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Analysis of the Pattern Leading to Final Descent in a Hailstone Sequence

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Abstract: This paper analyses and presents the pattern in hailstone sequence for any positive integer that makes it possible to predict a particular number in the sequence, from where after multiple descents and ascents the sequence will start to finally descent to the endless cycle 4, 2, 1.

MSC: 11B99.

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1. Introduction

The Collatz Conjecture [1, 2] is a conjecture in mathematics that concerns a sequence defined as follows: start with any positive integer n. Then each term is obtained from the previous term as follows: if the previous term is even, the next term is one half the previous term. Otherwise, the next term is 3 times the previous term plus 1. The conjecture is that no matter what value of n, the sequence will always reach 1. The conjecture is named after Lothar Collatz, who introduced the idea in 1937, two years after receiving his doctorate. It is also known as the $3n + 1$ conjecture. One way to depict the conjecture is that iteration of any positive integer, (A_p) as depicted below, will lead to a sequence that will eventually converge to 1.

- A_p is equal to $\frac{1}{2}A_{p-1}$, if A_{p-1} is even.
- A_p is equal to $3A_{p-1} + 1$, if A_{p-1} is odd.

The sequence of numbers involved is referred to as the hailstone sequence [3] or hailstone numbers because the values are usually subject to multiple descents and ascents. It is conjectured that all hailstone sequences eventually descend and falls to 1 for all starting n.

2. Creating hailstone sequence for a positive integer

Sequence for any positive integer **n** can be created through iteration stated above. for instance:

For $n = 3$, related hailstone sequence is 3, 10, 5, 16, 8, 4, 2, 1. To understand the underlying patterns in a hailstone sequence, each no from 1-200 was sequenced using a spreadsheet program in Microsoft excel. To illustrate, please find below the output from the excel spreadsheet for some of the numbers sequenced.

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No.	Count of Sequence	Highest No. in sequence	Step No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
3	7	16		10	5	16	8	4	2	1												
4	2	2		2	1																	
6	8	16		3	10	5	16	8	4	2	1											
7	16	52		22	11	34	17	52	26	13	40	20	10	5	16	8	4	2	1			
9	19	52		28	14	7	22	11	34	17	52	26	13	40	20	10	5	16	8	4	2	1
10	6	16		5	16	8	4	2	1													
11	14	52		34	17	52	26	13	40	20	10	5	16	8	4	2	1					

Figure 1. **Sample output of Excel spreadsheet program**

Also, please refer to the graph highlighting the no. of steps for each sequence to converge to 1.

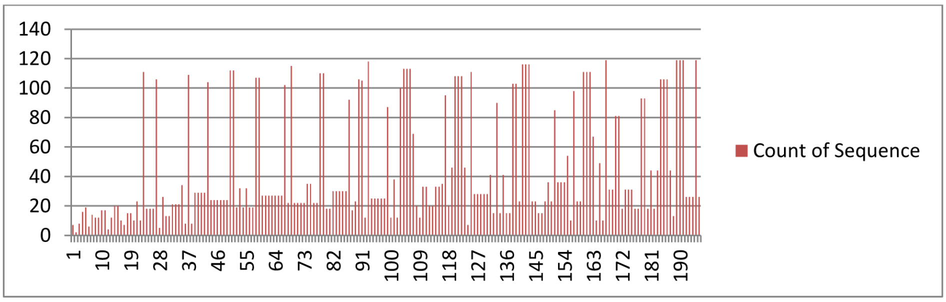

Figure 2. **Step Count of Hailstone Sequence for nos (1-200)**

It is evident from the data that each no. generates a sequence that has multiple ascends and descends, before it finally start to converge to the endless cycle 4, 2, 1.

3. A Pattern in the Hailstone Sequence Indicating Final Descent

To understand any underlying pattern that may exist in any hailstone sequence indicating final descent, analysis of sequences for nos. between 1- 200 was done, and it revealed the following pattern

Theorem 3.1. *The final descent of the hailstorm sequencing pattern in collatz conjecture starts from a positive even integer* $Z = (3n + 1) = 2^{(x+1)}$ *(where x is a positive odd integer).*

Proof. Let's assume n is a positive odd integer, therefore the next no. in sequence will be $(3n+1)$. Let Z be the no, from where the series start descent and terminates on 1. Therefore for any preceding odd no. n,

$$Z = (3n + 1) \tag{1}$$

Also consider that $Z = (3n + 1)$ is even and therefore can be represented as

$$Z = 2Y, \quad \text{where Y is a positive integer} \tag{2}$$

From (1) and (2) we can deduce that $3n + 1 = 2Y$

$$Y = \frac{(3n + 1)}{2} \tag{3}$$

Since n is an odd number, n can take any value out of the set $n = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, \ldots\}$. The corresponding values for Y will be from $Y = \{\mathbf{2}, 5, \mathbf{8}, 11, 14, 17, 20, 23, 26, 29, \mathbf{32}, \ldots\}$. And therefore, the Corresponding values for Z will be from $Z = \{\mathbf{4}, 10, \mathbf{16}, 22, 28, 34, 40, 46, 52, 58, \mathbf{64} \ldots\}$. However since $Z = 3n + 1$ is even, and the descent start from Z, next numbers in the sequence are also even i.e. $\frac{(3n+1)}{2}$, $(3n+1)4$ and $\frac{(3n+1)}{8}\dots$ till the sequence reaches 1. This means that only values of $Z = (3n+1)$ that will satisfy this condition will be from $Z = \{4, 16, 64, 256, \dots\}$, and corresponding values of Y will be from

$$Y = \{2, 8, 32, 128, \dots\} \tag{4}$$

This means that Y can be stated as

$$Y = 2^x \quad \text{(where x is a positive odd integer).} \tag{5}$$

From (2) and (5), we can derive that $Z = 2Y = 2^{(x+1)}$ (where x is a positive odd integer). □

Corollary 3.2. *For a Hailstone sequence, an odd no. from the sequence* $\{n, 4n+1, 4(4n+1)+1, \dots\}$*, where* $n = 1$*, will immediately proceed an even no* $Z = (3n+1)$*, from where the hailstone sequence finally start to descend.*

Proof. From (1) and (3), we can infer that

$$n = \frac{(Z-1)}{3} = \frac{(2Y-1)}{3} \tag{6}$$

And therefore for the values of $Y = \{2, 8, 32, 128, \dots\}$, the values of n that will satisfy (6) will be $n = \{1, 5, 21, 85, 341 \dots\}$, which can be described as a sequence $\{n, 4n+1, 4(4n+1)+1, \dots\}$, where $n = 1$.

Alternately: From (2) and (5), $Z = 2Y = 2^{(x+1)}$ (where x is positive odd integer and therefore $x+1$ will be even) Therefore Z can be also be written as $Z = 4p$, where $p = 1, 2, 3, 4, 5, \dots$ is a positive integer Since $n = \frac{(Z-1)}{3}$ or $n = \frac{(4^p-1)}{3}$, where $p = 1, 2, 3, 4, 5, \dots$ and therefore $n = 1, 5, 21, 85, 341, \dots$ which can be described as a sequence $n, 4n+1, 4(4n+1)+1, \dots$. □

Corollary 3.3. *The digital sum of the even number Z from where the descent start* $[D(Z)]$ *minus 1, will always be divisible by 3.*

Proof. From (1), $Z = (3n+1)$ and therefore $Z - 1 = 3n$.

$$D(Z) = 1 + (Z-1)|mod\ 9|$$

To illustrate, let's consider some random numbers, say 25 and 137. □

Example 3.4. $D(25) = 1 + 24|mod\ 9| = 1 + 6 = 7$.

Example 3.5. $D(137) = 1 + 136|mod\ 9| = 1 + 1 = 2$.

Since,

$$D(Z) = 1 + (Z-1)|mod\ 9|$$
$$D(Z) - 1 = (Z-1)|mod\ 9| \tag{7}$$

Therefore, for $D(Z) - 1$ to be divisible by 3, $(Z-1)|mod\ 9|$ has to be divisible by 3. Now $(Z-1)|mod\ 9| = 3n|mod\ 9|$ and is therefore divisible by 3. We can therefore infer that the digital sum of the number Z, from where the descent start $[D(z)]$ minus 1, will be divisible by 3.

4. Conclusion

This potentially proves that for every hailstone sequence, the final descent of sequence will start from an even number $(Z = 3n + 1)$, which can be represented as $2^{(x+1)}$ (where x is a positive odd integer). This number will be proceeded by a positive odd integer n that belongs to the sequence $\{n, 4n + 1, 4(4n + 1) + 1, \dots\}$, where $n = 1$. The digital sum of the number $Z[D(Z)]$ minus 1 i.e. $(D(Z) - 1)$ would always be divisible by 3.

References

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