Int. J. Math. And Appl., 6(1–E)(2018), 1159–1163
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of *Mathematics And its Applications*

Fixed Point Theorem For Pair of Weakly Compatible Mappings Using CLR_T Property

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Abstract: The purpose of this paper is to prove a common fixed point theorem in a fuzzy metric space using the concept of weakly compatible mappings and CLR_T property.

MSC: 54H25, 47H10.

Keywords: Fixed point, weakly compatible mappings, CLR_T property.

1. Introduction

The notion of the fuzzy sets was introduced by Zadeh [14]. Kramosil and Michelek initiated the concept of Fuzzy metric spaces since then many authors proved fixed point theorem using Fuzzy metric space. In 1988, Grabic [5] obtained the Banach contraction principle in fuzzy version. George and Veeramani [4] modified the notion of fuzzy metric space and shown that every metric induces a fuzzy metric. Recently Sintunavarat and kuman [11] initiated an interesting property Common Limit in the Range property (CLR property). Aim of this paper to prove a common fixed point for four self maps with the concept of weakly compatible maps and CLR_T property.

2. Definitions and Preliminaries

Definition 2.1. *A binary operation* $*: [0,1]\times[0,1] \to [0,1]$ *is called continuous t-norm if * satisfies the following conditions:*

(1). $*$ *is commutative and associative*

(2). $*$ *is continuous*

(3). $a * 1 = a$ *for all* $a \in [0,1]$

(4). $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$ *for all* $a, b, c, d \in [0,1]$.

Definition 2.2. *A 3-tuple* $(X, M, *)$ *is said to be fuzzy metric space if* X *is an arbitrary set, * is continuous t-norm and* M *is a fuzzy set on* $X^2 \times (0, \infty)$ *satisfying the following conditions for all* $x, y, z \in X,\ s, t > 0$.

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(FM-1) $M(x, y, 0) = 0$

(FM-2) $M(x, y, t) = 1$ *for all* $t > 0$ *if and only if* $x = y$

(FM-3) $M(x, y, t) = M(y, x, t)$

(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$

(FM-5) $M(x, y, .) : [0, \infty) \to [0, 1]$ *is left continuous*

Example 2.3 (Induced fuzzy metric space)**.** *Let* (X, d) *be a metric space defined* $a * b = \min\{a, b\}$ *for all* $x, y \in X$ *and* $t > 0$*,*

$$M(x, y, t) = \frac{t}{t + d(x, y)} \tag{a}$$

Then $(X, M, *)$ *is a fuzzy metric space. We call this fuzzy metric* M *induced by metric* d *the standard fuzzy metric. From the above example every metric induces a fuzzy metric but there exist no metric on* X *satisfying (a).*

Definition 2.4. *Let* $(X, M, *)$ *be a fuzzy metric space then a sequence* $\langle x_n \rangle$ *in* X *is said to be convergent to a point* $x \in X$*, if*

$$\lim_{n\to\infty} M(x_n, x, t) = 1 \; for \, all \, t > 0.$$

Definition 2.5. *A sequence* $\langle x_n \rangle$ *in* X *is called a Cauchy sequence if* $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ *for all* $t > 0$ *and* $p > 0$*.*

Definition 2.6. *A fuzzy metric space* $(X, M, *)$ *is said to be complete if every Cauchy sequence is convergent to a point in* X*.*

Lemma 2.7. *For all* $x, y \in X$*,* $M(x, y, .)$ *is non decreasing.*

Lemma 2.8. *l Let* $(X, M, *)$ *be a fuzzy metric space if there exists* $k \in (0, 1)$ *such that* $M(x, y, kt) \geq M(x, y, t)$ *then* $x = y$*.*

Proposition 2.9. *In the fuzzy metric space* $(X, M, *)$ *if* $a * a \geq a$ *for all* $a \in [0, 1]$ *then* $a * b = \min\{a, b\}$*.*

Definition 2.10. *Two self maps* S *and* T *of a fuzzy metric space* $(X, M, *)$ *are said to be compatible mappings if* $\lim_{n\to\infty} M(STx_n, TSx_n, t) = 1$*, whenever* $\langle x_n \rangle$ *is a sequence in* X *such that* $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ *for some* $z \in X$*.*

Definition 2.11. *Two self maps* S *and* T *of a fuzzy metric space* $(X, M, *)$ *are said to be weakly compatible if they commute at their coincidence point i.e if* $Su = Tu$ *for some* $u \in X$ *then* $STu = TSu$*.*

Definition 2.12. *Let* B *and* T *be two self maps defined on a metric space* (X, d)*. We say that the mappings* B *and* T *satisfy* CLR_T *property if there exists a sequence* $\langle x_n \rangle \in X$ *such that* $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = Tx$ *for some* $x \in X$*.*

3. Main Result

Theorem 3.1. *Let* $(X, M, *)$ *be a Fuzzy metric space with* $a * b = \min\{a, b\}$ *and let* A*,* B*,* S *and* T *are self maps of* X *satisfying the following condition conditions*

(1). $B(X) \subseteq S(X)$ *and the pair* (B, T) *satisfy* CLR_T *property*

(2). *the pairs* (A, S) *and* (B, T) *are weakly compatible*

(3). $$[M(Ax, By, kt)]^2 * M(Ax, By, kt)M(Ty, Sx, kt) \geq \{k_1 [M(By, Sx, 2kt) * M(Ax, Ty, 2kt)] + k_2 [M(Ax, Sx, kt) * M(By, Ty, kt)]\} M(Ty, Sx, t)$$

where for all x,y in X and k_1*,* $k_2 \geq 0$*,* $k_1 + k_2 \geq 1$

then the mappings A, B, S and T have a unique common fixed point in X.

Proof. Assume that the pairs (B,T) satisfy CLR_T property so there exists a sequence $\langle x_n\rangle \in X$ such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = Tx$. Since the condition $B(X) \subseteq S(X)$ implies there exists a sequence $\langle y_n\rangle \in X$ such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Sy_n$ implies $\lim_{n\to\infty} Sy_n = Tx$ this gives $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = Tx$. We show that $\lim_{n\to\infty} Ay_n = Tx$. Put $x = y_n$ and $y = x_n$ in (3).

$$\begin{aligned}
[M(Ay_n, Bx_n, kt)]^2 * M(Ay_n, Bx_n, kt)M(Tx_n, Sy_n, kt) &\geq \{k_1 [M(Bx_n, Sy_n, 2kt) * M(Ay_n, Tx_n, 2kt)] \\
&\quad + k_2 [M(Ay_n, Sy_n, kt) * M(Bx_n, Tx_n, kt)]\} M(Tx_n, Sy_n, t) \\
[M(Ay_n, Tx, kt)]^2 * M(Ay_n, Tx, kt)M(Tx, Tx, kt) &\geq \{k_1 [M(Tx, Tx, 2kt) * M(Ay_n, Tx, 2kt)] \\
&\quad + k_2 [M(Ay_n, Tx, kt) * M(Tx, Tx, kt)]\} M(Tx, Tx, t) \\
[M(Ay_n, Tx, kt)]^2 &\geq \{k_1 [M(Ay_n, Tx, 2kt)] + k_2 [M(Ay_n, Tx, kt)]\} \\
[M(Ay_n, Tx, kt)] &\geq \{k_1 + k_2\} \\
[M(Ay_n, Tx, kt)] &\geq 1
\end{aligned}$$

implies $\lim_{n\to\infty} Ay_n = Tx$ let $Tx = z$ then we have $\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Bx_n = z$. We prove $Bx = z$. Put $x = y_n$ and $y = x$ in (?)

$$\begin{aligned}
[M(Ay_n, Bx, kt)]^2 * M(Ay_n, Bx, kt)M(Tx, Sy_n, kt) &\geq \{k_1 [M(Bx, Sy_n, 2kt) * M(Ay_n, Tx, 2kt)] \\
&\quad + k_2 [M(Ay_n, Sy_n, kt) * M(Bx, Tx, kt)]\} M(Tx, Sy_n, t) \\
[M(z, Bx, kt)]^2 * M(z, z, kt)M(z, z, kt) &\geq \{k_1 [M(Bx, z, 2kt) * M(z, z, 2kt)] \\
&\quad + k_2 [M(z, z, kt) * M(Bx, z, kt)]\} M(z, z, t) \\
[M(z, Bx, kt)]^2 &\geq \{k_1 [M(Bx, z, 2kt)] + k_2 [M(Bx, z, kt)]\} \\
[M(z, Bx, kt)] &\geq \{k_1 + k_2\} \\
[M(z, Bx, kt)] &\geq 1
\end{aligned}$$

implies $Bx = z$ this gives $Tx = Bx = z$. Since the pair (B,T) is weakly compatible implies $BTx = TBx$ gives $Bz = Tz$. Also since the condition $B(X) \subseteq S(X)$ implies there exists a sequence $y \in X$ such that $Bx = Sy = z$. We next show that $z = Ay$. Put $y = x_n$ and $x = y$ in (?)

$$\begin{aligned}
[M(Ay, Bx_n, kt)]^2 * M(Ay, Bx_n, kt)M(Tx_n, Sy, kt) &\geq \{k_1 [M(Bx_n, Sy, 2kt) * M(Ay, Tx_n, 2kt)] \\
&\quad + k_2 [M(Ay, Sy, kt) * M(Bx_n, Tx_n, kt)]\} M(Tx_n, Sy, t) \\
[M(Ay, z, kt)]^2 * M(Ay, z, kt)M(z, Sy, kt) &\geq \{k_1 [M(z, Sy, 2kt) * M(Ay, z, 2kt)] \\
&\quad + k_2 [M(Ay, Sy, kt) * M(z, z, kt)]\} M(z, Sy, t) \\
[M(Ay, z, kt)]^2 * M(Ay, z, kt)M(z, z, kt) &\geq \{k_1 [M(z, z, 2kt) * M(Ay, z, 2kt)] + k_2 [M(Ay, z, kt)]\} M(z, z, t) \\
[M(Ay, z, kt)]^2 &\geq \{k_1 [M(Ay, z, 2kt)] + k_2 [M(Ay, z, kt)]\} \\
[M(Ay, z, kt)] &\geq 1
\end{aligned}$$

gives $Ay = z$ and this implies $Ay = Sy = z$. But the pair (A,S) is weakly compatible it follows that $ASy = Say$ implies $Az = Sz$. Now we prove that $Az = Bz$. Put $x = z$ and $y = z$ in (?).

$$[M(Az, Bz, kt)]^2 * M(Az, Bz, kt)M(Tz, Sz, kt) \geq \{k_1 [M(Bz, Sz, 2kt) * M(Az, Tz, 2kt)]$$

$$+k_2\left[M(Az,Sz,kt)*M(Bz,Tz,kt)\right]\}M(Tz,Sz,t)$$

$$[M(Az,Bz,kt)]^2*M(Az,Bz,kt)M(Bz,Az,kt)\geq\{k_1\left[M(Bz,Az,2kt)*M(Az,Bz,2kt)\right]$$
$$+k_2\left[M(Az,Az,kt)*M(Bz,Bz,kt)\right]\}M(Bz,Az,t)$$

$$[M(Az,Bz,kt)]^2*M(Az,Bz,kt)M(Bz,Az,kt)\geq\{k_1\left[M(Bz,Az,2kt)*M(Az,Bz,2kt)\right]$$
$$+k_2\left[M(Az,Az,kt)*M(Bz,Bz,kt)\right]\}M(Bz,Az,t)$$

$$[M(Az,Bz,kt)]^2\geq\{k_1\left[M(Bz,Az,2kt)\right]+k_2\}M(Bz,Az,t)$$

$$[M(Az,Bz,kt)]\geq\{k_1\left[M(Bz,Az,2kt)\right]+k_2\}$$

$$M(Az,Bz,kt)\geq\frac{k_2}{1-k_1}$$

$$M(Az,Bz,kt)\geq1$$

implies $Az=Bz$. Therefore $Az=Bz=Sz=Tz$. Now we show that $z=Az$. Put $x=z$ and $y=x$ in (?).

$$[M(Az,Bx,kt)]^2*M(Az,Bx,kt)M(Tx,Sz,kt)\geq\{k_1\left[M(Bx,Sz,2kt)*M(Az,Tx,2kt)\right]$$
$$+k_2\left[M(Az,Sz,kt)*M(Bx,Tx,kt)\right]\}M(Tx,Sz,t)$$

$$[M(Az,z,kt)]^2*M(Az,z,kt)M(z,Az,kt)\geq\{k_1\left[M(z,Az,2kt)*M(Az,z,2kt)\right]$$
$$+k_2\left[M(Az,Az,kt)*M(z,z,kt)\right]\}M(z,Az,t)$$

$$[M(Az,z,kt)]\geq\{k_1\left[M(z,Az,2kt)\right]+k_2\}$$

$$[M(Az,z,kt)]\geq\frac{k_2}{1-k_1}$$

$$[M(Az,z,kt)]\geq1$$

implies $Az=z$ therefore $Az=Bz=Sz=Tz=z$.

Uniqueness: Let $w(\neq z)$ be the common fixed point of A, B, S and T then we get $Aw=Bw=Sw=Tw=w$. Put $x=z$ and $y=w$ in (?).

$$[M(Az,Bw,kt)]^2*M(Az,Bw,kt)M(Tw,Sz,kt)\geq\{k_1\left[M(Bw,Sz,2kt)*M(Az,Tw,2kt)\right]$$
$$+k_2\left[M(Az,Sz,kt)*M(Bw,Tw,kt)\right]\}M(Tw,Sz,t)$$

$$[M(z,w,kt)]^2*M(z,w,kt)M(w,z,kt)\geq\{k_1\left[M(w,z,2kt)*M(z,w,2kt)\right]$$
$$+k_2\left[M(z,z,kt)*M(w,w,kt)\right]\}M(w,z,t)$$

$$[M(z,w,kt)]\geq\{k_1\left[M(w,z,2kt)\right]+k_2\}$$

$$[M(z,w,kt)]\geq\frac{k_2}{1-k_1}$$

$$[M(z,w,kt)]\geq1$$

Implies $z=w$. Which gives Self maps A,B,S and T have unique common fixed point. □

Example 3.2. *Let* $X=[0,2/3)$, $M(x,y,t)=\frac{t}{t+d(x,y)}$ *where* $d(x,y)=|x-y|$

$$Ax=\begin{cases}\frac{6}{10} & if\,0\leq x<\frac{1}{3}\\ \frac{1}{3} & if\,\frac{1}{3}\leq x<\frac{2}{3}\end{cases}\qquad Bx=\begin{cases}\frac{1}{4} & if\,0\leq x<\frac{1}{3}\\ \frac{1}{3} & if\,\frac{1}{3}\leq x<\frac{2}{3}\end{cases}\qquad Sx=Tx=\begin{cases}\frac{1}{6} & if\,0\leq x<\frac{1}{3}\\ \frac{2}{3}-x & if\,\frac{1}{3}\leq x<\frac{2}{3}\end{cases}$$

then $A(X) = \left\{\frac{6}{10}, \frac{1}{3}\right\}$, $B(X) = \left\{\frac{1}{4}, \frac{1}{3}\right\}$ *while* $S(X) = T(X) = \left\{\frac{1}{6} \cup \left(0, \frac{1}{3}\right]\right\}$ *so that the condition* $B(X) \subseteq S(X)$ *are satisfied. Clearly the pairs* (A, S) *and* (B, T) *are weakly compatible as they commute at coincident point* $1/3$*. Let a sequence* $x_n = \left(\frac{1}{3} + \frac{1}{n}\right)$ *for* $n \geq 1$*, then* $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \frac{1}{3}$ *and* $T\left(\frac{1}{3}\right) = \frac{1}{3}$ *which implies the pair* (B, T) *satisfies* CLR_T *property. The rational inequality holds for the values of* $0 \leq k_1 + k_2 \geq 1$ *where* $k_1, k_2 \geq 0$*. Therefore all the conditions of Theorem 3.1 are satisfied. Clearly* $1/3$ *is the unique common fixed point of* A*,* B*,* S *and* T*.*

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