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Group Action on Fuzzy R-Subgroup of a Near-Ring

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Abstract: Meldrum [7] studied new rings and their algebraic links with groups. Osman Kazanci [8] discussed intuitionistic Q-fuzzy R-subgroup in a near-ring, and found few contributions on their union and intersection. Cho and Jun [2] discussed intuitionistic fuzzy right (left) R-subgroup in a near-ring, and analyzed the usual algebraic properties like as the union, intersection, complement and decomposition of intuitionistic fuzzy right (left) R-subgroup in a near-ring. In this paper, group action on a right (respectively left) R subgroup and same type of fuzzy right (respectively left) R-subgroup of a near-ring R are studied. Few characterizations are obtained.

Keywords: Set action on a fuzzy set; group action on fuzzy R-subgroup on a near-ring.

1. Introduction

Kim and Jun [3] investigated fuzzy algebraic properties in fuzzy R-subgroup in a near-ring. They [4] studied on normal fuzzy R-subgroup, and its homomorphic image and pre-images in a near-ring. Kim and Jun [5] further extended their contributions in anti-fuzzy R-subgroup in a near-ring. They [6] again analyzed union and intersection of fuzzy R-subgroups in a near-ring. Solairaju and Nagarajan [9] introduced the new structures of Q-fuzzy groups and then they investigate the notion upper Q-fuzzy index with upper Q- fuzzy subgroups. They [10] initiated some contributions on Q-fuzzy left R-subgroup of near-ring under triangular norm. They [11] introduced the new structures of Q-fuzzy groups and then they investigate the notion upper Q-fuzzy index order with upper Q-fuzzy subgroups. On the other hand Osman Kazanci in [8] the notion of intuitionistic Q-fuzzy R-subgroup in a near ring is introduced and related properties are investigated.

2. Preliminary and Definitions

Definition 2.1 ([2])**.** *A near ring is a non-empty set R with two binary operations $+$ and $.$ satisfying the following axioms: (1). $(R,+)$ is a group; (2). $(R,.)$ is a semigroup; (3). $x.(y+z)=x.y+x.z$ for all x, y, z, $\setminus$ in R. Then It is called a left near-ring by (3). In this paper, it will use the word near-ring. Here xy denotes $x.y$; (2). $x.0=0$, and $x(-y)=-(xy)$ for x, y in R.*

Definition 2.2. *A two sided R-subgroup of a near-ring R is a subset H of R such that (1). $(H,+)$ is a subgroup of $(R,+)$; (2). $RH\subset H$; (3). $HR\subset H$. If H satisfies (1) and (2), then it is a left R-subgroup of R, while if H satisfies (1) and (3), then it is a right R-subgroup of R.*

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Definition 2.3. *A fuzzy set μ in a set R is a function $\mu : R \to [0,1]$.*

Definition 2.4. *Let G be any group. A mapping $\mu : G \to [0,1]$ is a fuzzy group if*

(FG1) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ *and*

(FG2) $\mu\left(x^{-1}\right) = \mu(x)$, *for all $x, y \in G$.*

Definition 2.5. *Let $(S,+)$ be a group, and G be a non-empty set. Then G acts on S if there exists a function $* : G \times S \to S$ (denoted $*(g,s) = gs$ for all $g \in G$, and $s \in S$) such that $es = s$ and $(g+h)*s = g*(h*s)$ for all s in S, and for all g, h in G.*

Definition 2.6. *A group (G,Δ) with identity 0 acts on a fuzzy group A under a near-ring $(R,+,.)$ if (GAFS1) the group G acts on R [there exists a function $* : G \times R \to R$ with the conditions $g*(h*s) = (g+h)*s$ and $e*s = s$ or all s in S, and for all g, h in G];*

(GAFG2) $A(x*(s-t) \geq \min\{A(x*s),\ A(x*t)\}$;

(GAFG3) $A(x*(st) \geq \min\{A(x*s),\ A(x*t)\}$;

(GAFG4) $A(x*s^{-1}) \geq A(x*s)$ *for all $x, y \in G$ and $s, t \in S$.*

Definition 2.7. *Let A be a fuzzy group on a near-ring $(R,+,.)$. Then G acts on a right (respectively left) R-fuzzy subgroup A of a near-ring R if. (1). The group (G,Δ) acts on R-fuzzy group A under a near-ring $(R,+,.)$;*

(GAFS5) $A(x*(sr)) \geq A(x*s)$ *(respectively $A(x*(rs)) \geq A(x*s)$) for all $r, x \in R$.*

Definition 2.8. *Let θ be a mapping from X to Y.*

(1). Let (G',Δ') a group acting on R'-fuzzy group B under a near-ring $(R',+',.')$. Then the inverse image of B under θ denoted by $\theta^{-1}(B)$ is fuzzy set in (G,Δ) defined by $\theta^{-1}(B) = \mu_{\theta^{-1}}(B)$, where $\mu_{\theta^{-1}(B)}(x) = \mu_B(\theta(x))$;

(2). Let (G,Δ) be a group acting on R-fuzzy group A under a near-ring (R, +, .). Then the image of A under θ denoted by $\theta(A)$, where $\mu_{\theta(A)}(y) = \{\sup \mu_{A(x)} : x \in \theta^{-1}(y)\ if\ \theta^{-1}(y) \neq 0;\ 0\ otherwise\}$. Then $\mu_{\theta^{-1}(B)}(x*s) = \mu_B(\theta(x)*s)$ for all s in S. Also $\mu_{\theta(A)}(y*s) = \{\sup \mu_A(x*s) : (x*s) \in (\theta^{-1}(y)*s)\ for\ all\ s\ in\ S\ if\ \theta^{-1}(y) \neq 0;\ = 0,\ otherwise\}$.

Definition 2.9. *Let a group (G, +) act on a fuzzy group A under (S,Δ). Then the upper level cut-set of A is defined by $U(A,t) = \{s \in S : \inf_{x\in G} A(x*s) \geq \alpha\}$.*

3. Group Action on Fuzzy Characteristic and Fuzzy Same Type R-subgroup

First the contribution on R-subgroup action a group is discussed:

Definition 3.1. *Let $\varphi : R \to R'$ be an near-ring endomorphism, and a group G acts on an R'-fuzzy-subgroup of a near-ring R'. An R-fuzzy set $A\varphi$ in R is defined by $A\varphi(x*s) = A(x*\varphi(s))$ for all s in R.*

Theorem 3.2. *Let $\varphi : R \to R'$ be an near-ring endomorphism. Let a group (G,Δ) acts on an R'-fuzzy subgroup A of a near-ring R'. Then (G,Δ) acts on R-fuzzy subgroup of R.*

Proof.

(GAFG1) The group (G, Δ) acts on R-fuzzy subset $A\varphi$ of R.

(GAFG2)
$$\begin{aligned} A\varphi(x*(s-t)) &= A(x*(\varphi(s-t))) \\ &= A(x*(\varphi(s)-\varphi(t))) \\ &\geq \min\{A(x*\varphi(s), A(x*\varphi(t)\} \quad = \min\{A\varphi(x*s), A\varphi(x*t)\} \end{aligned}$$

(GAFG3)
$$\begin{aligned} A\varphi(x*(st)) &= A(x*(\varphi(st))) \\ &= A(x*(\varphi(s).\varphi(t))) \\ &\geq \min\{A(x*\varphi(s), A(x*\varphi(t)\} \\ &= \min\{A\varphi(x*s), A\varphi(x*t)\} \end{aligned}$$

(GAFG4) $A\varphi(x*s^{-1}) = A(x*\varphi(s^{-1})) \geq A(x*\varphi(s)) = A\varphi(x*s)$.

Let r be in R.

$$A\varphi(x*(sr)) = A(x*\varphi(sr)) = A(x*\varphi(s).\varphi(r)) \geq A(x*\varphi(s)) = A\varphi(x*s).$$

Similarly

$$A\varphi(x*(rs)) = A(x*\varphi(rs)) = A(x*\varphi(r).\varphi(s)) \geq A(x*\varphi(s)) = A\varphi(x*s).$$

Hence $A\varphi$ is an R-fuzzy subgroup of R acted by the group G. □

Definition 3.3. *A group (G, Δ) with identity 0 acts on a R-fuzzy group A under a near-ring $(R, +, .)$. Then A is fuzzy characteristic if $A\varphi(x*s) = A(x*S)$ for all x in G, s in R, and φ in $Aut(R)$ that is the group of all automorphisms of R.*

Definition 3.4. *Let a group (G, Δ) act a family $\{A_i : i \in N\}$ of R-fuzzy subgroups of a near-ring $(R, +, .)$. Then* $\wedge_{i\in N} A_i(x) = \min\{A_1(x), A_2(x), A_3(x), \dots\} = \inf\{A_1(x), A_2(x), A_3(x), \dots\}$.

Theorem 3.5. *Let a group (G, Δ) act a family $\{A_i : i \in N\}$ of characteristic R-fuzzy subgroups of a near-ring $(R, +, .)$. Then $\wedge_{i\in N} A_i$ is characteristic.*

Proof. Assume that a group (G, Δ) act a family $\{A_i : i \in N\}$ of characteristic R-fuzzy subgroups of a near-ring $(R, +, .)$. Let φ be an automorphism. Let $x \in G$, and $s \in R$. It follows that $\wedge_{i\in N} A_i\varphi(s) = \min\{A_1\varphi(s), A_2\varphi(s), A_3\varphi(s), \dots\}$. It gets that

$$\begin{aligned} \varphi(\wedge_{i\in N} A_i)(x*s) &= (\wedge_{i\in N} A_i)(x*\varphi(s)) \\ &= (\wedge_{i\in N} A_i\varphi)(x*s) \\ &= \wedge_{i\in N} A_i(x*s), \quad \text{(since each Ai is characteristic).} \end{aligned}$$

Thus $\wedge_{i\in N} A_i$ is fuzzy characteristic. □

Definition 3.6. *A group (G, Δ) with identity 0 acts on a R-fuzzy group A under a near-ring $(R, +, .)$. Then a right (respectively left) R-subgroup H of R is characteristic right (respectively left) R-subgroup of R if $\varphi(H) = H$ for all φ in $Aut(R)$.*

Definition 3.7. *Let H be a non-empty subset of a near- ring R acted by a group (G, Δ), and A be a R-fuzzy set in R defined by: $A(x*s) = \alpha$, $s \in H$; β, $s \notin H$ such that $\alpha > \beta$ in [0, 1]. Then G acts on R-fuzzy right (respectively left) subgroup if and only if G acts on a right (respectively left) R-fuzzy subgroup H in R.*

Corollary 3.8. *If H is a characteristic right (respectively left) R-subgroup of R, and A is an R-fuzzy set Defined in 3.7, then A is an R-fuzzy characteristic right (respectively left) subgroup of r acted by G.*

Proof. Let $x \in G$, $s \in R$, and φ be an automorphism in $Aut(R)$. If $s \in H$, then $\varphi(x) \in \varphi(H) = H$ and so $A\varphi(x * s) = A(x * \varphi(s)) = \alpha = A(x * s)$. Otherwise $s \notin H$. Then $A\varphi(x * s) = A(x * \varphi(s)) = \beta = A(x * s)$. Thus G acts on a right (resp. left) R-fuzzy subgroup H in R. □

Lemma 3.9. *A group (G, Δ) act on a right (respectively left) R-fuzzy subgroup A in a near-ring R. Let $s \in R$, and $x \in G$. Then $A(x * s) = t$ if and only if $x \in A_t$, and $x \notin A_s$ for all $s > t$ in [0, 1].*

4. Properties on Isomorphic Characteristic Fuzzy R-subgroups

The following results are discussed:

Theorem 4.1. *A group (G, Δ) act on a right (respectively left) R-fuzzy subgroup A in a near-ring R. Then A is a fuzzy characteristic if and only if each level right (respectively left) R-subgroup of A is characteristic.*

Proof. **Case (1):** A is a fuzzy characteristic. Let t be in Image of A. Let $\varphi \in Aut(R)$, and x in G. Then $A_t = U(A, t) = \{s \in S : \inf_{x \in G} A(x * s) \geq \alpha\}$. Now $s \in A_t$ implies that $A\varphi(x * s) = A(x * s) \geq t$ for all x in G, and so $A(x * \varphi(s)) \in A_t$ for all x in G. This show that $\varphi(A_t) \subseteq A_t$. On the other hand let $s \in A_t$ and $v \in R$ with $\varphi(v) = s$. Then $A(x * v) = A\varphi(x * v) = A(x * \varphi(v)) = A(x * s) \geq t$ for all x in G. This gives that v is in A_t. So $x * s = \varphi(x * v) \in \varphi(A_t)$ for all x in G, and also $A_t \subseteq \varphi(A_t)$. Hence $\varphi(At) = A_t$ implies that A_t is characteristic.
Case (2): Let $s \in R$, x in G, and φ be an automorphism in $Aut(R)$. Assume that $A(x * u) = t$ for all x in G implies that by Lemma 3.9, $u \in A_t$, and further $u \notin A_s$ for all $s > t$ in [0, 1]. It follows from hypothesis that $\varphi(x * u) \in \varphi(A_t) = A_t$. This gives that $A\varphi(x * u) = A(x * \varphi(u)) \geq t$. Let $\alpha = A\varphi(x * u)$ and suppose that $\alpha > t$. Then $A(x * u) \in A_\alpha = \varphi(A_\alpha)$. It gets from injectivity of φ that $x * u \in A_\alpha$, a contradiction. Hence $A\varphi(x * s) = A(x * \varphi(u)) = t = A(x * u)$ showing that A is a fuzzy characteristic acted by G. □

Note 4.2. *A new R-fuzzy right (respectively left) R-subgroup B is constructed by using a given R-fuzzy right (respectively left) R-subgroup A acted by a group (G, Δ). Let $i \geq 0$ be a real number. If $u \in [0, 1]$, then u^i means the non-negative i^{th} power in case $i > 1$. Define $A^i : R \to [0, 1]$ by $A^i(x * s) = A(x * s)^i$. for all x in G. If g acts R-fuzzy right (respectively left) subgroup in R, then G acts R-fuzzy right (respectively left) subgroup A^i in R.*

Theorem 4.3. *A group (G, Δ) act on a right characteristic (respectively left) R-fuzzy subgroup A in a near-ring R. Then G acts a right characteristic (respectively left) R-subgroup A^i of R for all positive integer $i > 0$.*

Proof. Let $\varphi \in Aut(R)$, $x \in G$, and $s \in R$. It follows that

$$\begin{aligned}
(A^i)^\varphi(x * s) &= (A^i)(x * \varphi(s)) \\
&= A(x * \varphi(s))^i \\
&= (A(x * s))^i \\
&= (A^i)(x * s) \ \textit{for all}\ i > 0.
\end{aligned}$$

□

Definition 4.4. *Let μ and λ be R-fuzzy right (respectively left) subgroup of R acted by a group (G, Δ). Then μ is R-fuzzy same type with λ if there exist $\varphi \in Aut(R)$ such that $\mu(x * s) = \mu(x * \varphi(s))$ for all x in G, and s in R.*

Since all R-fuzzy right (respectively left) subgroups of same type with μ, λ, δ in R acted by a group (G, Δ), then (1). μ is R-fuzzy with same type with μ itself (2). if μ is R-fuzzy same type with λ, then λ is R-fuzzy same type with μ, and (3). if μ is R-fuzzy same type with λ and λ is R-fuzzy same type with δ, then μ is R-fuzzy same type with δ.

Theorem 4.5. *Let A, B be two R-fuzzy right (respectively left) subgroups in R such that A is R-fuzzy same type with B. Then A is isomorphic to B.*

Proof. Since A is R-fuzzy same type with B, there exists $\varphi \in Aut(R)$ such that $A(x * s) = B(x * \varphi(s))$ for all s in R, and for all x in G. Let $\psi : A(R) \to B(R)$ such that $\psi(A(x * s)) = B(x * \varphi(s))$ for all s in R, and for all x in G. Then for every u, v in R, it gets that $\psi(Ax * (u + v)) = B(x * \varphi(x + y))$ and $\psi(x * A(uv)) = B(x * \varphi(xy)) = B(x * \varphi(u).\varphi(v))$ for all x in G. If $\psi(A(x * u)) = \psi(A(x * v))$ for all u, v in R, and for all x in G, then it follows that $B(x * \varphi(u)) = B(x * \varphi(v))$, and so $A(x * u) = A(x * v)$ for all x in G showing that ψ is injective. □

Theorem 4.6. *Let a group (G, Δ) act two R-fuzzy right (respectively left) subgroups A, B in R. If $Ao\varphi = B$, then $\psi(A) = B$ for some φ, ψ in $Aut(R)$.*

Proof. Since $Ao\varphi = B$ for some φ in $Aut(R)$. Thus $B(x * \varphi(s)) = B(x * s)$ for all s in R, and for all x in G. It follows that $\varphi^{-1}(A)(x * s) = \sup_{v \in (s)} A(x * v) = A(x * \varphi(u)) = B(x * u)$ for all u in R, and for all x in G. Take $\psi = \varphi^{-1}$. Then $\psi \in Aut(R)$. and $\psi(A) = B$. □

Corollary 4.7. *Let a group (G, Δ) act two R-fuzzy right (respectively left) subgroups A, B in R. If there exist $\varphi \in Aut(R)$ such that $A_t = \varphi(B_t)$ for all t in [0, 1]. Then G acts R-fuzzy subgroup A same type with B.*

Proof. Suppose that $\varphi \in Aut(R)$ such that $A_t = \varphi(B_t)$ for all t in [0, 1]. Let u be in A_t, and x in G. Then $A(x * u) = t$. It implies that $\varphi^{-1}(x * u) \in \varphi^{-1}(A_t) = B_t$. So $B((x * \varphi^{-1}(u)) > t = A(x * u)$. Putting $B(x * \varphi^{-1}(u)) = \alpha$ implies that $\varphi^{-1}(x * u) \in B_\alpha$. Thus $x * u \in \varphi(B_\alpha) = A_\alpha$ by assumption. It follows that $A(x * u) \geq \alpha = B(x * \varphi^{-1}(u))$. Hence $A(x * u) = B(x * \varphi^{-1}(u))$ for all u in R, and x in G. Since $\varphi^{-1} \in Aut(R)$, then A is R-fuzzy of same type with B. □

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