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Solution of Blasius Equation by Adomian Decomposition Method and Differential Transform Method

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Abstract: The efficient semi-numerical schemes combining the features of Adomian polynomials (ADM) and Differential Transform method have been presented to solve the well-known non-linear Blasius equation. A numerical method for solving two forms of Blasius equation is proposed. The Blasius equation is a third order nonlinear ordinary differential equation, which arises in the problem of the two-dimensional laminar viscous flow over a half-infinite domain. The approaches are based on differential transform method and Adomian Decomposition method. In these schemes, the solution takes the form of a convergent series with easily computable components. The obtained series solution is combined to handle the boundary condition at infinity for only one of these forms. The numerical results demonstrate the validity and applicability of the methods and a comparison is made with both the methods.

Keywords: Differential transform method, Adomain Decomposition method, Blasius equation.

1. Introduction

It is well known that the Blasius equation is one of the basic equations of fluid dynamics. Blasius equation is a third order nonlinear ordinary differential equation which describes the velocity profile of the fluid in the boundary layer theory on a half-infinite interval. The Blasius differential equation arises in the theory of fluid boundary layer mechanics, and in general must be solved numerically as reported in [1, 3, 6, 7]. In the study of Prandtl boundary layer problems relevant to the motion of an incompressible viscous fluid, solutions of self-similar form naturally give rise to such equations as the Blasius equation. It describes the steady two-dimensional boundary layer that forms on a semi-infinite plate which is held parallel to a constant unidirectional flow [2]. Concerning the Blasius equation, many researchers have been attempted and much progress has been made so far [4] and [5]. We have been inspired by the recent work of Abbasbandy [1] to come up with a modified decomposition technique to solve the Blasius equation and our solution is consistent with the solution obtained and discussed by Abbasbandy [1], Liao [5] and Ishimura & Naoyuki [4].

2. The Blasius Equation

The Blasius Equation is a famous problem arising from boundary layer theory of fluid mechanics. This equation emerged when Blasius developed a method in which the boundary layer equations are reduced to ordinary differential equations [8].

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This well-known equation is a third order, nonlinear differential equation,

$$f'''(\eta) + f''(\eta).f(\eta) = 0 \tag{1}$$

on $0 \leq \eta < \infty$ satisfying the boundary conditions

$$f(0) = 0,\ f'(0) = 0, f'(\infty) = 1 \tag{2}$$

This equation describes the velocity profile in the boundary layer when one considers the movement of an incompressible, viscous fluid along a semi-infinite plate [9, 10]. The generalize boundary conditions as

$$f(0) = -\alpha,\ f'(0) = -\beta, f'(\infty) = 1 \tag{3}$$

where α and β are constants. According to Guedda [11], in the event of $f(0) = -\alpha$, α represents a suction/injection parameter where $-\alpha > 0$ represents suction and $-\alpha < 0$ corresponds to Injection of the fluid. The initial condition, $f'(0) = -\beta$, indicates the slip condition at the wall [12]. The case where $\beta = 0$ represents no-slip. Another parameter considered when evaluating the Blasius equation is the initial value of the second derivative, $f''(0)$. The value of $f''(0)$ is a significant parameter in the boundary layer theory which gave rise to the equation. According to Weyl [13], "the value $f''(0)$ is the essential factor in the formula for the skin friction along the immersed plate". Due to its importance, a portion of this work is focused on accurately determining this parameter. Much work has been done on the Blasius equation by ADM and DTM, although no exact solution is known. Solutions for the equation have been developed by many approaches. Blasius gave a power series solution [14]. Numerical methods, such as the Runge-Kutta method or the shooting method, have also been used [15, 16]. Other techniques used include perturbation methods [10, 14], the homotopy analysis method [14], and the differential transformation method [17]. Recently, Wang [18] presented a solution utilizing the Adomian Decomposition Method to solve the classical Blasius equation. This method proves to be reliable and demonstrates many advantages. In the Adomian decomposition method, the solution is expanded as an infinite series and is determined by a series of successive calculations. The partial sum of this series at any point provides an approximate solution, which can be improved by adding additional terms. Hashim [19] provided corrections to the numerical values in Wang's [18] article and also showed that the accuracy of the numerical solution can be improved by using Pade approximations. Despite the errors in the numerical values, the methodology in Wang [18] appears to be reliable. In this paper, we will apply Wang's [18] methodology to the Blasius equation [20].

$$f'''(\eta) + f''(\eta).f(\eta) = 0 \tag{4}$$

on $0 \leq \eta < \infty$ with

$$f(0) = -\alpha,\ f'(0) = -\beta, f'(\infty) = 1 \tag{5}$$

3. Description of Adomian Decomposition Method

In reviewing the basic methodology involved, a general nonlinear differential equation will be used for simplicity. Consider

$$Fy = g \tag{6}$$

The linear term is decomposed as

$$F = L + R \tag{7}$$

where L is easily invertible operator and R is the remainder of the linear operator. For convenience L is taken as higher order derivative. Thus the equation may be written as

$$Ly + Ry + Ny = g \tag{8}$$

Where Ny is corresponds to the nonlinear terms. Solving Ly from equation (8), we have

$$Ly = g - Ry - Ny \tag{9}$$

Because L is invertible, the equivalent expression is $L^{-1}(Ly) = y(t) - y(0) - ty'(0)$. If L is a second order operator, then L^{-1} is a two times integration operator $L^{-1} = \int\int (*)\, dt_1 dt_2$ and $L^{-1}(Ly) = y(t) - y(0) - ty'(0)$. Then the equation (9) becomes

$$y(t) - y(0) - ty'(0) = L^{-1}(g) - L^{-1}(Ry) - L^{-1}(Ny)$$
$$y(t) = y(0) - ty'(0) + L^{-1}(g) - L^{-1}(Ry) - L^{-1}(Ny) \tag{10}$$

Therefore, y can be presented as series

$$y(x) = \sum_{n=0}^{\infty} y_n \tag{11}$$

Let

$$y_0 = y(t_0) - ty'(t_0) + L^{-1}(g) \tag{12}$$

and $y_n\ (n < 0)$ is to be determined. The nonlinear term Ny will be decomposed by the infinite series of Adomain polynomials

$$Ny = \sum_{n=0}^{\infty} A_n \tag{13}$$

A_n are called as Adomain polynomials, and depend only on the y components and make a rapid convergent series, which can be determined by

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{n=0}^{\infty} \lambda^n y_n\right)\right]_{\lambda=0} \tag{14}$$

Substituting equation (12) and (14) into (10), we obtain

$$\sum_{n=0}^{\infty} y_n = y_0 - L^{-1}\left[R\sum_{n=0}^{\infty} y_n\right] - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right) \tag{15}$$

Consequently, the recursive relationship is found to be

$$\begin{aligned} y_0 &= y(t_0) - ty'(t_0) + L^{-1}(g) \\ y_1 &= -L^{-1}R(y_0) - L^{-1}(A_0) \\ y_2 &= -L^{-1}R(y_1) - L^{-1}(A_1) \\ y_{n+1} &= -L^{-1}R(y_n) - L^{-1}(A_n) \end{aligned} \tag{16}$$

Since

$$\begin{aligned} y_0 &= g(x) \\ y_{n+1} &= L^{-1}R(y_n) - L^{-1}(A_n) \end{aligned}$$

Based on the Adomian decomposition method, we consider the solution *y(t)* as $y = \lim_{n\to\infty} \phi_n$, where $(n+1)$ term approximation of the solution defined as the following form

$$\phi_n = \sum_{k=0}^{n} y_k(t), \quad n > 0$$

3.1. Adomian Decomposition Method of Blasius equation

We begin by introducing a differential operator, L, as

$$L = \frac{d}{d\eta^3}^3 \tag{17}$$

Then the inverse operator is

$$L^{-1} = \int_0^\eta \int_0^\eta \int_0^\eta (.)\, d\eta d\eta d\eta \tag{18}$$

The Blasius equation is then written as $f''' = -f''f$. Therefore,

$$Lf = Nf \tag{19}$$

where

$$Nf = -f''f \tag{20}$$

Operating with L^{-1} yields

$$f(\eta) = f(0) - f'(0)\eta - \frac{1}{2}f'(0)\eta^2 - L^{-1}(-f''f) \tag{21}$$

Using the boundary conditions and taking $f''(0) = P$.

3.2. A Brief Description of Differential Transform Method

Transformation of the k^{th} derivative of function $f(t)$ in one variable is defined as follows

$$F(k) = \frac{1}{k!}\left[\frac{\partial^k f(t)}{\partial t^k}\right]_{t=t_0} \tag{22}$$

$f(t)$ is the original function and $F(k)$ is the transformed function. Differential inverse transform of $F(k)$ is defined as follows

$$f(t) = \sum_{k=0}^{\infty} F(k)(t - t_0)^k \tag{23}$$

The following theorems that can be deduced from equations (22) and (23) are given below:

1. If $f(x) = g(x) \pm h(x)$, then $F(K) = G(k) \pm H(k)$.
2. If $f(x) = cg(x)$, then $F(K) = cG(k)$, c is a constant.
3. If $f(x) = \frac{d^n g(x)}{dx^n}$ then $F(K) = \frac{(k+n)!}{k!}G(k+n)$.
4. If $f(x) = g(x)h(x)$ then $F(K) = \sum_{i=0}^{k} G(i)H(k-i)$.
5. If $f(x) = x^n$ then $F(K) = \delta(k-n) = \delta(k-n) = \begin{cases} 1, & \text{for } k = n \\ 0, & \text{for } k \neq n \end{cases}$.
6. If $f(x) = \int_0^x g(t)dt$ then $F(K) = \frac{G(k-1)}{k}$, where $k \geq 1$.
7. If $f(x) = e^{\beta x}$ then $F(K) = \frac{\beta^k}{k!}$.
8. If $f(x) = \sin(\omega x + \alpha)$ then $F(K) = \frac{\omega^k}{k!}\sin(\frac{k\pi}{2} + \alpha)$.
9. If $f(x) = \cos(\omega x + \alpha)$ then $F(K) = \frac{\omega^k}{k!}\cos(\frac{k\pi}{2} + \alpha)$.

4. Solution of Blasius Problem by ADM and DTM

In this study, we applied both semi analytical methods in the non-linear Blasius problem [20] which names are Adomian decomposition method and differential transform method.

4.1. Analysis by Adomian Decomposition Method (ADM)

Consider the Blasius equation [20]

$$f^{'''}(\eta)+\frac{1}{2}f(\eta)f^{''}(\eta)=0,\qquad 0\leq\eta\leq 1 \tag{24}$$

subjected to the condition

$$f(0)=0,\quad f'(0)=1,\quad f''(\infty)=0 \tag{25}$$

To solve the differential equations, rearranging Eq.(24) making the highest derivative to remain on RHS and rewrite as

$$f^{'''}(\eta)=-\frac{1}{2}ff^{''}(\eta) \tag{26}$$

By the ADM, the linear and nonlinear terms can be decomposed by an infinite series of polynomials given by

$$ff^{''}=\sum_{0}^{\infty}A_n \tag{27}$$

To solve this equation (26), Adomian polynomials can be derived as follows

$$\begin{aligned}A_0&=f_0f_0^{''},\\ A_1&=f_1f_0^{''}+f_0f_1^{''},\\ A_2&=f_2f_0^{''}+f_1f_1^{''}+f_0f_2^{''}\end{aligned}$$

And so on, the rest of the polynomials can be constructed in similar manner. By using equation (24) given in Section 3, we have

$$\begin{aligned}f_0&=\eta+\frac{P}{2}\eta^2\\ f_1&=-\frac{1}{48}P\eta^4-\frac{1}{240}P^2\eta^5\\ f_2&=\frac{1}{960}P\eta^6+\frac{1}{2016}P^2\eta^7+\frac{1}{16128}P^3\eta^8\\ f_3&=-\frac{1}{21504}Pt^8-\frac{41}{967680}P^2\eta^9-\frac{173}{14515200}P^3\eta^{10}-\frac{173}{159667200}P^4\eta^{11}\end{aligned}$$

The next step is to determine the value of $f''(0)=P$ which could then be returned to the original series solution, equation

$$f(\eta)=\sum_{0}^{\infty}f_n=f_0+f_1+f_2+f_3+\cdots+f_n$$

This provides an approximate solution to the Blasius equation.

$$\begin{aligned}f(\eta)&=\eta+0.500000P\eta^2-0.0208333333P\eta^4+0.00104166667P\eta^6-0.00004650334433P\eta^8\\ &-0.004166667P^2\eta^5+0.0004960317P^2\eta^7+0.000062004P^3\eta^8-0.0000423694P^2\eta^9\\ &-0.0000119185P^3\eta^{10}-0.0000010835P^4\eta^{11}\end{aligned} \tag{28}$$

Using the boundary condition from the transformed Blasius equation, $f(0)=0$, we are able to determine that $P=0.5227030798$.

4.2. Analysis by Differential Transform Method (DTM)

Taking the differential transform of Equation (24) as given in section 3, we obtain

$$F(K+3) = -\frac{1}{2}\frac{1}{(k+1)(k+2)(k+3)}\left(\sum_{i=0}^{k}(k-i+1)(k-i+2)F(i)F(k-i+2)\right) \tag{29}$$

By applying the DTM into Equation (25), differential transform of initial value are thus determined into a recurrence equation that finally leads to the solution of a system of algebraic equations. The differential transform of the initial value is as follows

$$F(0) = 0,\ F(1) = 1,\ F(2) = \frac{P}{2} \tag{30}$$

Moreover, by recursive method (29), we can calculate another all values of $F[k]$. From this recurrence relation (29), obtain the terms of the series are

$$F[3] = 0,\ F(4) = -\frac{1}{48}P^2,\ F(5) = -\frac{1}{240}P^2,\ F(6) = \frac{P^2}{960},\ F(7) = \frac{11}{20160}P^2,$$
$$F(8) = \frac{11}{161280}P^3 - \frac{1}{21504}P,\ F(9) = -\frac{43}{967680}P^2,\ F(10) = \frac{1}{552960}P - \frac{5}{387072}P^3$$

Finally, using inverse transformation method, we obtained the approximate solution is

$$f(\eta) = F[0]\eta^0 + F[1]\eta^1 + F[2]\eta^2 + F[3]\eta^3 + \dots$$
$$f(\eta) = \eta + \frac{P}{2}\eta^2 - \frac{a}{48}\eta^4 - \left(\frac{1}{240}P^2\right)\eta^5 + \left(\frac{P^2}{960}\right)\eta^6 + \left(\frac{11}{20160}P^2\right)\eta^7 + \left(\frac{11}{161280}P^3 - \frac{1}{21504}P\right)\eta^8$$
$$- \left(\frac{43}{967680}P^2\right)\eta^9 + \left(\frac{1}{552960}P - \frac{5}{387072}P^3\right)\eta^{10}$$
$$f(\eta) = \eta + 0.50000P\eta^2 - 0.0208333334P\eta^4 - 0.0041666667P^2\eta^5 + 0.00104166647P^2\eta^6 - 0.000046503P\eta^8$$
$$+ 0.0005456349P^2\eta^7 + 0.0000682044P^3\eta^8 - 0.0000444362P^2\eta^9 + 0.0000018084P\eta^{10} - 0.0000129174P^3\eta^{10} \tag{31}$$

Table 1 shows the ADM solution and the DTM solution for $\alpha = 0$, $\beta = 1$. This further demonstrates that the truncated ADM solution is a good approximation of the DTM solution. We can also see that the solution is convergent in a small region after which it diverges quickly. However, it should be noted that the boundary layer is also a small region. Therefore, the approximation generated by ADM and DTM solutions which provide an acceptable approximation. The following table shows comparative analysis of $f(\eta)$ between ADM and DTM.

η	$f(\eta)$ ADM	$f(\eta)$ Ref. [20] DTM	ERROR
0	0	0	0
0.1	0.102612415608	0.102612415595	0.00000000013
0.2	0.210436310404	0.210436308848	0.000000001556
0.3	0.323431091423	0.323431064914	0.000000026509
0.4	0.441528252985	0.441528055184	0.000000197801
0.5	0.564631188334	0.564630249921	0.000000093841
0.6	0.692615543669	0.692612201945	0.000000334117
0.7	0.825330128806	0.825320370213	0.000000975859
0.8	0.962598370135	0.962573734296	0.000002463587
0.9	1.104220255667	1.104164629225	0.000055626458
1.0	1.249974677955	1.249859704471	0.000011497322

Table 1. Comparison of ADM and DTM solution with $f''(0) = P = 0.5227030798$

5. Conclusion

For the present study, we referred the work of Ertrka and Momani [20]. In this work, authors [20] have discussed the differential transform method with Pade' approximations was applied to compute approximate solutions for two forms of Blasius equation. In this paper, both semi analytical methods were applied by author in the non-linear Blasius problem which names are Adomian decomposition method and differential transform method. It was obtained their results for two cases by using these methods. Their results were presented in the Table 1. The results are shown that all of these methods are powerful and efficient technique for finding semi analytical solutions for Blasius problem in the fluid mechanics. The decomposition method introduces a significant improvement in the fields over existing techniques. Moreover, the decomposition method in away is easier, more convenient and more efficient. Also a comparative study has been conducted between the DTM and the ADM. Computations of this chapter have been carried out using computer package of Mathematica 7. From this work, the conveniences of the methods were observed in that it involves direct application to the problem and is easily performed. The results obtained were found to be quite accurate. The main disadvantage of ADM that was noted was the small region of convergence for the solution. Although several researchers have claimed to improve this with Pade' approximants as an after-treatment, this work shows that while their use does improve accuracy, it does not appear to improve region of convergence.

The Blasius equation could also be examined further. As stated before, an alternate transformation or definition for the operators could be investigated. This problem could be used for examining the convergence issues and effectiveness of after treatment techniques. In general, the ADM and DTM are believed to be easy and convenient tools with wide applicability. However, some discretion are recommended for these use as there are certain conditions which limit the quality of the solution generated.

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