Int. J. Math. And Appl., 6(1–E)(2018), 1133–1149
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Haar Wavelet Collocation Method for the Numerical Solution of Integral and Integro-Differential Equations

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Abstract: Haar wavelet collocation method is developed for the numerical solution of nonlinear Fredholm, Volterra, mixed Volterra-Fredholm integral and integro-differential equations. The method is tested on some of illustrative examples and made a comparison with the exact solution and existing methods. It shows that the proposed method yields better results than the others. Hence, the proposed scheme is a new alternative approach and efficient numerical method for the solution of nonlinear integral and integro-differential equations.

Keywords: Haar wavelet, Collocation method, Integral equations, Integro-differential equations.

1. Introduction

Integral and integro-differential equations find its applications in various fields of science and engineering. There are several numerical methods for approximating the solution of integral and integro-differential equations are known and many different basic functions have been used. In numerical analysis solving integral equations are reducing it to a system of equations. There are various methods to solve integral and integro-differential equations such as Adomian decomposition method, successive substitutions, Laplace transformation method, Picard's method, etc [1].

Wavelets theory is a relatively new and an emerging tool in applied mathematical research area. It has been applied in a wide range of engineering disciplines; particularly, signal analysis for waveform representation and segmentations, time-frequency analysis and fast algorithms for easy implementation. Wavelets permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms [2, 3]. Since 1991 the various types of wavelet method have been applied for the numerical solution of different kinds of integral equations, a detailed survey on these papers can be found in [4].

The solutions are often quite complicated and the advantages of the wavelet method get lost. Therefore any kind of simplification is welcome. One possibility for it is to make use of the Haar wavelets, which are mathematically the simplest wavelets. In the previous work, system analysis via Haar wavelets was led by Chen and Hsiao [5], who first derived a Haar operational matrix for the integrals of the Haar function vector and put the applications for the Haar

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analysis into the dynamic systems. Recently, Haar wavelet method is applied for different type of problems. Also, Haar wavelet method is applied for different kind of integral equations, which among the Lepik et al. [6–12] presented the solution for differential and integral equations. Babolian et al. [13] applied for solving nonlinear Fredholm integral equations.

Aziz et al. [14] have introduced a new algorithm for the numerical solution of nonlinear Fredholm and Volterra integral equations. Islam et. al [15], Mishra et. al [16] have applied the Haar wavelet method for solving nonlinear volterra integro-differential equations. Ramane et al. [23] have applied a new Hosoya polynomial of path graphs for the numerical solution of Fredholm integral equations. In fact the applications of the Haar wavelet collocation method based on Leibnitz rule in the numerical analysis field is not new and on the other side possessing some of the well known advantages such as:

(1). It is accurate, needless effort to achieve the results,

(2). It is possible to pick any point in the interval of integration and as well the approximate solutions and their derivatives will be applicable.

(3). The method does not require discretization of the variables, and it is not affected by computation round off errors and one is not faced with necessity of large computer memory and time.

(4). It is of global nature in terms of the solutions obtained as well as its ability to solve other mathematical, physical, and engineering problems.

In this paper, we applied the Haar wavelet collocation method (HWCM) based on Leibnitz rule for the numerical solution of integral and integro-differential equations.

2. Properties of Haar Wavelets

2.1. Haar wavelets

The scaling function $h_1(t)$ for the family of the Haar wavelet is defined as

$$h_1(t) = \begin{cases} 1 & for\ t \in [0,\ 1) \\ 0 & otherwise \end{cases} \tag{1}$$

The Haar Wavelet family for $t \in [0, 1)$ is defined as,

$$h_i(t) = \begin{cases} 1 & for\, t \in [\alpha, \beta), \\ -1 & for\, t \in [\beta, \gamma), \\ 0 & elsewhere, \end{cases} \tag{2}$$

where $\alpha = \frac{k}{m}$, $\beta = \frac{k+0.5}{m}$, $\gamma = \frac{k+1}{m}$, where $m = 2^l$, $l = 0, 1, \ldots, J$, J is the level of resolution; and $k = 0, 1, \ldots, m-1$ is the translation parameter. Maximum level of resolution is J. The index i in (2) is calculated using $i = m + k + 1$. In case of minimal values $m = 1$, $k = 0$ then $i = 2$. The maximal value of i is $N = 2^{J+1}$. Let us define the collocation points $t_j = \frac{j-0.5}{N}$, $j = 1, 2, \ldots, N$. Haar coefficient matrix $h\,(i, j) = h_i(t_j)$ which has the dimension $N \times N$. For instance, $J = 3 \Rightarrow N = 16$, then we have

$$h\,(16,16)=\begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&-1&-1&-1&-1&-1&-1&-1&-1\\
1&1&1&1&-1&-1&-1&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&-1&-1&-1&-1\\
1&1&-1&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&-1&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&-1&-1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&-1&-1\\
1&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&-1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&-1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&-1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&-1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&-1
\end{pmatrix}$$

Any function $f(t)$ which is square integrable in the interval (0, 1) can be expressed as an infinite sum of Haar wavelets as

$$f(t)=\sum_{i=1}^{\infty}a_i h_i(t) \tag{3}$$

The above series terminates at finite terms if $f(t)$ is piecewise constant or can be approximated as piecewise constant during each subinterval. Given a function $f(t)\in L^2(R)$ a multi-resolution analysis (MRA) of $L^2(R)$ produces a sequence of subspaces $V_j, V_{j+1}, \ldots$ such that the projections of $f(t)$ onto these spaces give finer and finer approximations of the function $f(t)$ as $j\to\infty$.

2.2. Operational Matrix of Haar Wavelet

The operational matrix P which is an N square matrix is defined by

$$P_{1,i}(t)=\int_0^t h_i(s)\,ds \tag{4}$$

often, we need the integrals

$$P_{r,i}(t)=\underbrace{\int_A^t\int_A^t \ldots \int_A^t h_i(s)\;ds^r}_{r-times}=\frac{1}{(r-1)!}\int_A^t (t-s)^{r-1}h_i(s)\,ds \tag{5}$$

$r=1,2,\ldots n$ and $i=1,2,\ldots,N$. For $r=1$, corresponds to the function $P_{1,i}(t)$, with the help of (2) these integrals can be calculated analytically; we get

$$P_{1,i}\,(t)=\begin{cases} t-\alpha & for \quad t\in[\alpha,\beta)\\ \gamma-t & for \quad t\in[\beta,\gamma)\\ 0 & Otherwise\end{cases} \tag{6}$$

$$P_{2,i}(t) = \begin{cases} \frac{1}{2}(t-a)^2 & for \quad t \in [\alpha, \beta) \\ \frac{1}{4m^2} - \frac{1}{2}(\gamma - t)^2 & for \quad t \in [\beta, \gamma) \\ \frac{1}{4m^2} & for \quad t \in [\gamma, 1) \\ 0 & Otherwise \end{cases} \tag{7}$$

In general, the operational matrix of integration of r^{th} order is given as

$$P_{r,i}(t) = \begin{cases} \frac{1}{r!}(t-\alpha)^r & for \quad t \in [\alpha, \beta) \\ \frac{1}{r!}\{(t-\alpha)^r - 2(t-\beta)^r\} & for \quad t \in [\beta, \gamma) \\ \frac{1}{r!}\{(t-\alpha)^r - 2(t-\beta)^r + (t-\gamma)^r\} & for \quad t \in [\gamma, 1) \\ 0 & Otherwise \end{cases} \tag{8}$$

For instance, $J = 3 \Rightarrow N = 16$, then we have

$$P_{1,i}(16,16) = \frac{1}{32}\begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 19 & 21 & 23 & 25 & 27 & 29 & 31 \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 15 & 13 & 11 & 9 & 7 & 5 & 3 & 1 \\ 1 & 3 & 5 & 7 & 7 & 5 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 5 & 7 & 7 & 5 & 3 & 1 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 3 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$P_{2,i}(16,16)=\frac{1}{2048}\begin{pmatrix}
1 & 9 & 25 & 49 & 81 & 121 & 169 & 225 & 289 & 361 & 441 & 529 & 625 & 729 & 841 & 961\\
1 & 9 & 25 & 49 & 81 & 121 & 169 & 225 & 287 & 343 & 391 & 431 & 463 & 487 & 503 & 511\\
1 & 9 & 25 & 49 & 79 & 103 & 119 & 127 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 9 & 25 & 49 & 79 & 103 & 119 & 127\\
1 & 9 & 23 & 31 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32\\
0 & 0 & 0 & 0 & 1 & 9 & 23 & 31 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 9 & 23 & 31 & 32 & 32 & 32 & 32\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 9 & 23 & 31\\
1 & 7 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8\\
0 & 0 & 1 & 7 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8\\
0 & 0 & 0 & 0 & 1 & 7 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 8 & 8 & 8 & 8 & 8 & 8\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 8 & 8 & 8 & 8\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 8 & 8\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7
\end{pmatrix}$$

3. Haar Wavelet Collocation Method (HWCM) of Solution

In this section, we present a Haar wavelet collocation method (HWCM) based on Leibnitz rule for the numerical solution of nonlinear integral and integro-differential equations of the form,

(i). Nonlinear Volterra integral equations

$$u(t)=f(t)+\lambda\int_a^t k_1(t,s,u(s))ds$$

(ii). Nonlinear Fredholm integral equations

$$u(t)=f(t)+\int_0^1 k_2(t,s,u(s))\,ds\,,$$

(iii). Nonlinear Fredholm-Hammerstein integral equations

$$u(t)=f(t)+\int_0^1 k_2(t,s)g(s,u(s))\,ds\,,$$

(iv). Nonlinear Volterra-Fredholm integral equations

$$u(t)=f(t)+\int_0^t k_1(t,s,u(s))ds\;+\int_0^1 k_2(t,s,u(s))\,ds\;,$$

(v). Nonlinear Volterra-Fredholm-Hammerstein integral equations

$$u(t)=f(t)+\int_0^t k_1(t,s)g(s,u(s))ds\;+\int_0^1 k_2(t,s)g(s,u(s))\,ds\;,$$

(vi). Nonlinear Volterra-integro-differential equations

$$u'(t)=f(t)+\int_0^t k_1(t,s,u(s))ds$$

(vii). Nonlinear Fredholm integro-differential equations

$$u'(t) = f(t) + \int_0^1 k_2(t, s, u(s))ds$$

(viii). Nonlinear Fredholm-Hammerstein integro-differential equations

$$u'(t) = f(t) + \int_0^1 k_2(t, s)g(s, u(s))ds$$

where $k_1(t, s, u(s))$ and $k_2(t, s, u(s))$ is a nonlinear function defined on $[0, 1] \times [0, 1]$ are the known function is called the kernel of the integral equation and $f(t)$ is also a known function, while the unknown function $u(t)$ represents the solution of the integral equation. Basic principle is that for conversion of the integral equation into equivalent differential equation with initial conditions. The conversion is achieved by the well-known Leibnitz rule [1]. Numerical computational Procedure is as follows,

Step 1: Differentiating above equations w.r.t t using Leibnitz rule, we get differential equations with subject to initial conditions $u(0) = \beta$, $u'(0) = \gamma$.

Step 2: Applying Haar wavelet collocation method. Let us assume that,

$$u''(t) = \sum_{i=1}^{N} a_i h_i(t) \tag{9}$$

Step 3: By integrating (9) twice and substituting the initial conditions, we get,

$$u'(t) = \gamma + \sum_{i=1}^{N} a_i p_{1,i}(t) \tag{10}$$

$$u(t) = \beta + \gamma t + \sum_{i=1}^{N} a_i p_{2,i}(t) \tag{11}$$

Step 4: Substituting (9)-(11) in the differential equation, which reduces to the nonlinear system of N equations with N unknowns and then the Newton's method is used to obtain the Haar coefficients a_i, $i = 1, 2, \ldots, N$. Substituting Haar coefficients in (11) to obtain the required approximate solution of equation.

4. Numerical Experiments

In this section, we consider the some of the examples to demonstrate the capability of the present method and error function is presented to verify the accuracy and efficiency of the following numerical results:

$$Error\ function = E_{\max} = \|u_e(t_i) - u_a(t_i)\|_\infty = \sqrt{\sum_{i=1}^{n} (u_e(t_i) - u_a(t_i))^2}$$

where u_e and u_a are the exact and approximate solution respectively.

Example 4.1. *Let us consider the Nonlinear Volterra-Fredholm integral equation [17],*

$$u(t) = \frac{1}{6}t + \frac{1}{2}t\exp(-t^2) + \int_0^t ts\exp(-u^2(s))\,ds + \int_0^1 t\,u^2(s)\,ds\ ,\ 0 \le x, t \le 1 \tag{12}$$

with the initial conditions $u(0) = 0$, $u'(0) = 1$. Which has the exact solution $u(t) = t$. We applied the present technique and solved Equation (12) as follows, successively differentiating Equation (12) twice w.r.t t and using Leibnitz rule, we get differential equation,

$$u''(t) - \frac{2t^3}{\exp(t^2)} + \frac{3t}{\exp(t^2)} - t\exp(-u^2(t)) - 2t\exp(-u^2(t)) - t^2\exp(-u^2(t))(-2\,u(t)\,u'(t)) = 0 \tag{13}$$

Let us assume that,

$$u''(t) = \sum_{i=1}^{N} a_i h_i(t) \tag{14}$$

integrating Equation (14) twice,

$$\begin{aligned} u'(t) - u'(0) &= \sum_{i=1}^{N} a_i p_{1,i}(t) \\ u'(t) &= \sum_{i=1}^{N} a_i p_{1,i}(t) + 1 \qquad (15) \\ u(t) - u(0) &= \sum_{i=1}^{N} a_i p_{2,i}(t) + t \\ u(t) &= \sum_{i=1}^{N} a_i p_{2,i}(t) + t \qquad (16) \end{aligned}$$

substituting Equation (14)-Equation (16) in Equation (13), we get the system of N equations with N unknowns,

$$\sum_{i=1}^{N} a_i h_i(t) - \frac{2t^3}{\exp(t^2)} + \frac{3t}{\exp(t^2)} - t\exp\left(-\left(\sum_{i=1}^{N} a_i p_{2,i}(t) + t\right)^2\right) - \exp\left(-\left(\sum_{i=1}^{N} a_i p_{2,i}(t) + t\right)^2\right) \cdot 2t$$
$$- t^2 \exp\left(-\left(\sum_{i=1}^{N} a_i p_{2,i}(t) + t\right)^2\right) \cdot \left(-2 \cdot \left(\sum_{i=1}^{N} a_i p_{2,i}(t) + t\right) \cdot \left(\sum_{i=1}^{N} a_i p_{1,i}(t) + 1\right)\right) = 0. \tag{17}$$

Solving Equation (17) using Newton's Method to obtain the Haar wavelet coefficients a_i for $N = 16$ i.e., $[-2.33e-09 \;\; 2.32e-09 \;\; 1.22e-12 \;\; 4.67e-09 \;\; 3.17e-12 \;\; 6.85e-12 \;\; -1.32e-11 \;\; 9.21e-09 \;\; 3.88e-12 \;\; -5.13e-12 \;\; 1.77e-11 \;\; -1.00e-11 \;\; 2.10e-12 \;\; 4.41e-11 \;\; 1.80e-11 \;\; 1.61e-08]$. Substituting a_i's, in Equation (16) obtain the approximate solution are given in table 1 and figure 1 shows the comparison with exact and existing method. Maximum error analysis is presented in table 2.

t	Exact	HWCM for N = 8	Method[17]	Error (HWCM)	Error[17]
0	0	0.00000000	0.00000000	5.25e-12	0.00e+00
0.1	0.1	0.10000000	0.06616916	4.65e-13	3.38e-02
0.2	0.2	0.20000000	0.13333039	4.56e-14	6.66e-02
0.3	0.3	0.30000000	0.20355479	4.91e-14	9.64e-02
0.4	0.4	0.40000000	0.27915784	1.31e-13	1.20e-01
0.5	0.5	0.50000000	0.36269547	6.73e-14	1.37e-01
0.6	0.6	0.60000000	0.45678660	8.33e-14	1.43e-01
0.7	0.7	0.70000000	0.56375872	4.21e-13	1.36e-01
0.8	0.8	0.80000000	0.68523040	1.09e-12	1.14e-01
0.9	0.9	0.90000000	0.82191762	1.14e-11	7.80e-02
1	1	1.00000000	0.97404036	7.18e-11	2.59e-02

Table 1. **Numerical results of the Example 4.1**

N	$E_{\max}\,(HWCM)$
4	1.73e-11
8	2.43e-11
16	3.03e-11
32	4.34e-11
64	5.70e-11
128	6.80e-11

Table 2. **Maximum error analysis of the Example 4.1**

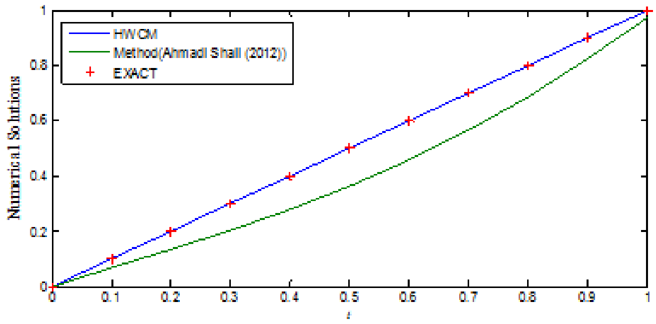

Figure 1. **Comparison of HWCM with exact and existing method of the Example 4.1**

Example 4.2. *Secondly, consider the Nonlinear Volterra-Fredholm-Hammerstein integral equation [18],*

$$u(t) = \frac{t}{2} - \frac{t^4}{12} - \frac{1}{3} + \int_0^1 (t+s)\,u(s)\,ds + \int_0^t (t-s)\,u^2(s)\,ds\,,\ 0 \le t\,, s \le 1 \tag{18}$$

Initial condition's: $u(0) = 0$, $u'(0) = 1$. *Exact solution of Equation* (18) *is* $u(t) = t$.

Differentiating Equation (18) twice w.r.t t and using Leibnitz rule which reduces to the differential equation,

$$u'(t) = \frac{1}{2} - \frac{1}{3}t^3 + \int_0^1 u(s)\,ds + \int_0^t u^2(s)\,ds \tag{19}$$

$$u''(t) - [u(t)]^2 + t^2 = 0 \tag{20}$$

Assume that,

$$u''(t) = \sum_{i=1}^{N} a_i h_i(t) \tag{21}$$

integrating Equation (21) twice,

$$u'(t) = \sum_{i=1}^{N} a_i p_{1,i}(t) + 1 \tag{22}$$

$$u(t) = \sum_{i=1}^{N} a_i p_{2,i}(t) + t \tag{23}$$

substituting Equation (21)-(23) in Equation (20), we get the system of N equations with N unknowns,

$$\sum_{i=1}^{N} a_i h_i(t) - \left[\sum_{i=1}^{N} a_i p_{2,i}(t) + t\right]^2 + t^2 = 0. \tag{24}$$

Solving Equation (24) using Newton's method to find Haar wavelet coefficients a_i's for $N = 16$, i.e., $[-1.93e-11\ \ 1.93e-11\ \ 8.48e-19\ \ 3.70e-11\ \ -2.56e-18\ \ 1.08e-12\ \ 6.99e-11\ \ -5.24e-20\ \ -1.59e-18\ \ -7.87e-18\ \ 8.86e-18\ \ 4.92e-13\ \ 1.53e-12\ \ 2.60e-12\ \ 9.36e-11]$. Substituting a_i's, in Equation (23) and obtained the required HWCM solution with exact solution is presented in table 3. Error analysis is shown in table 4. This justifies the efficiency of the HWCM.

t(/32)	Exact	(HWCM)	Error (HWCM)
1	0.03125	0.03125	0
3	0.09375	0.09375	0
5	0.15625	0.15625	0
7	0.21875	0.21875	0

$t(/32)$	Exact	(HWCM)	Error (HWCM)
9	0.28125	0.28125	0
11	0.34375	0.34375	0
13	0.40625	0.40625	0
15	0.46875	0.46875	0
17	0.53125	0.53125	0
19	0.59375	0.59375	4.44e-16
21	0.65625	0.65625	4.44e-15
23	0.71875	0.71875	1.41e-14
25	0.78125	0.78125	3.82e-14
27	0.84375	0.84375	7.70e-14
29	0.90625	0.90625	1.67e-13
31	0.96875	0.96875	5.29e-13

Table 3. **Comparision of exact and approximate solution of the Example 4.2**

N	$E_{\max}$(HWCM)
4	1.93e-13
8	1.68e-13
16	5.29e-13
32	6.66e-13
64	6.55e-13
128	9.38e-13

Table 4. **Error analysis of the Example 4.2**

Example 4.3. *Next, consider the Nonlinear Volterra Integral equation [14],*

$$u(t) = \frac{3}{2} - \frac{1}{2}e^{-2t} - \int_0^t [(u(s))^2 + u(s)]ds, \ \ 0 \le t \le 1,$$

with initial conditions $u(0) = 1$. *Which has the exact solution* $u(t) = e^{-t}$.

$$u(t) = \frac{3}{2} - \frac{1}{2}e^{-2t} - \int_0^t [(u(s))^2 + u(s)]ds, \ \ 0 \le t \le 1, \tag{25}$$

Successively differentiating Equation (25) w.r.t t and using Leibnitz rule reduces to the differential equation,

$$u'(t) = e^{-2t} + (u(t)^2 + u(t)) \tag{26}$$

$$u'(t) - (u(t)^2 + u(t)) - e^{-2t} = 0 \tag{27}$$

Assume that,

$$u'(t) = \sum_{i=1}^{N} a_i h_i(t) \tag{28}$$

integrating Equation (28),

$$u(t) = \sum_{i=1}^{N} a_i p_{1,i}(t) + 1 \tag{29}$$

substituting Equation (28) and (29) in (27), we get the system of N equations with N unknowns.

$$\sum_{i=1}^{N} a_i h_i(t) - \left(\left(\sum_{i=1}^{N} a_i p_{1,i}(t) + 1 \right)^2 + \left(\sum_{i=1}^{N} a_i p_{1,i}(t) + 1 \right) \right) - e^{-2t} = 0 \tag{30}$$

solving (30) using Matlab to find Haar wavelet coefficients a_i's, for $N = 16$ i.e, $[-0.63 \ \ -0.16 \ \ -0.10 \ \ -0.06 \ \ -0.06 \ \ -0.04 \ \ -0.03 \ \ -0.03 \ \ -0.03 \ \ -0.03 \ \ -0.02 \ \ -0.02 \ \ -0.02 \ \ -0.02 \ \ -0.01 \ \ -0.01]$. Substituting a_i's, in Equation (29) and obtained the required HWCM solution compared with exact solutions is shown in table 6. Error analysis is given in table 5, which justifies the efficiency of the HWCM.

N	$E_{\max}$(HWCM)
4	5.30e-3
8	1.60e-3
16	4.38e-4
32	1.15e-4
64	2.96e-5
128	7.52e-6

Table 5. **Error analysis of the Example 4.3.**

t=(/32)	HWCM	Exact
1	0.9697	0.9692
3	0.9109	0.9105
5	0.8556	0.8553
7	0.8037	0.8035
9	0.7550	0.7548
11	0.7092	0.7091
13	0.6662	0.6661
15	0.6259	0.6258
17	0.5879	0.5879
19	0.5523	0.5523
21	0.5188	0.5188
23	0.4874	0.4874
25	0.4578	0.4578
27	0.4301	0.4301
29	0.4040	0.4040
31	0.3795	0.3796

Table 6. **Comparison of Exact and HWCM for N= 16 of the Example of 4.3**

Example 4.4. *Next, consider the Nonlinear Fredholm Integral equation [14],*

$$u(t) = -t^2 - \frac{t}{3}(2\sqrt{2}-1) + 2 + \int_0^1 ts\,\sqrt{u(s)}\;ds, \qquad 0 \le t \le 1,$$

with initial conditions $u(0) = 2$, $u'(0) = 0$. Which has the exact solution $u(t) = 2 - t^2$.

$$u(t) = -t^2 - \frac{t}{3}(2\sqrt{2}-1) + 2 + \int_0^1 ts\,\sqrt{u(s)}\;ds, \qquad 0 \le t \le 1, \tag{31}$$

Successively differentiating Equation (31) twice w.r.t t, using Leibnitz rule reduces to the differential equation,

$$u'(t) = -2t - \frac{1}{3}(2\sqrt{2}-1) + \int_0^1 s\,\sqrt{u(s)}ds, \tag{32}$$

$$u''(t) + 2 = 0 \tag{33}$$

Assume that,

$$u''(t) = \sum_{i=1}^{N} a_i h_i(t) \tag{34}$$

integrating Equation (34) twice,

$$u'(t) = \sum_{i=1}^{N} a_i p_{1,i}(t) \tag{35}$$

$$u(t) = \sum_{i=1}^{N} a_i p_{2,i}(t) + 2 \tag{36}$$

Substituting Equation (34)-(36) in Equation (33), we get the system of N equations with N unknowns,

$$\sum_{i=1}^{N} a_i h_i(t) + 2 = 0. \tag{37}$$

Solving Equation (37) using Matlab to find Haar wavelet coefficients a_i's. For $N = 16$ i.e., $[-2.00 \;\; 0.00 \;\; 0.00 \;\; 0.00 \;\; 0.00 \;\; 0.00 \;\; 0.00 \;\; 0.00 \;\; 0.00 \;\; 0.00 \;\; 0.00 \;\; 0.00 \;\; 0.00 \;\; 0.00 \;\; 0.00 \;\; 0.00]$. Substituting a_i's, in Equation (36) and obtained the accurate HWCM solutions is compared with the exact solutions is shown in table 7, which justifies the efficiency of the HWCM.

t=(/32) HWCM	Exact
11.9990	1.9990
31.9912	1.9912
51.9756	1.9756
71.9521	1.9521
91.9209	1.9209
111.8818	1.8818
131.8350	1.8350
151.7803	1.7803
171.7178	1.7178
191.6475	1.6475
211.5693	1.5693
231.4834	1.4834
251.3896	1.3896
271.2881	1.2881
291.1787	1.1787
311.0615	1.0615

Table 7. Comparison of Exact and HWCM for $N = 16$ of the Example 4.4

Example 4.5. *Next, consider the Nonlinear Fredholm Integro-differential equation [19],*

$$u'(t) = 1 - \frac{1}{2}t - \frac{t}{2e} + \int_0^1 ts\exp(-(u(s))^2)ds, \;\; 0 \le t \le 1, \tag{38}$$

with initial conditions $u(0) = 0$, $u'(0) = 1$, $u''(0) = 0$. Which has the exact solution $u(t) = t$.

Differentiating Equation (38) twice w.r.t t and using Leibnitz rule reduces to differential equation,

$$u''(t) = -\frac{1}{2} - \frac{1}{2e} + \int_0^1 s\exp(-(u(s))^2)ds$$
$$u'''(t) = 0 \tag{39}$$

assume that,

$$u'''(t) = \sum_{i=1}^{N} a_i h_i(t) \tag{40}$$

integrating Equation (40) thrice, we get

$$u''(t) = \sum_{i=1}^{N} a_i p_{1,i}(t) \tag{41}$$

$$u'(t) = \sum_{i=1}^{N} a_i p_{2,i}(t) + 1 \tag{42}$$

$$u(t) = \sum_{i=1}^{N} a_i p_{3,i}(t) + t \tag{43}$$

substituting Equations (40)-(43) in the differential equation Equation (39), we get the system of N equations with N unknowns,

$$\sum_{i=1}^{N} a_i h_i(t) = 0. \tag{44}$$

Solving (44) using Newton's Method to obtain Haar wavelet coefficients a_i's. substituting these coefficients in Equation (43) and obtained the required HWCM solutions, which gives the accurate solutions is presented in figure 2. This justifies the efficiency of the HWCM.

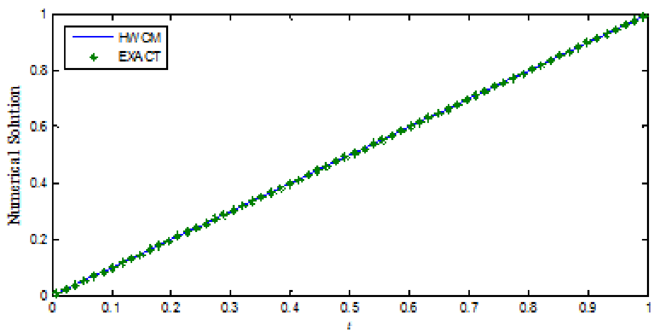

Figure 2. **Comparison of HWCM with exact solution for $N = 64$ of the Example 4.5**

Example 4.6. *Next, consider the Nonlinear Fredholm-Hammerstein Integro-differential equation [20],*

$$u'(t) = 2t + \frac{1}{8}(-\pi + \log(4)) + \int_0^1 s \arctan(u(s))ds, \quad 0 \le t \le 1 \tag{45}$$

with initial conditions $u(0) = 0$, $u'(0) = 0$. Which has the exact solution $u(t) = t^2$.

Differentiating Equation (45) w.r.t t and using Leibnitz rule reduces to differential equation,

$$u''(t) = 2 \tag{46}$$

assume that,

$$u''(t) = \sum_{i=1}^{N} a_i h_i(t) \tag{47}$$

integrating Equation (47) twice, we get

$$u'(t) = \sum_{i=1}^{N} a_i p_{1,i}(t) \tag{48}$$

$$u(t) = \sum_{i=1}^{N} a_i p_{2,i}(t) \tag{49}$$

substituting Equations (47)-(49) in the differential equation Equation (46), we get the system of N equations with N unknowns,

$$\sum_{i=1}^{N} a_i h_i(t) - 2 = 0. \tag{50}$$

Solving Equation (50) using Newton's Method to obtain Haar wavelet coefficients a_i's substituting these coefficients in Equation (49) and obtained the required HWCM solutions, which gives the accurate solutions is presented in figure 3. This justifies the efficiency of the HWCM.

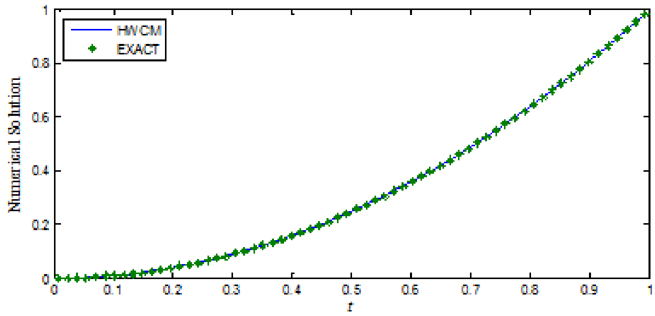

Figure 3. **Comparison of HWCM with exact solution for $N = 64$ of the Example 4.6**

Example 4.7. *Next, consider the nonlinear Volterra integro-differential equation [21],*

$$u'(t) = 1 + \int_0^t u(s)\,u'(s)ds, \;\; 0 \leq t \leq 1 \tag{51}$$

with initial conditions $u(0) = 0$, $u'(0) = 1$. *Which has the exact solution* $u(t) = \sqrt{2}\tan\left(\frac{t}{\sqrt{2}}\right)$.

Differentiating Equation (51) w.r.t t, using Leibnitz rule, we get the differential equation,

$$u''(t) - u(t)\,u'(t) = 0 \tag{52}$$

assume that,

$$u''(t) = \sum_{i=1}^{N} a_i h_i(t) \tag{53}$$

integrating Equation (53) twice,

$$u'(t) = \sum_{i=1}^{N} a_i p_{1,i}(t) + 1 \tag{54}$$

$$u(t) = \sum_{i=1}^{N} a_i p_{2,i}(t) + t \tag{55}$$

substituting Equations (53)-(55) in Equation (52), we get the system of N equations with N unknowns,

$$\sum_{i=1}^{N} a_i h_i(t) - \left(\sum_{i=1}^{N} a_i p_{2,i}(t) + t\right) \cdot \left(\sum_{i=1}^{N} a_i p_{1,i}(t) + 1\right) = 0. \tag{56}$$

Solving Equation (56) using Newton's Method to obtain Haar wavelet coefficients a_i's for $N = 32$ i.e., [0.7303 -0.4579 -0.1448 -0.3576 -0.0648 -0.0825 -0.1291 -0.2408 -0.0315 -0.0336 -0.0379 -0.0451 -0.0564 -0.0737 -0.1005 -0.1429 -0.0157 -0.0159 -0.0164 -0.0172 -0.0182 -0.0197 -0.0214 -0.0237 -0.0265 -0.0300 -0.0343 -0.0396 -0.0462 -0.0545 -0.0650 -0.0783]. Substituting a_i's, in Equation (55) yields the HWCM solution. In table 9 and figure 4, the HWCM solution is compared with the exact solution and ADM. Further, error analysis of HWCM is shown in table 8.

N	$E_{\max}$(HWCM)
4	1.67e-03
8	4.56e-04
16	1.19e-04

N	$E_{\max}$(HWCM)
32	3.04e-05
64	7.84e-06
128	2.13e-06

Table 8. **Error analysis of HWCM of the Example 4.7**

t	Exact	ADM	HWCM	Error (ADM)	Error (HWCM)
0	0	0	0	0	0
0.0312	0.0312551	0.0312001	0.0312576	5.5e-05	2.5e-06
0.0625	0.0625407	0.0625013	0.0625458	3.9e-05	5.1e-06
0.0938	0.0938876	0.0938065	0.0938953	8.1e-05	7.7e-06
0.1250	0.1253265	0.1250200	0.1253368	3.0e-04	1.0e-05
0.1562	0.1568889	0.0156250	0.1569019	1.4e-01	1.3e-05
0.1875	0.1886064	0.1876030	0.1886221	1.0e-03	1.5e-05
0.2188	0.2205114	0.2189910	0.2205299	1.5e-03	1.8e-05
0.2500	0.2526371	0.2503260	0.2526585	2.3e-03	2.1e-05
0.2812	0.2850175	0.2817230	0.2850418	3.3e-03	2.4e-05
0.3125	0.3176876	0.3132980	0.3177150	4.4e-03	2.7e-05
0.3438	0.3506837	0.3449710	0.3507143	5.7e-03	3.0e-05
0.3750	0.3840435	0.3766600	0.3840775	7.4e-03	3.4e-05
0.4062	0.4178061	0.4084910	0.4178436	9.3e-03	3.7e-05
0.4375	0.4520125	0.4405900	0.4520538	1.1e-02	4.1e-05
0.4688	0.4867056	0.4728850	0.4867508	1.3e-02	4.5e-05
0.5000	0.5219305	0.5053030	0.5219799	1.6e-02	4.9e-05
0.5312	0.5577348	0.5379810	0.5577886	1.9e-02	5.3e-05
0.5625	0.5941686	0.5710610	0.5942273	2.3e-02	5.8e-05
0.5938	0.6312855	0.6044810	0.6313492	2.6e-02	6.3e-05
0.6250	0.6691419	0.6381770	0.6692112	3.1e-02	6.9e-05
0.6562	0.7077985	0.6723040	0.7078737	3.5e-02	7.5e-05
0.6875	0.7473198	0.7070280	0.7474014	4.0e-02	8.1e-05
0.7188	0.7877751	0.7422990	0.7878637	4.5e-02	8.8e-05
0.7500	0.8292390	0.7780680	0.8293351	5.1e-02	9.6e-05
0.7812	0.8717916	0.8145190	0.8718960	5.7e-02	1.0e-04
0.8125	0.9155197	0.8518530	0.9156331	6.3e-02	1.1e-04
0.8438	0.9605172	0.8900430	0.9606405	7.0e-02	12e-04
0.8750	1.0068862	0.9290630	1.0070204	7.7e-02	1.3e-04
0.9062	1.0547377	0.9691440	1.0548841	8.5e-02	1.4e-04
0.9375	1.1041932	1.0105500	1.1043529	9.3e-02	1.5e-04
0.9688	1.1553854	1.0532800	1.1555599	1.0e-01	1.7e-04
1.0000	1.2084602	1.0973700	1.2086514	1.1e-01	1.9e-04

Table 9. **Comparison of numerical results with exact and existing method of the Example 4.7, for** $N = 32$**.**

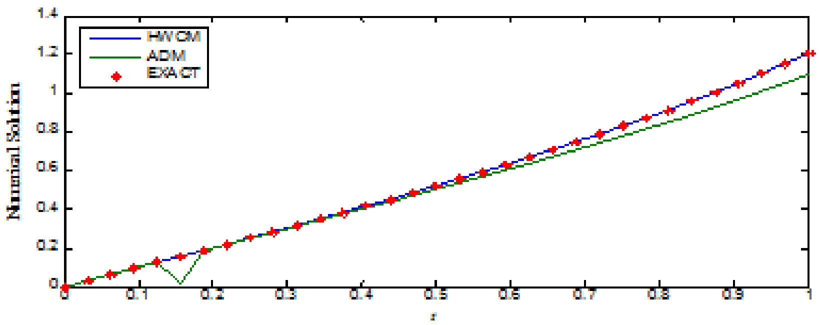

Figure 4. **Comparison of HWCM, ADM with exact solution for** $N = 32$ **of the Example 4.7**

Example 4.8. *Lastly, consider the Nonlinear Fredholm-Hammerstein Integral equation [22],*

$$u(t) + \int_0^1 e^{t-2s}\ [u(s)]^3\ ds =\ e^{t+1},\ \ 0 \le t \le 1, \tag{57}$$

with initial conditions $u(0) = 1$*. Which has the exact solution* $u(t) = e^t$*.*

Differentiating Equation (57) w.r.t t and using Leibnitz rule, its equivalent differential equation,

$$u'(t) = e^{t+1} - \int_0^1 e^{t-2s}\ [u(s)]^3 ds \tag{58}$$

$$u'(t) - u(t) = 0 \tag{59}$$

Let us assume that,

$$u'(t) = \sum_{i=1}^{N} a_i h_i(t) \tag{60}$$

integrating Equation (60), we get

$$u(t) = 1 + \sum_{i=1}^{N} a_i p_{1,i}(t) \tag{61}$$

substituting Equation (60) and Equation (61) in the differential equation Equation (59), we get the system of N equations with N unknowns,

$$\sum_{i=1}^{N} a_i h_i(t) - \left(1 + \sum_{i=1}^{N} a_i p_{1,i}(t)\right) = 0. \tag{62}$$

Solving Equation (62) using Newton's Method to obtain Haar wavelet coefficients a_i's for $N = 16$ i.e., [1.7192 -0.4212 -0.1615 -0.2662 -0.0710 -0.0911 -0.1170 -0.1503 -0.0333 -0.0377 -0.0428 -0.0485 -0.0549 -0.0622 -0.0705 -0.0799]. Substituting a_i's, in (61) and obtained the required HWCM solutions is presented in table 10 and figure 5, compared with exact and existing solutions. Error analysis is shown in table 11, which justifies the efficiency of the HWCM.

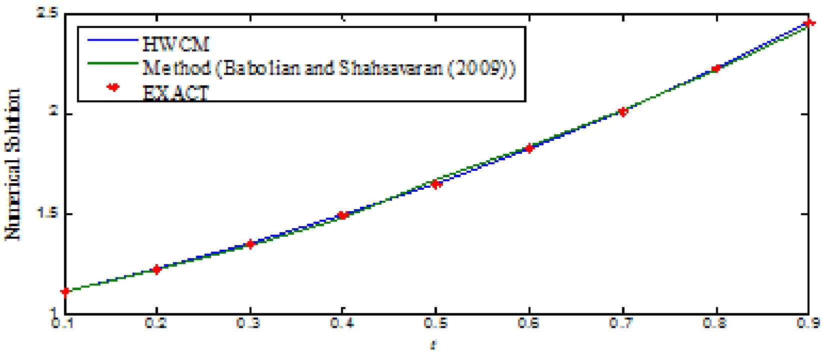

Figure 5. **Comparison of HWCM solution with exact solution and existing method**

t	Exact	HWCM	Method [22]	Error (HWCM)	Error [22]
0.1	1.105170918	1.105314848	1.107217811	1.4e-04	2.0e-03
0.2	1.221402758	1.221571768	1.218102916	1.6e-04	3.3e-03
0.3	1.349858808	1.350056580	1.341165462	1.9e-04	8.7-03
0.4	1.491824698	1.492055414	1.474918603	2.3e-04	1.6e-02

t	Exact	HWCM	Method [22]	Error (HWCM)	Error [22]
0.5	1.648721271	1.648989674	1.667402633	2.6e-04	1.8e-02
0.6	1.822118800	1.822430264	1.833861053	3.1e-04	1.1e-02
0.7	2.013752707	2.014113322	2.016679830	3.6e-04	2.9e-03
0.8	2.225540928	2.225957586	2.217456630	4.1e-04	8.1e-03
0.9	2.459603111	2.460083612	2.437978177	4.8e-04	2.1e-02

Table 10. Comparison of Exact and HWCM of the Example 4.8, for $N = 32$.

N	$E_{\max}$(HWCM)
4	3.0e-02
8	8.1e-03
16	2.1e-03
32	5.4e-04
64	1.3e-04
128	3.4e-05

Table 11. Maximum error analysis of the Example 4.8

5. Conclusion

In the present work, Haar wavelet collocation method based on Leibnitz rule is applied to obtain the numerical solution of nonlinear integral and integro-differential equations of the second kind. The Haar wavelet function and its operational matrix were employed to solve the resultant integral and integro-differential equations. The numerical results are obtained by the proposed method have been compared with existing method. The integral and integro-differential equations are converted to differential equations with initial conditions, then we reduces to a system of algebraic equations. HWCM are mathematically simple and easy to use, then the required less computational complexity and provide more quantitatively reliable results. Illustrative examples clearly depict the validity and applicability of the technique and error analysis shows that, as the level of resolution N increases, gives the better accuracy.

Acknowledgement

The authors thank for the financial support of UGC's UPE Fellowship vide sanction letter D. O. No. F. 14-2/2008(NS/PE), dated-19/06/2012 and F. No. 14-2/2012(NS/PE), dated 22/01/2013.

References

[1] A.M. Wazwaz, *Linear and nonlinear integral equations methods and applications*, Springer, (2011).

[2] C.K. Chui, *Wavelets: A mathematical tool for signal analysis*, in: SIAM Monographs on Mathematical Modeling and Computation, SIAM, Philadelphia, Pennsylvania, (1997).

[3] G. Beylkin, R. Coifman and V. Rokhlin, *Fast wavelet transforms and numerical algorithms*, I. Commun. Pure Appl. Math., 44(1991), 141183.

[4] U. Lepik and E. Tamme, *Application of the Haar wavelets for solution of linear integral equations*, Ant. TurkDynam. Sys. Appl. Proce., (2005), 395407.

[5] C.F. Chen and C.H. Hsiao, *Haar wavelet method for solving lumped and distributed parameter systems*, IEEE Proc. Pt. D., 144(1)(1997), 87-94.

[6] U. Lepik, *Numerical solution of differential equations using Haar wavelets*, Math. Comput. Simul., 68(2005), 127-143.

[7] U. Lepik and E. Tamme, *Application of the Haar wavelets for solution of linear integral equations*, Ant. TurkDynam. Sys. Appl. Proce., (2005), 395407.

[8] U. Lepik, *Haar wavelet method for nonlinear integro-differential equations*, Appl. Math. Comp., 176(2006), 324-333.

[9] U. Lepik, *Application of the Haar wavelet transform to solving integral and differential Equations*, Proc. Estonian Acad Sci. Phys. Math., 56(2007), 28-46.

[10] U. Lepik and E. Tamme, *Solution of nonlinear Fredholm integral equations via the Haar wavelet method*, Proc. Estonian Acad. Sci. Phys. Math., 56(2007), 1727.

[11] U. Lepik, *Numerical solution of evolution equations by the Haar wavelet method*, Appl. Math. Comput. 185(2007), 695704.

[12] U. Lepik, *Solving fractional integral equations by the Haar wavelet method*, Appl. Math. Comp., 214(2009), 468-478.

[13] E. Babolian and A. Shahsavaran, *Numerical solution of nonlinear Fredholm integral equations of the second kind using Haar wavelets*, J. Comp. Appl. Math., 225(2009), 8795.

[14] I. Aziz and S. Islam, *New algorithms for the numerical solution of nonlinear Fredholm and Volterra integral equations using Haar wavelets*, J. Comp. Appl. Math., 239(2013), 333345.

[15] S. Islam, I. Aziz and M. Fayyaz, *A new approach for numerical solution of integro-differential equations via Haar wavelets*, Int. J. Comp. Math., 90(9)(2013), 1971-1989.

[16] V. Mishra, H. Kaur and R.C. Mittal, *Haar wavelet algorithm for solving certain differential, integral and integro-differential equations*, Int. J. of Appl. Math. Mech., 8(6)(2012), 69-82.

[17] J. Ahmadi Shali, A.A. Joderi Akbarfam and G. Ebadi, *Approximate solutions of Nonlinear Volterra-Fredholm Integral equations*, Int. J. Non. Sci., 14(2012), 425-433.

[18] N.H. Sweilam, M.M. Khader and W.Y. Kota, *On the Numerical Solution of Hammerstein Integral Equations using Legendre Approximation*, Inter. J. Appl. Math. Res., 1(1)(2012), 65-76.

[19] A. Shahsavaran and A. Shahsavaran, *Application of lagrange interpolation for Nonlinear integro differential equations*, Appl. Math. Sci., 6(2012), 887-892.

[20] F. Mirzaee and E. Hadadiyan, *A collocation method to the solution of Nonlinear fredholm-hammerstein integral and integro-differential equations*, J. Hyper., 2(1)(2013), 72-86.

[21] S.M. El-Sayed and M.R. Abdel-Aziz, *A comparison of Adomians decomposition method and Wavelet-Galerkin method for solving integro-differential equations*, Appl. Math. Comp., 136(2003), 151159.

[22] E. Babolian and A. Shahsavaran, *Numerical solution of nonlinear Fredholm integral equations of the second kind using Haar wavelets*, J. Comp. Appl. Math., 225(2009), 8795.

[23] H.S. Ramane, S.C. Shiralashetti, R.A. Mundewadi and R.B. Jummannaver, *Numerical solution of Fredholm Integral Equations Using Hosoya Polynomial of Path Graphs*, American Journal of Numerical Analysis, 5(1)(2017), 11-15.