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Applications of Random Fixed Point Theorem to Integral Equation

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Abstract: In this paper random fixed point theorem has been applied seeking to prove the existence and uniqueness of volterra type integral equation and we show how the proper conditions guarantee the uniqueness of the solution of volterra type integral equation in complete metric space.

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1. Introduction

Random fixed point theory is a stochastic generalization of classical fixed point theory for deterministic mappings. Probabilistic functional analysis has emerged one of the important mathematical disciplines in view of its role in analyzing probabilistic model in applied science. The study of fixed points of random operators forms a central topic in this area. In 1950 the prague school of probabilistic initiated its study. A. T. Bharucha-Reid [1] had published the survey article on it, since then many interesting random fixed point results and its applications have appeared in the literature of random fixed point theory.

In recent years the theory of random equation have been attracted many researchers in various mathematical applications like in engineering, biological and in physical sciences have risen to random stochastic integral equation. In 1972 A. T. Bharucha-Reid [1] had given much importance to random integral equation in his book called random fixed point theory. The application of fixed point theory in different branches of mathematics, statistics, engineering and economics associated with the theory of integral equations and differential equations. Stochastic is also called random integral equations are very important in the study of many physical phenomena in engineering and technology, life sciences and in different fields.

Another milestone of study of random equations had been noted in prague school of probabilistic by spacek [2], R. Subramanium and Balchandram [12] are responsible to establish existence criterion of solution of a very wide class of random integral equations of Volterra type.

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2. Preliminaries and Hypothesis

Banach contraction theorem [7] is one of the most useful result in metric fixed point theory and it is widely used in most of the fixed point theorem in all analysis, because of contraction condition on the mapping which is very simple and easy to verify. The contraction condition requires only completeness assumption on the underlying metric space and because it finds almost canonical applications in the integral equations.

Definition 2.1 ([4]). *A nonempty set X and a map $T : X \to X$ is said to have a fixed point $t \in X$ such that $T(t) = t$.*

Example 2.2. *Let X is the set of all positive real numbers and $T : X \to X$ be a mapping defined by $T(t) = \frac{7}{9}t$ then $t = 0$ is the only one fixed point of T, and if $T(t) = t^2$ then $t = 0$ and $t = 1$ are two fixed point of T. If $t \in X = R$ be a metric space with usual metric and $T : X \to X$ be a mapping defined by $T(t) = (1 + t)^{1/3}$ for all $t \in X$, so to find the fixed point of T we solve the equation $t^3 - t - 1 = 0$.*

Definition 2.3 ([3]). *A mapping $T : X \to X$ in metric space (X, d) is said to be Lipschitz continuous if there exist a constant $\eta > 0$ such that $d(T(m), T(n)) \leq \eta d(m, n)$ for all $m, n \in X$, then we have the following conditions*

(1). If $\eta = 1$ then T is said to be non-expansive.

(2). T is called contraction if $\eta \in (0, 1)$.

(3). And if $d(T(m), T(n)) \leq d(m, n)$ for all $m \neq n$ then T is said to be contractive.

Therefore η is called Lipschitz constant of T.

Hence we say that T is contraction $\Rightarrow$ T is contractive $\Rightarrow$ T is non-expansive $\Rightarrow$ T is Lipschitz, but the converse is not true. We say that T is Continuous on metric space (X, d) therefore it is uniformly continuous.

Remark 2.4 ([10]). *Every lipschitz continuous function is continuous and every contraction mapping is contractive but its converse may not be true.*

Example 2.5. *Consider a metric space $X = [0, \infty)$ with the usual metric in the mapping $T : X \to X$ defined by $T(t) = \frac{1}{1+t^2}$ then T is contractive but not contraction.*

Definition 2.6 ([8]). *A mapping $T : X \to X$ is said to be Lipschitzian in a metric space (X, d) if there exist a constant $\alpha(T)$ called Lipschitz constant such that $d(T(m), T(n)) \leq \alpha(T)d(m, n)$, for all $m, n \in X$.*

Definition 2.7. *A Lipschitzian mapping with Lipschitzian constant $\alpha(T) < 1$ is called contraction.*

Remark 2.8. *Lipschitzian mapping is contraction mapping.*

Theorem 2.9 ([7] BCMT). *Let the mapping $T : X \to X$ be a contraction mapping in a complete metric space (X, d) then T has a unique fixed point.*

Remark 2.10. *By above Banach contraction theorem if X is not complete then T may not having fixed point.*

Example 2.11. *Let the mapping $T : X \to X$ in $X = (0, 1)$ defined as $T(x) = \frac{x}{2}$ then X is not a complete metric space with usual metric and T does not have any fixed point, in fact $T(0) = 0$ does not belongs to X.*

Remark 2.12. *By above Theorem 2.9 if T is not contraction then it may not have a fixed point.*

Example 2.13. *Consider a metric space $X = [1, \infty)$ with usual metric and the mapping $T : X \to X$ given by $T(x) = x + \frac{1}{x}$ then X is a complete metric space but T is not a contraction mapping, in fact $|T(x) - T(y)| = |x - y| \left(x - \frac{1}{xy}\right) < |x - y|$, for all $x, y \in X$. Therefore T is contractive and hence T does not have any fixed point.*

Theorem 2.14. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that for some integer k, $T^k = T * T * T * ... * T$ is a contraction mapping, then T has unique fixed point.*

Proof. By the above Theorem 2.9, T^k has a unique fixed point $t \in X$, i. e. $T^k(t) = t$ then $T(t) = T[T^k(t)] = T^k[T(t)]$ hence $T(t)$ is a fixed point of T^k. Let us assume that l is another fixed point of T then $T(l) = l$ and so $T^k(l) = l$. Hence by the uniqueness of fixed point of T^k we get $t = l$ therefore t belongs to X is a unique fixed point of T. □

3. Main Result

Our main result guarantees that the existence of a fixed point for a contractive mapping as follows.

Theorem 3.1. *Let $T : X \to X$ be a contractive mapping in a compact metric space (X, d) then T has a unique fixed point t for any $t \in T$, then $\lim_{n\to\infty} T^k(t) = v$ i.e. the successive iterates $T(t), T^2(t), T^3(t), ..., T^k(t)$ converge to the unique fixed point v of T.*

Proof. Let us define the mapping $\sigma : X \to [0, \infty)$ such that $\sigma(t) = d(t, T(t)) \ \ \forall \ \ t \in X$. Then σ is continuous indeed by contractiveness of T we have,

$$\begin{aligned} |\sigma(t) - \sigma(n)| &= |d(t, T(t)) - d(n, T(n))| \\ &\leq |d(t, T(t)) - d(T(t), n| + d(T(t), n) - d(T(n), n) \\ &\leq d(t, n) + d(T(t), T(n)) \\ &< 2d(t, n). \end{aligned}$$

Let $\gamma > 0$ be given that $|\sigma(t) - \sigma(n)| < 2d(t, n) < \gamma$ and $d(t, n) < \delta = \frac{\gamma}{2}$. Therefore T is continuous and hence σ is bounded below. Since X is continuous and σ is continuous there exist minimizer v belongs to X of σ such that $\sigma(v) \leq \sigma(n)$, for all n in X then we show that v is a fixed point of T. Suppose contrary that v is not a fixed point of T, and then $T(v) \neq v$ then by contractiveness of T we have $\sigma(T(v) = d(T(v), T^2(v)) < d(v, T(v)) = \sigma(v)$. Which contracts that v is a minimizer of σ and hence v is a fixed point of T. Let m belongs to X, if $T^n(m) \neq v$ then $d(T^{k+1}(t), v) = d(T^{k+1}(t), T(v)) < d(T^k(t), v)$, for all $k \in N$. Therefore $d(T^{k+1}(t), v)$ is strictly decreasing sequence of nonnegative real numbers and so converges to infimum. Suppose $\mu = \lim_{n\to\infty} d(T^{k+1}(t), v)$ and since $T^k(t)$ is a sequence of points of a compact metric space then there exist a subsequence $T^{nk}(t)$ which converges to some point say $n \in X$. Since $T^k(t)$ is decreasing.

$$d(n, v) = \lim_{n\to\infty} d(T^{nk}(t), v) = \lim_{n\to\infty} d(T^{nk+1}(t), v) = \mu.$$

If $\mu \neq 0$ then $n \neq v$ and hence we have

$$\begin{aligned} \mu = d(n, v) &> d(T(n), T(v)) = d(T(n), v) \\ &= \lim_{n\to\infty} d(T(T^{nk}(t)), v) = d(T^{nk+1}(t), v) \\ &= \mu. \end{aligned}$$

This is a contraction and hence $\mu = 0$. Therefore $\lim_{n\to\infty} T^k(t) = v$. □

By using the above theorem here we discuss the existence and uniqueness of the volterra integral equation.

Definition 3.2. *Let K be a continuous function on closed interval a, b to itself and φ be the continuous function on [a, b] then consider the equation such as $f(t) = \varphi(t) + \lambda \int_a^t K(t,n)f(n)dn$, for all $t \in [a,b]$. Where λ is a parameter. Then the above equation is called as Volterra integral equation.*

Integral equations are occurs in science, engineering & technology and in various research fields, the initial and boundary value problems are transferred in Volterra or Fredholm integral equation.
Let p, q be the real numbers $p < q$ and assume (t,n) tends to $K(t,n)$ be a measurable function such that t, n lies between p and q.

Theorem 3.3. *The integral equation $T(f(t)) = \varphi(t) + \lambda \int_a^t K(t,n)f(n)dn$. Where $\varphi \in L^2([a,b])$ and suppose that $\int_p^q \int_p^q |K(t,n)|^2 dtdn$ then the integral equation has a unique solution for $|\lambda| \leq \|K\|^{-1}_{L^2([p,q]\times[p,q])}$.*

Proof. Let us assume the function J is defined as $J(t) = \varphi(t) + \lambda \int_p^q K(t,n)f(n)dn$, where $f \in L^2([a,b])$. Then by triangular inequality and linearity property assume φ lies in $L^2([a,b])$. Hence we show that $J(t) = \varphi(t) + \lambda \int_p^q K(t,n)f(n)dn$ belongs to $L^2([a,b])$. Then by Holders inequality we have,

$$\begin{aligned}
\int_p^q |J(t)|^2 dt &\leq \int_p^q \left(\int_p^q |K(t,n)| \, |f(n)| \, dn \right)^2 dt \\
&\leq \int_p^q \left(\int_p^q |K(t,n)|^2 dn \right) \left(\int_p^q |f(n)|^2 dn \right) dt \\
&\leq \int_p^q \int_p^q |K(t,n)|^2 dndt \left(\int_p^q |f(n)|^2 dn \right) \\
&< \infty.
\end{aligned}$$

Now we take the mapping $S : L^2([a,b]) \to L^2([a,b])$ such that $sf = J$ and let δ be the L^2 metric then f_1, f_2 are in $L^2([a,b])$ hence by using Holder's inequality we have

$$\begin{aligned}
\delta(Sf_1, Sf_2) &= |\lambda| \left(\int_p^q \left| \int_p^q K(t,n)(f_1(n) - f_2(n))dn \right|^2 dx \right)^{\frac{1}{2}} \\
&\leq |\lambda| \left(\int_p^q \left(\int_p^q |K(t,n)| \, |f_1(n) - f_2(n)| \, dn \right)^2 dx \right)^{\frac{1}{2}} \\
&\leq |\lambda| \left(\int_p^q \left(\int_p^q |K(t,n)|^2 dn \right) \left(\int_p^q |f_1(n) - f_2(n)|^2 dn \right) dx \right)^{\frac{1}{2}} \\
&= |\lambda| \left(\int_p^q \int_p^q |K(t,n)|^2 dndx \right)^{\frac{1}{2}} \delta(f_1, f_2).
\end{aligned}$$

If $|\lambda| \leq \|K\|^{-1}_{L^2([p,q]\times[p,q])}$ which show that S is a contraction mapping. □

Theorem 3.4. *The Volterra equation for each $\lambda \in R$, has a unique solution f that is continuous on $[a,b]$.*

Proof. Let $X = C[a,b]$ be the set of all continuous real valued functions defined on interval a, b with uniform metric. Since K is continuous then there exist a constant $p > 0$ such that $|K(t,n)| \leq p$ for all t, n belongs to closed interval a, b. let us define the transformation $T : f \to T(f)$on X such as

$$T(f(t)) = \varphi(t) + \lambda \int_a^t K(t,n)f(n)dn$$

Then we have for all $f,\ g$ in X. Therefore

$$\begin{aligned}|T(f(t)) - T(g(t))| &= \left|\lambda \int_a^t K(t,n)\,|f(n) - g(n)|\,dn\right| \\ &\leq |\lambda|\,p(t-a)d(f,g)\forall t \in [a,b].\end{aligned}$$

Since $T^2(f) - T^2(g) = T(T(f) - T(g))$, we have

$$\begin{aligned}\left|T^2(f(t)) - T^2(g(t))\right| &= \left|\lambda \int_a^t K(t,n)\,|T(f(n)) - T(g(n))|\,dn\right| \\ &\leq |\lambda| \int_a^t K(t,n)\,|\lambda|\,k(n-a)d(f,g)dn \\ &\leq |\lambda|^2 k^2 \int_a^t (n-a)dnd(f,g) \\ &\leq \frac{|\lambda|^2 k^2 (t-a)^2}{2} d(f,g).\end{aligned}$$

Hence continuing this iterative process we obtain $|T^n(f(t)) - T^n(g(t))| \leq \frac{|\lambda|^n k^n (t-a)^n}{n!} d(f,g)$, for all $t \in [a,b]$. Therefore $|T^n(f) - T^n(g)| \leq \frac{|\lambda| k (t-a)^n}{n!} d(f,g)$. As $\frac{r^n}{n!} \to 0$ as $n \to \infty$ for any r is real number. Hence we conclude that there exist n such that T^n is a contraction mapping. By taking n sufficiently large to have $\frac{[|\lambda| k(t-a)]^n}{n!} < 1$. Then by the Theorem 3.1 there exist a unique solution f in X satisfying $T(f) = f$, and obviously if $T(f) = f$ the we solve f by using above Theorem 3.1. □

4. Conclusion

In this paper we see the most interesting applications of fixed point theory under the metric space is become function space. We discuss the existence and uniqueness of volterra integral equation.

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References

[1] A. T. Bharucha-Reid, *Fixed Point Theorem in Probabilistic Analysis*, Bulletin of American Mathematical Society, 82(1976), 641-657.

[2] Spacek and Z. Gleichungen, *Zufallige Gleichungen*, Czechoslovak Mathematical Journal, 5(80)(1955), 462-466.

[3] Suhas Patil and U. P. Dolhare , *Random Fixed Point Theorems for Contraction Mappings in Metric Space*, Int. Jou. of Sci. & Res., 5(10)(2016), 1172-1176.

[4] Suhas Patil and U. P. Dolhare, *Random Fixed Point Theorems for Compatible Mappings in Metric Space*, International Journal of Mathematical Archive, 8(6)(2017), 1-5.

[5] R. A. Rashwan and D. M. Albaqeri, *A Common Random Fixed Point Theorem and Application to Random Integral Equations*, Int. J. Applied Mathematical Research., 3(1)(2014), 71-80.

[6] H. K. Pathak, M. S. Khan and R. Tiwari, *A Common Fixed Point Theorem and its applications to Non-linear Integral Equations*, Int. Journal of Comp. & Math. with Applications, 53(2007), 961-971.

[7] Suhas Patil and U. P. Dolhare, *A Note on Development of Metric Fixed Point Theory*, Int. Jou. Adv. Reserch, 4(8)(2016), 1729-1734.

[8] Ta Ngoc Anh, *Random equations and applications to General Random Fixed Point Theorems*, New Zealand J. Mathematics, 41(2011), 17-24.

[9] O. Hans, *Reduzierenda Zufallige Transformations*, Czechoslovak Math. J., 7(82)(1957), 154-158.

[10] Suhas Patil and U. P. Dolhare, *Random Fixed Point Theorems for Multi-Valued Contraction Mapping In Complete Metric Space*, Int. J. of Current Research, 12(2016), 42856-42860.

[11] Suhas Patil and U. P. Dolhare, *Some Random Fixed Point Theorems For Multi-Valued Contraction Mapping In Complete Metric Space*, Int. J. of Current Research, 12(2016), 42856-42860.

[12] R. Subramaniam and K. Balachandran, *Existance of solution of general nonlinear stochastic integral equations*, Indian J. of Pure and Applied Mathematics, 28(1997), 775-789.

[13] Suhas Patil and U. P. Dolhare, *Unique Fixed Point Theorems for Non-Self Contraction Mapping in Banach Space*, International Journal of Mathematics and its Applications, 5(3-A)(2017), 29-32.

[14] Suhas Patil and U. P. Dolhare, *Some Common Fixed Point Theorems in Metric Space via Weakly Commuting Mapping*, Int. J. of Applied and Pure Science and Agriculture, 2(10)(2016), 32-38.

[15] R. Subramaniam and K. Balachandran, *Existance of solution of general nonlinear stochastic integral equations*, Indian J. of Pure and Applied Mathematics, 28(1997), 775-789.