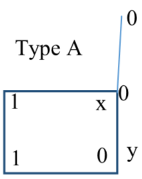

Fig 4.2 $v_f(0,1)$ = (3,2), $ef(0,1)$ = (3,2)

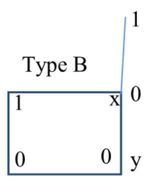

Fig 4.3 $v_f(0,1)$ = (3,2), $ef(0,1)$ = (2,3)

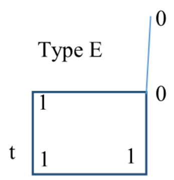

Fig 4.4 $v_f(0,1)$ = (2,3), $e_f(0,1)$ = (3,2)

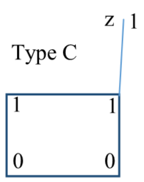

Fig 4.5 $v_f(0,1)$ = (2,3), $e_f(0,1)$ = (3,2)

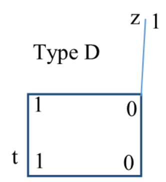

Fig 4.6 $v_f(0,1)$ = (2,3), $e_f(0,1)$ = (2,3)

We take one point union on vertex ‘a’ ‘b’,’c’ or at ‘d’ (see fig 4.1 above). In that case we get structure 1, structure 2, structure 3, structure 4 respectively. All these structures will be pairwise non-isomorphic.

To obtain **structure 1** we fuse Type A and Type B at point ‘x’ on it. To obtain labeled copy of $G^{(k)}$ Type A is used when $i \equiv 1(mod\ 2)$ and Type B label is used when $i \equiv 0(mod\ 2)$, $i = 1, 2, ..., k$.

To obtain **structure 2** we fuse Type A and Type B at point ‘y’ on it. To obtain labeled copy of $G^{(k)}$ Type A is used when $i \equiv 1(mod\ 2)$ and Type B label is used when $i \equiv 0(mod\ 2)$, $i = 1, 2, ..., k$.

To obtain **structure 3** we fuse Type C and Type D at point ‘z’ on it. To obtain labeled copy of $G^{(k)}$ Type C is used when

$i \equiv 1(mod\ 2)$ and Type D label is used when $i \equiv 0(mod\ 2)$, $i = 1, 2, ..., k$.

To obtain **structure 4** we fuse Type E and Type D at point 't' on it. To obtain labeled copy of $G^{(k)}$ Type E is used when $i \equiv 1(mod\ 2)$ and Type D label is used when $i \equiv 0(mod\ 2)$, $i = 1, 2, ..., k$.

For **structure 1** and **structure 2** the number distribution is : On vertices $v_f(0,1) = (3+4x, 2+4x)$ and on edges we have $e_f(0,1) = (3+5x, 2+5x)$ when m is of type $2x+1$, $x = 0,1,2,...$ and when m is of type $2x$, $x = 1,2,...$ we have number distribution as: on vertices $v_f(0,1) = (1+4x, 4x)$ and on edges we have $e_f(0,1) = (5x, 5x)$. For **structure 3** and **structure 4 and structure 5** the number distribution is : On vertices $v_f(0,1) = (2+4x, 3+4x)$ and on edges we have $e_f(0,1) = (3+5x, 2+5x)$ when k is of type $2x+1$, $x = 0,1,2,...$ and when k is of type $2x$, $x = 1,2,...$ we have number distribution as: on vertices $v_f(0,1) = (4x, 4x+1)$ and on edges we have $e_f(0,1) = (5x, 5x)$. Thus the graph is cordial. □

Theorem 3.2. *$G^{(k)}$ is cordial where $G = tail(C_4, P_3)$.*

Proof. Define A function $f : V(G) \to \{0,1\}$ as follows. It introduces different types of labeling units as given below. We combine them suitably to obtain a labeled copy of $G^{(k)}$.

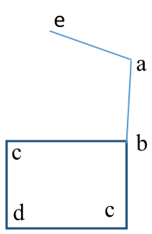

Fig 4.7 Tail(C_4,P_2)

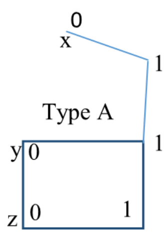

Fig 4.8 v_f(0,10=(3,3), e_f(0,1) = (3,3))

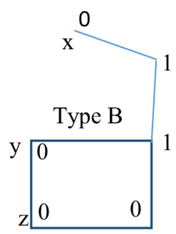

Fig 4.9 v_f(0,1)=(4,2), e_f(0,1) = (3,3))

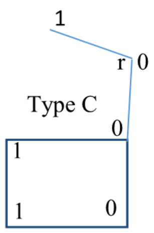

Fig 4.10 v_f(0,10 = (3,3), e_f(0,1) = (3,3))

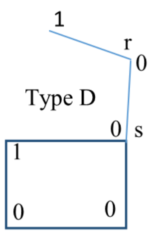

Fig 4.11 v_f(0,10=(4,2), e_f(0,1) = (3,3))

We take one point union on vertex 'e' , 'a', 'b', 'c' , 'd' (see fig 4.7 above). In that case we get structure 1, structure 2, structure 3, structure 4 and structure 5 respectively. All these structures will be pairwise non-isomorphic.

To obtain **structure 1** we fuse Type A and Type B at point 'x' on it. To obtain labeled copy of $G^{(k)}$ Type A is used when $i \equiv 1(mod\ 2)$ and Type B label is used when $i \equiv 0(mod\ 2)$, $i = 1, 2, ..., k$.

To obtain **structure 2** we fuse Type C and Type D at point 'r' on it. To obtain labeled copy of $G^{(k)}$ Type C is used when $i \equiv 1(mod\ 2)$ and Type D label is used when $i \equiv 0(mod\ 2)$, $i = 1, 2, ..., k$.

To obtain **structure 3** we fuse Type C and Type D at point 's' on it. To obtain labeled copy of $G^{(k)}$ Type C is used when $i \equiv 1(mod\ 2)$ and Type D label is used when $i \equiv 0(mod\ 2)$, $i = 1, 2, ..., k$.

To obtain **structure 4** we fuse Type A and Type B at point 'y' on it. To obtain labeled copy of $G^{(k)}$ Type A is used when $i \equiv 1(mod\ 2)$ and Type B label is used when $i \equiv 0(mod\ 2)$, $i = 1, 2, ..., k$.

To obtain **structure 5** we fuse Type A and Type B at point 'z' on it. To obtain labeled copy of $G^{(k)}$ Type A is used when $i \equiv 1(mod\ 2)$ and Type B label is used when $i \equiv 0(mod\ 2)$, $i = 1, 2, ..., k$.

For all structures above the number distribution is : On vertices $v_f(0,1) = (3+5x, 3+5x)$ for k is of type $2x+1$, $x = 0,1,2,...$ and when k is of type $2x$, $x = 1,2,...$ and we have vertices distribution as $v_f(0,1) = (1+5x, 5x)$ and on edges we have $e_f(0,1) = (3k, 3k)$. Thus the graph is cordial. □

Theorem 3.3. *$G^{(k)}$ is cordial where $G = tail(C_4, 2 - P_2)$.*

Proof. In this case two copies of P_2 are fused at the same vertex of C_4. Define A function $f : V(G) \to \{0,1\}$ as follows. It introduces different types of labeling units as given below. We combine them suitably to obtain a labeled copy of $G^{(k)}$.

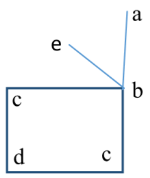

Fig 4.12 Tail(C_4,P_2)

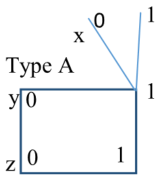

Fig 4.13 $v_f(0,1)$=(3,3), $e_f(0,1)$ = (3,3))

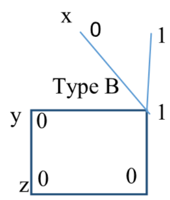

Fig 4.14 $v_f(0,1)$=(4,2), $e_f(0,1)$ = (3,3))

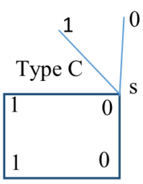

Fig 4.15 $v_f(0,1)$=(3,3), $e_f(0,1)$ = (3,3))

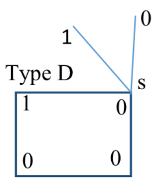

Fig 4.16 $v_f(0,1)$=(4,2), $e_f(0,1)$ = (3,3))

We take one point union on vertex 'e' or 'a', 'b', 'c' , 'd' (see fig 4.12 above). In that case we get structure 1, structure 2, structure 3 and structure 4 respectively. All these structures will be pairwise non-isomorphic.

To obtain **structure 1** we fuse Type A and Type B at point 'x' on it. To obtain labeled copy of $G^{(k)}$ Type A is used when $i \equiv 1(mod\ 2)$ and Type B label is used when $i \equiv 0(mod\ 2)$, $i = 1,2,...,k$.

To obtain **structure 2** we fuse Type C and Type D at point 's' on it. To obtain labeled copy of $G^{(k)}$ Type C is used when $i \equiv 1(mod\ 2)$ and Type D label is used when $i \equiv 0(mod\ 2)$, $i = 1,2,...,k$.

To obtain **structure 3** we fuse Type A and Type B at point 'y' on it. To obtain labeled copy of $G^{(k)}$ Type A is used when $i \equiv 1(mod\ 2)$ and Type B label is used when $i \equiv 0(mod\ 2)$, I = 1,2,..k.

To obtain **structure 4** we fuse Type A and Type B at point 'z' on it. To obtain labeled copy of $G^{(k)}$ Type A is used when $i \equiv 1(mod\ 2)$ and Type B label is used when $i \equiv 0(mod\ 2)$, $i = 1,2,...,k$.

For all structures above the number distribution is: On vertices $v_f(0,1) = (3+5x, 3+5x)k$ is of type $2x+1$, $x = 0,1,2,...$ and when k is of type $2x$, $x = 1,2,...$ we have vertices distribution as $v_f(0,1) = (1+5x, 5x)$ and on edges we have $e_f(0,1) = (3k, 3k)$. This label number distribution is same as that observed in Theorem 4.2 for $G^{(k)}$ where $G = tail(C_4, P_3)$. Thus the graph is cordial. □

Theorem 3.4. *$G^{(k)}$ is cordial where $G = tail(C_4, P_2, P_3)$.*

Proof. In this case a copy of P_2 and a copy of P_2 are fused at the same vertex of C_4 (fig 4.17). It has 7 edges and 7 vertices. One point union of this graph has 6 non isomorphic structures as we can use 6 different vertices to obtain one point union. Define a function $f : V(G) \to \{0,1\}$ as follows. It introduces different types of labeling units as given below.

We combine them suitably to obtain a labeled copy of $G^{(k)}$.

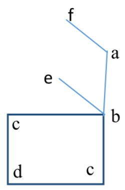

Fig 4.17 Tail(C_4,P_2)

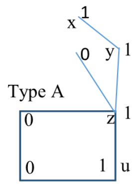

Fig 4.18 $v_f(0,1) = (3,4)$, $e_f(0,1) = (4,3)$)

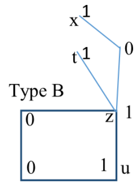

Fig 4.19 $v_f(0,1) = (3,4)$, $e_f(0,1) = (3,4)$)

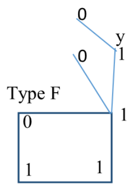

Fig 4.20 $v_f(0,1) = (3,4)$, $e_f(0,1) = (3,4)$

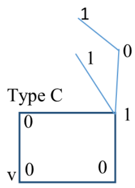

Fig 4.21 $v_f(0,1) = (4,3)$, $e_f(0,1) = (3,4)$)

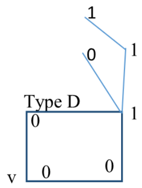

Fig 4.22 $v_f(0,1) = (4,3)$, $e_f(0,1) = (4,3)$)

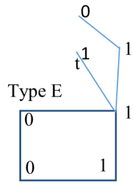

Fig 4.23 $v_f(0,1) = (3,4)$, $e_f(0,1) = (4,3)$)

We take one point union on any of the vertices 'f' ,'a', 'b','e', 'c' or 'd' (see fig 4.17 above). In that case we get structure 1, structure 2, structure 3 , structure 4, structure 5 and structure 6 respectively. All these structures are pairwise non-isomorphic.

To obtain **structure 1** we fuse Type A and Type B at point 'x' on it. To obtain labeled copy of $G^{(k)}$ Type A is used when $i \equiv 1(mod\ 2)$ and Type B label is used when $i \equiv 0(mod\ 2)$, $i = 1, 2, ..., k$.

To obtain **structure 2** we fuse Type A and Type F at point 'y' on it. To obtain labeled copy of $G^{(k)}$ Type A is used when $i \equiv 1(mod\ 2)$ and Type F label is used when $i \equiv 0(mod\ 2)$, $i = 1, 2, ..., k$.

To obtain **structure 3** we fuse Type A and Type B at point 'z' on it. To obtain labeled copy of $G^{(k)}$ Type A is used when $i \equiv 1(mod\ 2)$ and Type B label is used when $i \equiv 0(mod\ 2)$, $i = 1, 2, ..., k$.

To obtain **structure 4** we fuse Type E and Type B at point 't' on it. To obtain labeled copy of $G^{(k)}$ Type E is used when $i \equiv 1(mod\ 2)$ and Type B label is used when $i \equiv 0(mod\ 2)$, $i = 1, 2, ..., k$.

To obtain **structure 5** we fuse Type A and Type B at point 'u' on it. To obtain labeled copy of $G^{(k)}$ Type A is used when $i \equiv 1(mod\ 2)$ and Type B label is used when $i \equiv 0(mod\ 2)$, $i = 1, 2, ..., k$.

To obtain **structure 6** we fuse Type C and Type D at point 'v' on it. To obtain labeled copy of $G^{(k)}$ Type C is used when $i \equiv 1(mod\ 2)$ and Type D label is used when $i \equiv 0(mod\ 2)$, $i = 1, 2, ..., k$.

For **structure 1** through **structure 5** the label number distribution is as follows: For k is of type $2x+1$ we have on vertices $v_f(0,1) = (3+6x, 4+6x)k$ is of type $2x+1$, $x = 0, 1, 2, ...$ and when k is of type $2x$, $x = 1, 2, ...$ we have vertices distribution as $v_f(0,1) = (6x, 6x+1)$ and on edges we have $e_f(0,1) = (7x+4, 7x+3)$ when $k = 2x+1$, $x = 0, 1, ...$ and when k is of type $2x+1$, $x = 0, 1, 2...$ and when $k = 2x$, $x = 1, 2, ...$; $e_f(0,1) = (7x, 7x)$. For **structure 6** the label number distribution is as follows: For k is of type $2x+1$ we have on vertices $v_f(0,1) = (4+6x, 3+6x)k$ is of type $2x+1$, $x = 0, 1, 2, ...$ and when k is of type $2x$, $x = 1, 2, ...$ we have vertices distribution as $v_f(0,1) = (6x+1, 6x)$. And on edges we have $e_f(0,1) = (7x+3, 7x+4)$ and when k is of type $2x+1$, $x = 0, 1, 2...$ and when $m = 2x$, $x = 1, 2, ...$; $e_f(0,1) = (7x, 7x)$. The graph is cordial. □

Theorem 3.5. *$G^{(k)}$ is cordial where $G = tail(C_4, P_4)$.*

Proof. In this case a copy of P_4 is fused at the vertex of C_4. It has 7 edges and 7 vertices. One point union of this graph has 6 non isomorphic structures as we can use 6 different vertices to obtain one point union. Define a function $f : V(G) \rightarrow \{0, 1\}$ as follows. It introduces different types of labeling units as given below. We combine them suitably to obtain a labeled copy of $G^{(k)}$.

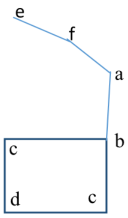

Fig 4.24 Tail(C_4,P_2)

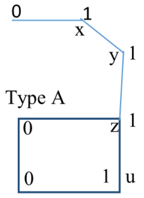

Fig 4.25 $v_f(0,1) = (3,4)$, $e_f(0,1) = (4,3))$

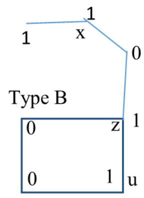

Fig 4.26 $v_f(0,1) = (3,4)$, $e_f(0,1) = (3,4))$

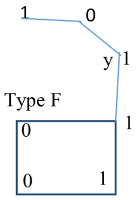

Fig 4.27 $v_f(0,1) = (3,4)$, $e_f(0,1) = (3,4)$

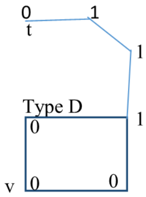

Fig 4.28 $v_f(0,1) = (4,3)$, $e_f(0,1) = (4,3))$

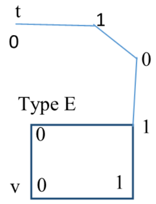

Fig 4.29 $v_f(0,1) = (4,3)$, $e_f(0,1) = (3,4))$

We take one point union on any of the vertices 'e', 'f' ,'a', 'b', 'c' or 'd' (see fig 4.24 above). In that case we get structure 1, structure 2, structure 3, structure 4, structure 5 and structure 6 respectively. All these structures will be pairwise non-isomorphic.

To obtain **structure 1** we fuse Type E and Type D at point 't' on it. To obtain labeled copy of $G^{(k)}$ Type E is used when $i \equiv 1(mod\ 2)$ and Type D label is used when $i \equiv 0(mod\ 2)$, $i = 1, 2, ..., k$.

To obtain **structure 2** we fuse Type A and Type B at point 'x' on it. To obtain labeled copy of $G^{(k)}$ Type A is used when $i \equiv 1(mod\ 2)$ and Type B label is used when $i \equiv 0(mod\ 2)$, $i = 1, 2, ..., k$.

To obtain **structure 3** we fuse Type A and Type F at point 'y' on it. To obtain labeled copy of $G^{(k)}$ Type A is used when $i \equiv 1(mod\ 2)$ and Type F label is used when $i \equiv 0(mod\ 2)$, $i = 1, 2, ..., k$.

To obtain **structure 4** we fuse Type A and Type B at point 'z' on it. To obtain labeled copy of $G^{(k)}$ Type A is used when $i \equiv 1(mod\ 2)$ and Type B label is used when $i \equiv 0(mod\ 2)$, $i = 1, 2, ..., k$.

To obtain **structure 5** we fuse Type A and Type B at point 'u' on it. To obtain labeled copy of $G^{(k)}$ Type A is used when $i \equiv 1(mod\ 2)$ and Type B label is used when $i \equiv 0(mod\ 2)$, $i = 1, 2, ..., k$.

To obtain **structure 6** we fuse Type E and Type D at point 'v' on it. To obtain labeled copy of $G^{(k)}$ Type E is used when $i \equiv 1(mod\ 2)$ and Type D label is used when $i \equiv 0(mod\ 2)$, $i = 1, 2, ..., k$.

For structure 2 through structure 5 the label number distribution is as follows:We have on vertices $v_f(0,1) = (3+6x, 4+6x)$ for k is of type $2x+1$, $x = 0, 1, 2, ...$ and when k is of type $2x$, $x = 1, 2, ...$ we have vertices distribution as $v_f(0,1) = (6x, 6x+1)$ and on edges we have $e_f(0,1) = (7x+4, 7x+3)$ when $k = 2x+1$, $x = 0, 1, ...$ and when k is of type $2x+1$, $x = 0, 1, 2...$ and when $k = 2x$, $x = 1, 2, ...$; $e_f(0,1) = (7x, 7x)$. For **structure 1** and **structure 6** the label number distribution is as follows: For k is of type $2x+1$ we have on vertices $v_f(0,1) = (4+6x, 3+6x)k$ is of type $2x+1$, $x = 0, 1, 2, ...$ and when k is of type $2x$, $x = 1, 2, ...$ we have vertices distribution as $v_f(0,1) = (6x+1, 6x)$. And on edges we have $e_f(0,1) = (7x+3, 7x+4)$ and when k is of type $2x+1$, $x = 0, 1, 2...$ and when $k = 2x$, $x = 1, 2, ...$; $e_f(0,1) = (7x, 7x)$. The graph is cordial. □

Theorem 3.6. *$G = tail(C_4, 3 - P_2$. $G^{(k)}$ is cordial for all structures.*

Proof. In this case three copies of P_2 are fused at the same vertex of C_4. Define A function $f : V(G) \to \{0, 1\}$ as follows.

It introduces different types of labeling units as given below. We combine them suitably to obtain a labeled copy of $G^{(k)}$.

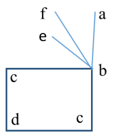

Fig 4.30 Tail(C_4,P_2)

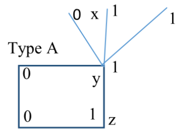

Fig 4.31 $v_f(0,1)=(3,4)$, $e_f(0,1) = (4,3)$)

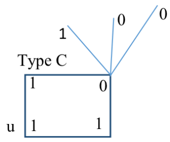

Fig 4.32 $v_f(0,1)=(3,4)$, $e_f(0,1) = (4,3)$)

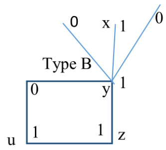

Fig 4.33 $v_f(0,1)=(3,4)$, $e_f(0,1) = (3,4)$)

We take one point union on vertex 'e' or 'f'or 'a', 'b', 'c' , 'd' (see fig 4.30 above). In that case we get structure 1, structure 2, structure 3 and structure 4 respectively. All these structures will be pairwise non-isomorphic. To obtain **structure 1** we fuse Type A and Type B at point 'x' on it. To obtain labeled copy of $G^{(k)}$ Type A is used when $i \equiv 1(mod\ 2)$ and Type B label is used when $i \equiv 0(mod\ 2)$, $i = 1, 2, ..., k$. To obtain **structure 2** we fuse Type A and Type B at point 'y' on it. To obtain labeled copy of $G^{(k)}$ Type A is used when $i \equiv 1(mod\ 2)$ and Type B label is used when $i \equiv 0(mod\ 2)$, $i = 1, 2, ..., k$. To obtain **structure 3** we fuse Type A and Type B at point 'z' on it. To obtain labeled copy of $G^{(k)}$ Type A is used when $i \equiv 1(mod\ 2)$ and Type B label is used when $i \equiv 0(mod\ 2)$, $i = 1, 2, ..., k$. To obtain **structure 4** we fuse Type C and Type B at point 'u' on it. To obtain labeled copy of $G^{(k)}$ Type C is used when $i \equiv 1(mod\ 2)$ and Type B label is used when $i \equiv 0(mod\ 2)$, $i = 1, 2, ..., k$. For all structures above the number distribution is : On vertices $v_f(0, 1) = (3 + 6x, 4 + 6x)$ where k is of type $2x + 1$, $x = 0, 1, 2, ...$ and on edges we have $e_f(0, 1) = (4 + 7x, 3 + 7x)$ when k is of type $2x$, $x = 1, 2, ...$ we have vertices distribution as $v_f(0, 1) = (6x, 6x + 1)$ and on edges we have $e_f(0, 1) = (7x, 7x)$. Thus the graph is cordial. □

Theorem 3.7. *$G^{(k)}$ is cordial where $G = tail(C_4, P_2, P_4)$.*

Proof. Define a function $f : V(G) \rightarrow \{0, 1\}$ as follows. It introduces four types of labeling units as given below. Type C and Type B are not cordial while Type A is cordial. We combine them suitably to obtain a labeled copy of $G^{(k)}$.

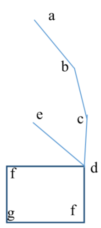

Fig 4.34 Tail(C_4,P_2)

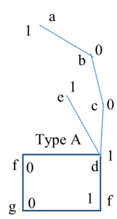

Fig 4.35 $v_f(0,1)= (4,4)$, $e_f(0,1)=(4,4)$

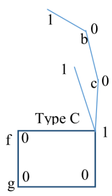

Fig 4.36 $v_f(0,1)=$ (5,3), $e_f(0,1)=(4,4)$

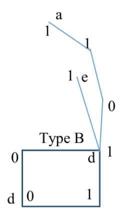

Fig 4.37 $v_f(0,1)= (3,5)$ $e_f(0,1)=(4,4)$

We take one point union on any of the vertices 'a', 'b', 'c' , 'd', 'e', 'f', and 'g' (see fig 4.34 above). In that case we get structure 1, structure 2, structure 3, structure 4, structure 5, structure 6 and structure 7 respectively. All these structures are pairwise non-isomorphic. In structure 2, type A and type C are used repeatedly. The one point union is taken at vertex 'b' on it. Type A is used as i^{th} copy if $i \equiv 1(mod\ 4)$ and copy C is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$, in construction of $G^{(k)}$. In structure 3, type A and type C are used repeatedly. The one point union is taken at vertex 'c' on it. Type A is used as i^{th} copy if $i \equiv 1(mod\ 4)$ and copy C is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$, in construction of $G^{(k)}$. In structure 6, type A and type C are used repeatedly. The one point union is taken at vertex 'f' on it. Type A is used as i^{th} copy if $i \equiv 1(mod\ 4)$ and copy C is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$ in construction of $G^{(k)}$. In structure 7, type A and type C are used repeatedly. The one point union is taken at vertex 'g' on it. Type A is used as i^{th} copy if $i \equiv 1(mod\ 4)$ and copy C is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$ in construction of $G^{(k)}$. For above structures the label distribution is given by: on vertices $v_f(0,1) = (4+7x, 4+7x)$ and on edges $e_f(0,1) = (4x, 4x)$ when $k = 2x+1$, $x = 0, 1, 2...$ and when k is of type $k = 2x$, on vertices we have $v_f(0,1) = (7x+1, 7x)$ and on edges $e_f(0,1) = (4x, 4x)$. In structure 1, type A and type B are used repeatedly. The one point union is taken at vertex 'a' on it.Type A is used as i^{th} copy if $i \equiv 1(mod\ 2)$ and copy B is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$ in construction of $G^{(k)}$. In structure 4, type A and type B are used repeatedly. The one point union is taken at vertex 'd' on it. Type A is used as i^{th} copy if $i \equiv 1(mod\ 2)$ and copy B is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$ in construction of $G^{(k)}$. In structure 5, type A and type B are used repeatedly. The one point union is taken at vertex 'e' on it. Type A is used as i^{th} copy if $i \equiv 1(mod\ 2)$ and copy B is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$ in construction of $G^{(k)}$. For above structures the label distribution is given by: on vertices $v_f(0,1) = (4+7x, 4+7x)$ and on edges $e_f(0,1) = (4x, 4x)$ when $k = 2x+1$, $x = 0, 1, 2...$ and when k is of type $k = 2x$, on vertices we have $v_f(0,1) = (7x, 7x+1)$ and on edges $e_f(0,1) = (4x, 4x)$. □

Theorem 3.8. *$G^{(k)}$ is cordial where $G = tail(C_4, P_5)$.*

Proof. Define a function $f : V(G) \to \{0, 1\}$ as follows. It introduces four types of labeling units as given below. Type C and Type B are not cordial while Type A is cordial. We combine them suitably to obtain a labeled copy of $G^{(k)}$.

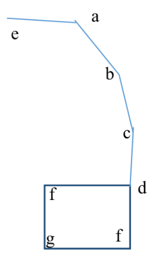

Fig 4.38 Tail(C_4,P_2)

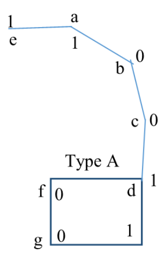

Fig 4.39 $v_f(0,1)$= (4,4), $e_f(0,1)$=(4,4)

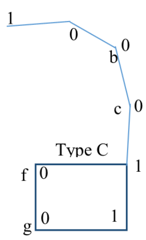

Fig 4.40 $v_f(0,1)$= (5,3), $e_f(0,1)$=(4,4)

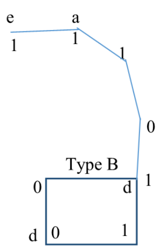

Fig 4.41 $v_f(0,1)$= (3,5) $e_f(0,1)$=(4,4)

We take one point union on any of the vertices 'e', 'a', 'b', 'c' , 'd', 'f',and 'g' (see fig 4.38 above). In that case we get structure 1, structure 2, structure 3, structure 4, structure 5, structure 6 and structure 7 respectively. All these structures are pairwise non-isomorphic. In structure 1, type A and type B are used repeatedly .The one point union is taken at vertex ‘e’ on it. Type A is used as i^{th} copy if $i \equiv 1(mod\ 4)$ and copy B is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$ in construction of $G^{(k)}$. In structure 2, type A and type B are used repeatedly .The one point union is taken at vertex ‘a’ on it. Type A is used as i^{th} copy if $i \equiv 1(mod\ 4)$ and copy B is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$ in construction of $G^{(k)}$. In structure 5, type A and type B are used repeatedly. The one point union is taken at vertex ‘d’ on it. Type A is used as i^{th} copy if $i \equiv 1(mod\ 4)$ and copy B is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$ in construction of $G^{(k)}$. For above structures the label distribution is given by: on vertices $v_f(0, 1) = (4+7x, 4+7x)$ and on edges $e_f(0, 1) = (4x, 4x)$ when $k = 2x + 1$, $x = 0, 1, 2, ...$ and when k is of type $k = 2x$, on vertices we have $v_f(0, 1) = (7x, 7x + 1)$ and on edges $e_f(0, 1) = (4x, 4x)$. In structure 3, type A and type C are used repeatedly. The one point union is taken at vertex ‘b’ on it. Type A is used as i^{th} copy if $i \equiv 1(mod\ 4)$ and copy C is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$ in construction of $G^{(k)}$. In structure 4, type A and type C are used repeatedly. The one point union is taken at vertex ‘c’ on it. Type A is used as i^{th} copy if $i \equiv 1(mod\ 4)$ and copy C is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$ in construction of $G^{(k)}$. In structure 6, type A and type C are used repeatedly. The one point union is taken at vertex ‘f’ on it. Type A is used as i^{th} copy if $i \equiv 1(mod\ 4)$ and copy C is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$ in construction of $G^{(k)}$. In structure 7, type A and type C are used repeatedly. The one point union is taken at vertex ‘g’ on it. Type A is used as i^{th} copy if $i \equiv 1(mod\ 4)$ and copy C is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$ in construction of $G^{(k)}$. For above structures the label distribution is given by: on vertices $v_f(0, 1) = (4 + 7x, 4 + 7x)$ and on edges $e_f(0, 1) = (4x, 4x)$ when $k = 2x + 1$, $x = 0, 1, 2, ...$ and when k is of type $k = 2x$, on vertices we have $v_f(0, 1) = (7x + 1, 7x)$ and on edges $e_f(0, 1) = (4x, 4x)$. The graph is cordial. □

Theorem 3.9. *$G^{(k)}$ is cordial where $G = tail(C_4, 2 - P_3)$.*

Proof. Define a function $f : V(G) \to \{0, 1\}$ as follows. It introduces four types of labeling units as given below. Type C and Type B are not cordial while Type A is cordial. We combine them suitably to obtain a labeled copy of $G^{(k)}$.

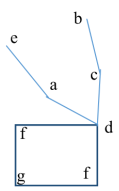

Fig 4.42 Tail(C_4,P_2)

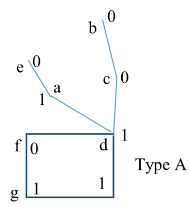

Fig 4.43 $v_f(0,1)$= (4,4), $e_f(0,1)$=(4,4)

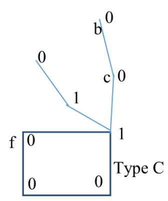

Fig 4.44 $v_f(0,1)$= (5,3), $e_f(0,1)$=(4,4)

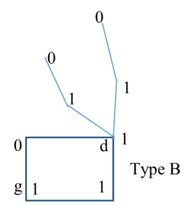

Fig 4.45 $v_f(0,1)$= (3,5) $e_f(0,1)$=(4,4)

We take one point union on any of the vertices ‘e’or ‘b’, ‘a’ or ‘c’ , ‘d’, ’f’ , ‘g’, (see fig 4.42 above). In that case we get structure 1, structure 2, structure 3, structure 4, structure 5, structure 6 and structure 7 respectively. All these structures are pairwise non-isomorphic. In structure 1, type A and type C are used repeatedly. The one point union is taken at vertex ‘b’ on it. Type A is used as i^{th} copy if $i \equiv 1(mod\ 4)$ and copy C is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$ in construction of $G^{(k)}$. In structure 2, type A and type C are used repeatedly. The one point union is taken at vertex 'c' on it. Type A is used as i^{th} copy if $i \equiv 1(mod\ 4)$ and copy C is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$ in construction of $G^{(k)}$. In structure 4, type A and type C are used repeatedly. The one point union is taken at vertex 'd' on it. Type A is used as i^{th} copy if $i \equiv 1(mod\ 4)$ and copy C is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$ in construction of $G^{(k)}$. For above structures the label distribution is given by: on vertices $v_f(0,1) = (4+7x, 4+7x)$ and on edges $e_f(0,1) = (4x, 4x)$ when $k = 2x+1$, $x = 0, 1, 2, ...$ and when k is of type $k = 2x$, on vertices we have $v_f(0,1) = (7x+1, 7x)$ and on edges $e_f(0,1) = (4x, 4x)$. In structure 3, type A and type B are used repeatedly. The one point union is taken at vertex 'd' on it. Type A is used as i^{th} copy if $i \equiv 1(mod\ 4)$ and copy B is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$ in construction of $G^{(k)}$. In structure 5, type A and type B are used repeatedly. The one point union is taken at vertex 'g' on it. Type A is used as i^{th} copy if $i \equiv 1(mod\ 4)$ and copy C is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$ in construction of $G^{(k)}$. For above structures the label distribution is given by: on vertices $v_f(0,1) = (4+7x, 4+7x)$ and on edges $e_f(0,1) = (4x, 4x)$ when $k = 2x+1$, $x = 0, 1, 2, ...$ and when k is of type $k = 2x$, on vertices we have $v_f(0,1) = (7x, 7x+1)$ and on edges $e_f(0,1) = (4x, 4x)$. The graph is cordial. □

Theorem 3.10. *$G^{(k)}$ is cordial where $G = tail(C_4, 2 - P_2, P_3)$.*

Proof. Define a function $f : V(G) \to \{0,1\}$ as follows. It introduces four types of labeling units as given below. Type C and Type B are not cordial while Type A is cordial. We combine them suitably to obtain a labeled copy of $G^{(k)}$.

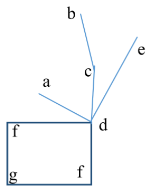
Fig 4.46 Tail(C_4,P_2)

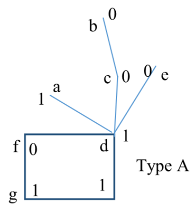

Fig 4.47 $v_f(0,1)$= (4,4), $e_f(0,1)$=(4,4)

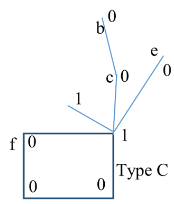

Fig 4.48 $v_f(0,1)$= (5,3), $e_f(0,1)$=(4,4)

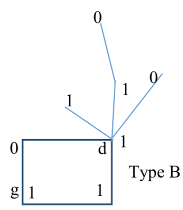

Fig 4.49 $v_f(0,1)$= (3,5) $e_f(0,1)$=(4,4)

We take one point union on any of the vertices 'b', 'a' or 'e', 'c','d', 'f' , 'g', (see fig 4.46 above). In that case we get structure 1, structure 2, structure 3, structure 4, structure 5, structure 6 and structure 7 respectively. All these structures are pairwise non-isomorphic. In structure 1, type A and type C are used repeatedly. The one point union is taken at vertex 'b' on it. Type A is used as i^{th} copy if $i \equiv 1(mod\ 4)$ and copy C is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$ in construction of $G^{(k)}$. In structure 2, type A and type C are used repeatedly. The one point union is taken at vertex 'e' on it. Type A is used as i^{th} copy if $i \equiv 1(mod\ 4)$ and copy C is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$ in construction of $G^{(k)}$. In structure 3, type A and type C are used repeatedly. The one point union is taken at vertex 'c' on it. Type A is used as i^{th} copy if $i \equiv 1(mod\ 4)$ and copy C is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$ in construction of $G^{(k)}$. In structure 5, type A and type C are used repeatedly. The one point union is taken at vertex 'f' on it. Type A is used as i^{th} copy if $i \equiv 1(mod\ 4)$ and copy C is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$ in construction of $G^{(k)}$. For above structures the label distribution is given by: on vertices $v_f(0,1) = (4+7x, 4+7x)$ and on edges $e_f(0,1) = (4x, 4x)$ when $k = 2x+1$,

$x = 0, 1, 2, ...$ and when k is of type $k = 2x$, on vertices we have $v_f(0, 1) = (7x + 1, 7x)$ and on edges $e_f(0, 1) = (4x, 4x)$. In structure 4, type A and type B are used repeatedly. The one point union is taken at vertex 'f' on it. Type A is used as i^{th} copy if $i \equiv 1(mod\ 4)$ and copy B is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$ in construction of $G^{(k)}$. In structure 6, type A and type B are used repeatedly. The one point union is taken at vertex 'g' on it. Type A is used as i^{th} copy if $i \equiv 1(mod\ 4)$ and copy C is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$ in construction of $G^{(k)}$. For above structures the label distribution is given by: on vertices $v_f(0, 1) = (4 + 7x, 4 + 7x)$ and on edges $e_f(0, 1) = (4x, 4x)$ when $k = 2x + 1$, $x = 0, 1, 2, ...$ and when k is of type $k = 2x$, on vertices we have $v_f(0, 1) = (7x, 7x + 1)$ and on edges $e_f(0, 1) = (4x, 4x)$. □

Theorem 3.11. *$G^{(k)}$ is cordial where $G = tail(C_4, 4 - P_2)$.*

Proof. Define a function $f : V(G) \to \{0, 1\}$ as follows. It introduces three types of labeling units as given below. Type C and Type B are not cordial while Type A is cordial. We combine them suitably to obtain a labeled copy of $G^{(k)}$.

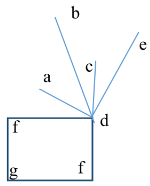

Fig 4.50 Tail(C_4,P_2)

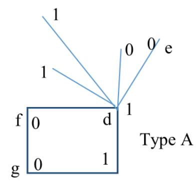

Fig 4.51 $v_f(0,1)$= (4,4), $e_f(0,1)$=(4,4)

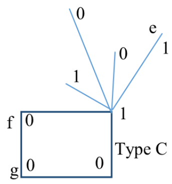

Fig 4.52 $v_f(0,1)$= (5,3), $e_f(0,1)$=(4,4)

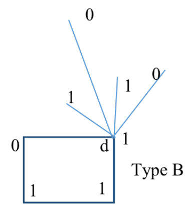

Fig 4.53 $v_f(0,1)$= (3,5) $e_f(0,1)$=(4,4)

We take one point union on any of the vertices (pendent vertices) 'a', or 'b', or 'c', or 'e', 'd', 'f' , 'g', (see fig 4.50 above). In that case we get structure 1, structure 2, structure 3, structure 4 respectively. All these structures are pairwise non-isomorphic. In Structure 1, type A and type C are used repeatedly. The one point union is taken at vertex 'e' on it. Type A is used as i^{th} copy if $i \equiv 1(mod\ 4)$ and copy C is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$ in construction of $G^{(k)}$. In structure 3, type A and type C are used repeatedly. The one point union is taken at vertex 'f' on it. Type A is used as i^{th} copy if $i \equiv 1(mod\ 4)$ and copy C is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$ in construction of $G^{(k)}$. In structure 4, type A and type C are used repeatedly. The one point union is taken at vertex 'g' on it. Type A is used as i^{th} copy if $i \equiv 1(mod\ 4)$ and copy C is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$ in construction of G$^{(k}$. For above structures the label distribution is given by: on vertices $v_f(0, 1) = (4 + 7x, 4 + 7x)$ and on edges $e_f(0, 1) = (4x, 4x)$ when $k = 2x + 1$, $x = 0, 1, 2, ...$ and when k is of type $k = 2x$, on vertices we have $v_f(0, 1) = (7x + 1, 7x)$ and on edges $e_f(0, 1) = (4x, 4x)$. In structure 2, type A and type B are used repeatedly. The one point union is taken at vertex 'd' on it. Type A is used as i^{th} copy if $i \equiv 1(mod\ 4)$ and copy B is used at i^{th} copy if $i \equiv 0(mod\ 2)$ for all $i = 1, 2, ..., k$ in construction of $G^{(k)}$. For structures 2 the label distribution is given by: on vertices $v_f(0, 1) = (4 + 7x, 4 + 7x)$ and on edges $e_f(0, 1) = (4x, 4x)$ when $k = 2x + 1$, $x = 0, 1, 2, ...$ and when k is of type $k = 2x$, on vertices we have $v_f(0, 1) = (7x, 7x + 1)$ and on edges $e_f(0, 1) = (4x, 4x)$. The graph is cordial. □

4. Conclusion

A tail graph $tail(G, P_m)$ is obtained by attaching a path P_m to a vertex on G. We have investigated and shown that one point union of k copies of $tail(G, P_m)$ for $G = C_4$ and $m = 2$ to 5 is cordial. We also have split P_m to obtain more tails keeping sum of edges fixed to $m - 1$. We expect that one point union of $tail(C_4, P_m)$ is cordial for all m.

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