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Acyclic Distance Closed Domination Critical Graphs

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Abstract: In a graph $G = (V, E)$, a set $S \subset V(G)$ is said to be an acyclic distance closed dominating set if (i) $\langle S\rangle$ is distance closed and (ii) $\langle S\rangle$ is acyclic. The cardinality of the minimum acyclic distance closed dominating set is called an acyclic distance closed domination number and it is denoted by $\gamma_{adcl}(G)$. In this paper, we discuss the critical concept in acyclic distance closed domination which deals with those graphs that are critical in the sense that their acyclic distance closed domination number drops when any missing edge is added. Also, we analyze some structural properties of those acyclic distance closed domination critical graphs.

MSC: 05C69.

Keywords: Acyclic distance closed dominating set, acyclic distance closed domination critical graphs.

1. Introduction

The concept of domination in graphs was introduced by Ore [7] in 1962. It is originated from the chess game theory which paved the way to the development of the study of various domination parameters and then relation to various other graph parameters. A set $D \subseteq V(G)$ is called a dominating set of G if every vertex in $V(G) - D$ is adjacent to some vertex in D and D is said to be a minimal dominating set if $D - \{v\}$ is not a dominating set for any $v \in D$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set. We call a set of vertices a γ-set if it is a dominating set with cardinality $\gamma(G)$. Different types of dominating sets have been studied by imposing conditions on dominating sets. A dominating set D is called an acyclic dominating set if $\langle D\rangle$ is acyclic. Also, the cardinality of a minimum acyclic dominating set of G is called the acyclic domination number of G and is denoted by $\gamma_a(G)$. The concept of dominating set and different types of dominating set are studied in [1].

Graphs which are critical with respect to a given property frequently play an important role in the investigation of that property. A graph G is said to be domination critical if for every edge $e \notin E(G)$, $\gamma(G + e) < \gamma(G)$. If G is a domination critical graph with $\gamma(G) = k$, we will say G is k-domination critical or just k-critical. The concept of domination critical graphs and their structural properties are studied in [2] and [8]. The critical concept in graphs plays an important role in the study of structural properties of graphs and hence it will be useful to study any communication model. In this paper, we introduced a new domination critical graph called acyclic distance closed domination critical graphs through which the structural properties of those graphs are studied.

The concept of ideal set is defined and studied in the doctoral thesis of Janakiraman [3] and the concept of ideal sets in graph theory is due to the related concept of ideals in ring theory in algebra. The ideals in a ring are defined with respect to

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the multiplicative closure property with the elements of that ring. Similarly, the ideal set in a graph is defined with respect to the distance property between the ideal set and the vertices of the graph. Thus, the ideal set of a graph G is defined as follows:

Let I be a vertex subset of G. Then I is said to be an ideal set of G if

(i). For each vertex $u \in I$ and for each $w \in V - I$, there exists at least one vertex $v \in I$ such that $d_{\langle I \rangle}(u, v) = d_G(u, w)$.

(ii). I is the minimal set satisfying (i).

The ideal set without the minimality condition is taken as a distance closed set in the present work. Thus, the distance closed dominating set of a graph G is defined as follows:

A subset $S \subseteq V(G)$ is said to be a distance closed dominating (D.C.D) set, if

(i). $\langle S \rangle$ is distance closed;

(ii). S is a dominating set.

The cardinality of a minimum D.C.D set of G is called the distance closed domination number of G and is denoted by γ_{dcl}. The definition and the extensive study of the above said distance closed domination and the acyclic distance closed domination in graphs are studied in [4]. A graph G is said to be a distance closed domination critical if for every edge $e \notin E(G)$, $\gamma_{dcl}(G + e) < \gamma_{dcl}(G)$. If G is a D.C.D critical graph with $\gamma_{dcl}(G) = k$, then G is said to be k-D.C.D critical. Also the structural properties of any k-D.C. D critical graph G with type (I) (structure having every minimum D.C.D set is a path of length k) and type (II) (structure having every minimum D.C.D set is a cycle of length k) are analyzed in [5] and [6].

2. Main Results

Continuing the above, we studied the critical concept of the acyclic distance closed domination in graphs while adding an edge in that graph. The acyclic distance closed domination critical graph is defined as follows. A graph G is said to be an acyclic distance closed domination (A.D.C.D) critical if for every edge $e \notin E(G)$, $\gamma_{adcl}(G+e) < \gamma_{adcl}(G)$. If G is an A.D.C.D critical graph with $\gamma_{adcl}(G) = k$, then G is said to be k-A.D.C. D critical. In a graph G, addition of an edge $e \notin E(G)$ will reduce the acyclic distance closed domination number of G implies that it will reduce the acyclic distance closed domination number.

Theorem 2.1. *If G is any k-A.D.C.D critical graph and if D is a minimum D.C.D set of G, then $\langle D \rangle$ is a path.*

Proof. Let G be any k-A.D.C.D critical graph and let D be a minimum D.C.D set of G.

Claim: $\langle D \rangle$ is a path.

Since D is an A.D.C.D set, $\langle D \rangle$ may be either a tree or a path. If $\langle D \rangle$ is a tree, then it has at least one vertex of degree greater than or equal to 3, say v. Now, addition of an edge between any two vertices in $N(v)$ will not reduce the A.D.C.D number of G. Hence, $\langle D \rangle$ cannot have a vertex of degree greater than or equal to 3 and hence $\langle D \rangle$ must be a path. □

Theorem 2.2. *There is no 2-A.D.C.D critical graph.*

Proof. $\gamma_{adcl}(G) = 1$ if and only if $G \cong K_1$. Also, there is no graph G in which $(G + e) \cong K_1$. Hence the result. □

Theorem 2.3. *A graph G is 3-A.D.C.D critical if and only if G is 3-D.C.D critical.*

Proof. Clearly, any 3-A.D.C.D critical graph is 3-D.C.D critical. Conversely, if G is a 3-D.C.D critical graph, then $\gamma_{dcl}(G) = 3$ and if D is a minimum D.C.D set of G, then $\langle D \rangle$ is a path. Also $\gamma_{dcl}(G+xy) = 2$, for every pair of non-adjacent vertices (x, y) of G and $\langle D^1 \rangle$ is a path, where D^1 is the minimum D.C.D set of $(G+xy)$. Therefore, $\gamma_{dcl}(G) = 3 = \gamma_{adcl}(G)$ and $\gamma_{dcl}(G+xy) = 2 = \gamma_{adcl}(G+xy)$, for every pair of non-adjacent vertices (x, y) of G. Hence the proof. □

Theorem 2.4. *G is 3-A.D.C.D critical if and only if the following hold good.*

(i). G is connected.

(ii). $\gamma_{adcl}(G) = 3$*;*

(iii). G has exactly one vertex of degree $p-1$*;*

(iv). For any two non-adjacent vertices at least one of them is of degree $p-2$*.*

Proof. Results (i) and (ii) are trivial. (iii) Since $\gamma_{adcl}(G) = 3$, G must have exactly one vertex with eccentricity equal to 1.

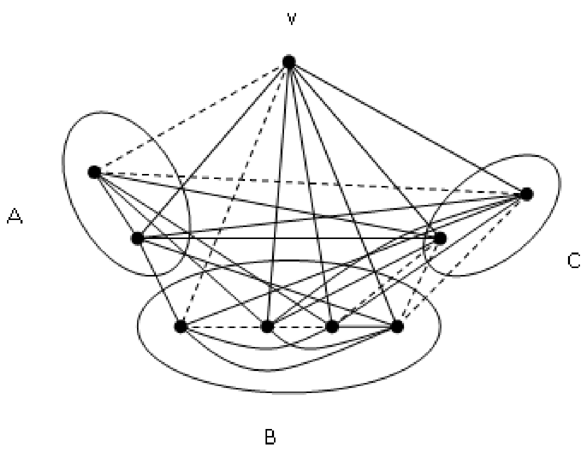

Figure 1. **Structure of 3-A.D.C.D critical graphs**

(iv). To prove, for every pair of non-adjacent vertices at least one of them is of degree $p-2$, let us take x, y be any two non-adjacent vertices. As G is a 3-A.D.C.D critical graph, the inclusion of the edge xy reduces the domination number to 2. Therefore, at least one of the vertices x or y must dominate all the vertices of $(G+xy)$. That is any one of x or y must have degree $p-1$ in $(G+xy)$. Hence, $d(x)$ or $d(y)$ must be equal to $p-2$ in G. Proof of the converse is trivial. □

Corollary 2.5. *If G is a 3 A.D.C.D critical graph, then* $\Delta(G) = p-1$ *and* $1 \leq \delta(G) \leq p-2$*.*

Proof. Without loss of generality, let us assume that G is a 3-A.D.C.D critical graph. Then any vertex in G is of degree at least 1. Hence, the degree of any vertex in $\bar{G}$ is of degree at most $p-2$ and hence $\bar{G}$ cannot be 3-A.D.C.D critical. □

Theorem 2.6. *Any 3-A.D.C.D critical graph has diameter equal to two.*

Proof. Let G be a 3-A.D.C.D critical graph. Then G must have exactly one vertex with eccentricity equal to 1. Hence, the radius of G is 1 and hence the diameter of G must be equal to two. □

Corollary 2.7. *Any 3-A.D.C.D critical graph with* $\delta \geq 2$ *is a block.*

Proposition 2.8. *There exists no graph G for which both G and* $\bar{G}$ *are 3-A.D.C.D critical.*

Proof. Without loss of generality, let us assume that G is a 3-A.D.C.D critical graph. Then any vertex in G is of degree at least 1. Hence, the degree of any vertex in $\bar{G}$ is of degree at most $p-2$ and hence $\bar{G}$ cannot be 3-A.D.C.D critical. □

Theorem 2.9. *There is no 4-A.D.C.D critical graph.*

Proof. Let G be a 4-A.D.C.D critical graph. Then G has a minimum A.D.C.D set D such that $|D| = 4$ and $\langle D \rangle$ is a path (by Theorem 5.3.1). Clearly, $\langle D \rangle$ has two pendant vertices and if we add an edge between that two pendant vertices, then it will not reduce the distance closed domination number of G, a contradiction to G is 4-A.D.C.D critical. Hence, $\langle D \rangle$ must be a cycle and hence G cannot be 4-A.D.C.D critical. □

Theorem 2.10. *A graph G is 5-A.D.C.D critical if and only if G is 5-D.C.D type (I) critical.*

Proof. Clearly, any 5-A.D.C.D critical graph is 5-D.C.D type (I) critical. Conversely, let G be a 5-D.C.D type (I) critical graph. Then G is of diameter 4 and the induced sub graph of any minimum D.C.D set D of G is a path on 5 vertices. Therefore, every minimum D.C.D set of G is a minimum A.D.C.D set of G.

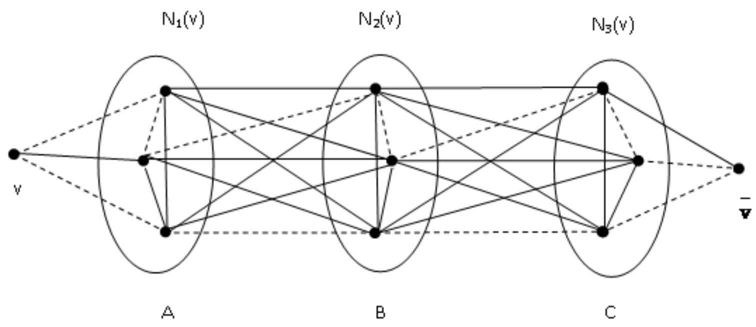

Figure 2. **Structure of 5-A.D.C.D critical graphs**

Also, addition of an edge between any two vertices u and v such that $e(u) = 4$, $e(v) = 3$ (or) $e(u) = 4$, $e(v) = 2$ (or) $e(u) = 3$, $e(v) = 3$ and vice versa results in a graph $(G + uv)$ with diameter 3. Clearly, $\gamma_{dcl}(G + uv) = 4$ and the induced sub graph of the minimum D.C.D set of $(G + uv)$ is a path of length 4. In case, if we add an edge between u and v such that $e(u) = 4$ and $e(v) = 4$, then $(G + uv)$ is 2 self-centered having at least one C_5 containing the edge uv. Therefore, $\gamma_{adcl}(G + uv) = 4$ and the induced sub graph of any minimum D.C.D set of $(G + uv)$ is a path of length 4. Hence, any minimum D.C.D set of $(G + uv)$ is a minimum A.D.C.D set of $(G + uv)$, for every pair of non-adjacent vertices u and v of G and hence G is 5-A.D.C.D critical. □

Theorem 2.11. *If G is a 5-A.D.C.D critical graph, then G is of radius 2 and diameter 4.*

Proof. If D is a minimum A.D.C.D set of a 5-A.D.C.D critical graph G, then D must contain the unique pair peripheral nodes of G and the induced subgraph of D, $\langle D \rangle$ is the diametral path on 5 vertices. Since, it is the distance preserving subgraph of G having diameter 4, G must have diameter equal to 4 and radius 2. □

Theorem 2.12. *If u is a cut vertex of a 5-A.D.C.D critical graph, then*

(i). $G - u$ can have at most two components;

(ii). One of the components is a clique.

Proof. Let u be a cut vertex of a 5-A.D.C.D critical graph G. Then, u must be in every A.D.C.D set $\{v, u_1, u_2, u_3, \bar{v}\}$ of G, where $u_1 \in A$, $u_2 \in B$ and $u_3 \in C$. Also, u can be any one of the vertex in $\{u_1, u_2, u_3\}$.

Case (1): $u = u_1$

If $u = u_1$, then $e(u) = 3$ and u is the only vertex in A. Also, $G - u$ has two components C_1 and C_2 such that $C_1 = \{v\}$ and $C_2 = B \cup C \cup \{\bar{v}\}$. Hence, in this case $G - u$ has two components and one of them is a clique.

Case (2): $u = u_2$

If $u = u_2$, then $e(u) = 2$ and u is the only vertex in B. Also, $G - u$ has two components C_1 and C_2 such that $C_1 = \{v\} \cup A$ and $C_2 = C \cup \{\bar{v}\}$. Hence, in this case $G - u$ has two components and both of them are cliques.

Case (3): $u = u_3$

If $u = u_3$, then $e(u) = 3$ and u is the only vertex in C. Also, $G - u$ has two components C_1 and C_2 such that $C_1 = \{v\} \cup A \cup B$ and $C_2 = \{\bar{v}\}$. Hence, in this case $G - u$ has two components and one of them is a clique.

Hence from all the three cases, we have the theorem. □

Theorem 2.13. *If G is a 5-A.D.C.D critical graph, then*

(i). G can have at most 2 pendant vertices.

(ii). G can have at most 3 cut vertices.

Proof.

(i). Let G be a 5-A.D.C.D critical graph. Then every vertex u in G belongs to at least one minimum A.D.C.D set D of G. Also, $\langle D \rangle$ is the diametral path and it contains exactly the two peripheral vertices $\{v, \bar{v}\}$ of G. Therefore, $d(u) \geq 2$, for every vertex u in $V(G) - \{v, \bar{v}\}$. Hence, the vertices $\{v, \bar{v}\}$ can be the pendant vertices of G and hence G can have at most 2 pendant vertices.

(ii). If u is a cut vertex of a 5-A. D.C.D critical graph G, then u must be in every A.D.C.D set $\{v, u_1, u_2, u_3, \bar{v}\}$ of G, where $u_1 \in A$, $u_2 \in B$ and $u_3 \in C$ and also u cannot be a peripheral node of G. Hence, u can be any one of the vertex in the set $\{u_1, u_2, u_3\}$ and hence G can have at most 3 cut vertices. □

Proposition 2.14. *Any 5-A.D.C.D critical graph, which is also a block, is Hamiltonian.*

Proof. For any vertex v in G, $\langle N_i(v) \rangle$, i = 1 to 4 is a clique and also $|A|$, $|B|$ and $|C| \geq 2$ (as G is a block). Hence, we can have a cycle that covers all the vertices of G (refer Figure 3.2) and hence G is Hamiltonian. □

Proposition 2.15. *There exists no graph G for which both G and $\bar{G}$ are 5-A.D.C.D critical.*

Proof. Without loss of generality, assume that G is a 5-A.D.C.D critical graph. Then, the diameter of G is 4 and $\bar{G}$ contains a dominating edge. Hence, $\gamma_{adcl}(\bar{G}) = 4$ and hence $\bar{G}$ cannot be 5-A.D.C.D critical. □

Theorem 2.16. *There is no 6-A.D.C.D critical graph.*

Proof. Let G be a 6-A.D.C.D critical graph. Then G has a minimum A.D.C.D set D such that $|D| = 6$ and $\langle D \rangle$ is path on 6 vertices. Clearly, $\langle D \rangle$ has two pendant vertices and if we add an edge between that two pendant vertices, then it will not reduce the distance closed domination number of G, a contradiction to G is 6-A.D.C.D critical. Hence, G cannot have an A.D.C.D set and hence G cannot be 6-A.D.C.D critical. □

2.1. Generalization of k-A.D.C.D critical graphs

From the above results, it can be generalized that for any k-A.D.C.D critical graph

(i). If k is odd, then k-A.D.C.D critical graph exists and a graph G is k-A.D.C.D critical if and only if G is k-D.C.D type (I) critical.

(ii). If k is even, then there exists no k-A.D.C.D critical graph.

(iii). It is obvious that if a graph G is k-A.D.C.D critical, then k must be odd.

The following are some of the structural properties of any generalized k-A.D.C.D critical graphs.

Theorem 2.17. *Any k-A.D.C.D critical graph is of diameter $k-1$ and radius $\frac{k-1}{2}$.*

Proof. Let D be a minimum A.D.C.D set of a k-A.D.C.D critical graph G. Then $\langle D\rangle$ is the diametral path on k vertices (distance preserving sub graph of G) with diameter $d = k-1$ and radius $r = \frac{k-1}{2}$ such that $d = 2r$ and hence in G also. □

Corollary 2.18. *Any k-A.D.C.D critical graph has at most two pendant vertices.*

Proof. Let G be a k-A.D.C.D critical graph and let D be a minimum A.D.C.D set of G. Then, clearly $\langle D\rangle$ is the diametral path of G and it has at most two pendant vertices and hence in G also. □

Theorem 2.19. *If G is a k-A.D.C.D critical graph, then G can have at most $(k-2)$ cut vertices.*

Proof. Let G be a k-A.D.C.D critical graph and let D be a minimum A.D.C.D set of G. Then, $\langle D\rangle$ is the diametral path of G and that must include all the cut vertices of G. That is, $\langle D\rangle$ is a path on k vertices having $(k-2)$ cut vertices (except the 2 pendant vertices). Thus, G can have at most $(k-2)$ cut vertices. □

Theorem 2.20. *Any k-A.D.C.D critical graph is diameter edge (addition) critical.*

Theorem 2.21. *Any k-A.D.C.D critical graph, which is also a block, is Hamiltonian.*

Proof. For any vertex v of a k-A.D.C.D critical graph G, $\langle N_i(v)\rangle$ and $\langle N_i(v) \cup N_{i+1}(v)\rangle$, i = 1 to $k-1$ are cliques and also $|N_i(v)| \geq 2$, where i = 1 to $k-1$. Hence, there exists a cycle that covers all the vertices of G and hence G is Hamiltonian. □

Theorem 2.22. *If k is even, then there exists no A.D.C.D critical graph for k.*

Proof. Let G be a k-A.D.C.D critical graph with k is even. If D is an A.D.C.D set of G, then $\langle D\rangle$ must be a path of length k (which is even). Now addition of an edge between the two end vertices of that path will induce an even cycle of length k and the A.D.C.D number of G will not be reduced, a contradiction. Hence, $\langle D\rangle$ must be a cycle and hence G cannot have an A.D.C.D set. Therefore, there exists no A.D.C.D critical graph for an even k. □

3. Conclusion

In this paper, the critical concept of acyclic distance closed domination is defined and the structural properties of k-A.D.C.D critical graphs are analyzed. Also, the acyclic distance closed domestic partition in graphs is defined and the structure of graphs with a given d_{adcl} is analyzed of distance closed restrained domination with respect to both distance closed and restrained property are analyzed and the structural properties of k-D.C.R. D critical graphs such as radius, diameter, number of cut vertices, clique components and Hamiltonian properties are studied. Since this concept deals the reduction in the cardinality of distance closed restrained dominating set for any addition of one new link in the original structure, it will be useful to study the communication model, which reduces it dominating parameters by simple addition of a link, which doesn't exist in the system. Hence this critical concept can be directly applied to the construction of a fault tolerant communication model.

References

[1] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, (1998).

[2] M.A. Henning, O.R. Oellermann and H.C. Swart, *Local edge domination critical graphs*, Discrete Math., 161(1996), 175-184.

[3] T.N. Janakiraman, *On some eccentricity properties of the graphs*, Ph.D thesis submitted to Madras University, (1991).

[4] T.N. Janakiraman, P.J.A. Alphonse and V. Sangeetha, *Acyclic Distance closed domination in graph*, International Journal of Engineering Science Advanced Computing and Bio-Technology, 2(1)(year?), 47-56.

[5] T.N. Janakiraman, P.J.A. Alphonse and V. Sangeetha, *Structural properties of k-Distance closed domination critical graphs for k=3 and 4*, International Journal of Engineering Science Advanced Computing and Bio-Technology, 2(2011), 150-161.

[6] T.N. Janakiraman, P.J.A. Alphonse and V. Sangeetha, *Structural properties of k-Distance closed domination critical graphs for k=5 and 6*, *International Journal of Engineering Science Advanced Computing and Bio-Technology*, 3(2012), 1-14.

[7] O. Ore, *Theory of Graphs*, Amer. Soc. Colloq. Publ. vol.38. Amer. Math. Soc., Providence, RI, (1962).

[8] D.P. Sumner and P. Blitch, *Domination critical graphs*, J. Combin. Theory ser. B, 34(1983), 65-76.