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A Class of *-Simple Type A I-Semigroups

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Abstract: *-simple type A I-semigroups as the generalized Bruck-Reilly *-extensions is studied and properties obtained. It is proved that a semigroup S is a *-simple type A I-semigroup if and only if it can be expressed as $S = GBR^*(T, \theta)$ where T is a finite chain of cancellative monoids. Thus the structure of *-simple type A I-semigroups is described and the results obtained is amplified in the light of studies on simple I-regular semigroups by Warne and that of *-simple type A ω-semigroups by Asibong-Ibe.

Keywords: Type A I-semigroups, cancellative monoids, generalized Bruck-Reilly *-extensions.

1. Introduction and Preliminaries

Earlier investigations in [8] studied *-bisimple type A I-semigroups and characterized them as the generalized Bruck-Reilly *-extensions of cancellative monoids. Their congruences were later studied in [7] while the results of [8] generalized those of regular I-bisimple semigroups obtained in [10], the study of *-simple type A I-semigroup undertaken here follow naturally from that of simple I-regular semigroups by Warne in [11]. The theory developed here draws inspiration from facts in [6, 9, 11] and [1]. In this section some basic facts on type A semigroups are presented. In section 2 we construct a *-simple type A I-semigroup from a sequence of cancellative monoids M_i $(i = 0, 1, \dots, d-1)$, a homomorphism θ by the generalized Bruck-Reilly *-extensions. The integer d is the number of distinct $\mathcal{D}^*$-classes in such a semigroup. Section 3 considers the structure theorem for *-simple type A I-semigroups which is invariably analogous to that of *-simple type A ω-semigroups. For a semigroup S $E(S)$ denotes the set of idempotents of S. Let S be a semigroup whose set $E(S)$ is non-empty. We define a partial order "$\leq$" on $E(S)$ such that $e \leq f$ if and only if $ef = fe = e$. Let I denote the set of all integers and let $\mathbb{N}^0$ denote the set of non-negative integers. A semigroup S is said to be an I-semigroup if and only if $E(S)$ is order isomorphic to I under the reverse of the partial order. Let S be a semigroup and let $a, b \in S$. Then the elements a and b are said to be $\mathcal{R}^*$-related written $a\mathcal{R}^*b$ if and only if for all $x, y \in S^1$, $xa = ya$ if and only if $xb = yb$. The relation $\mathcal{L}^*$ is defined dually. The join of the equivalence relations $\mathcal{R}^*$ and $\mathcal{L}^*$ is denoted by $\mathcal{D}^*$ and their intersection by $\mathcal{H}^*$. Thus $a\ \mathcal{H}^*b$ if and only if $a\mathcal{R}^*b$ and $a\mathcal{L}^*b$. In general $\mathcal{R}^* \circ \mathcal{L}^* \neq \mathcal{L}^* \circ \mathcal{R}^*$ (see [3]). Following Fountain [4], a semigroup is an abundant semigroup if every $\mathcal{L}^*$-class and every $\mathcal{R}^*$-class in S contain idempotents. An abundant semigroup S is said to be adequate [3] if $E(S)$ forms a semilattice. In an adequate semigroup every $\mathcal{L}^*$-class $\mathcal{R}^*$-class contains a unique idempotent. If a is an element in an adequate semigroup S, then $a^*(a^\dagger)$ denotes the unique idempotent in the $\mathcal{L}^*$-class L^*_a ($\mathcal{R}^*$-class R^*_a) containing a. Fountain in [2] introduced the concept of right type A semigroup as special type of right PP monoids which is e-cancellable

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for an idempotent. He followed it in [3] with introduction of type A as an adequate semigroup satisfying certain internal conditions. An adequate semigroup S is a type A semigroup if $ea = a(ea)^*$ and $ae = (ae)^\dagger a$ for all $a \in S$ and $e \in E(S)$. We conclude this section by defining the relation $\mathcal{J}^*$. Let S be a semigroup and I^* be an ideal of S. Then I^* is said to be a *-ideal if $L_a^* \subseteq I^*$ and $R_a^* \subseteq I^*$ for all $a \in I^*$. The smallest *-ideal containing an element 'a' is the principal *-ideal generated by 'a' and is denoted by $J^*(a)$. For $a, b \in S$, $a\mathcal{J}^*b$ if and only if $J^*(a) = J^*(b)$. The relations $\mathcal{J}^*$ contains $\mathcal{D}^*$. A semigroup S is said to be *-simple if the only *-ideal of S is itself. Clearly a semigroup is *-simple if all its elements are $\mathcal{J}^*$-related.

Lemma 1.1 ([3]). *Let S be a semigroup and $a, b \in S$. Then $b \in J^*(a)$ if and only if there are elements $a_0, a_1, \ldots, a_n \in S$, $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in S^1$ such that $a = a_0$, $b = a_n$ and $a_i\mathcal{D}^*x_ia_{i-1}y_i$, for $i = 1, 2, \ldots, n$.*

Other basic results discussed in [3] and [8] will be assumed. The notation used in this paper is similar to that in Fountain [3], Howie [5] and Asibong-Ibe [1]. Recently type A semigroups have been shown to be special type of restriction semigroups. In this case type A ω-semigroup will essentially be an ω-restriction semigroups. The idea developed here will prove useful in the study of restriction semigroups. However, we will in this work retain the term type A semigroups generally.

2. The *-Simple Type A I-Semigroup: Construction

Consider a chain of cancellative monoids $T = \bigcup_{i=0}^{d-1} M_i$. Each element $x_i \in T$ is necessarily in M_i for $0 \le i \le d-1$. An identity $e_i \in M_i$ is an idempotent in T. Thus in $T, e_i \in T$ form a chain of idempotents $e_0 > e_i > \cdots > e_{d-1}$. Let $\theta : T \to M_0$ be a monoid morphism and let $S = T \times I \times I$ (where I is the set of all integers) be the set of all ordered triples (x_i, m, n) where $m \in \mathbb{N}^0$, $n \in I$, $0 \le i \le d-1$ and $x_i \in T$. Define multiplication on S by the rule

$$(x_i, m, n)(y_j, p, q) = \begin{cases} \left(x_i.f_{n-p,p}^{-1}.y_j\theta^{n-p}.f_{n-p,q}, m, n+q-p\right) & \text{if } n \ge p \\ \left(f_{p-n,m}^{-1}.x_i\theta^{p-n}.f_{p-n,n}.y_j, m+p-n, q\right) & \text{if } n \le p \end{cases}$$

where θ^0 is the identity automorphism of T, and for $m \in \mathbb{N}^0$, $n \in I$, $f_{0,n} = e_i$, the identity of M_i, while for $m > 0$, $f_{m,n} = u_{n+1}\theta^{m-1}.u_{n+2}\theta^{m-2}\ldots u_{n+(m-1)}\theta.u_{n+m}$, and $f_{m,n}^{-1} = u_{n+m}^{-1}.u_{n+(m-1)}^{-1}\theta \ldots u_{n+2}^{-1}\theta^{m-2}.u_{n+1}^{-1}\theta^{m-1}$, where $\{u_n : n \in I\}$ is a collection of T with $u_n = e_i$ for $n > 0$. A routine calculation shows that $S = T \times I \times I$ is a semigroup. This semigroup constructed will be called the generalized Bruck-Reilly *-extension of the semillatice of cancellative monoid T determined by θ and will be denoted by $S = GBR^*(T, \theta)$ where $T = \bigcup_{i=0}^{d-1} M_i$. If for each i we now let $M_i = \{e_i\}$, a monoid with one element, we obtain the set $I \times I$ under the multiplication

$$(md+i, nd+i)(pd+j, qd+j) = \begin{cases} (md+i, (n+q-p)d+i) & \text{if } n \ge p \\ ((m+p-n)d+j, qd+j) & \text{if } n \le p \end{cases}$$

We denote $I \times I$ under the above multiplication by B_d^* and call it the extended bicyclic semigroup. Now let (x_i, m, n) be an idempotent in S. Then

$$(x_i, m, n) = (x_i, m, n)(x_i, m, n) = \begin{cases} \left(x_i.f_{n-m,m}^{-1}.x_i\theta^{n-m}.f_{n-m,n}, m, n-m+n\right) & \text{if } n \ge m \\ \left(f_{m-n,m}^{-1}.x_i\theta^{m-n}.f_{m-n,n}.x_i, m-n+m, n\right) & \text{if } n \le m \end{cases}$$

in which case $m = n$, $x_i^2 = x_i$.

Conversely, suppose $x_i^2 = x_i$ then certainly $(x_i, m, n)(x_i, m, n) = (x_i, m, n)$. Thus (x_i, m, n) is an idempotent if and only if $m = n$ and x_i is an idempotent in S.

Lemma 2.1. *Let $S = GBR^*(T,\theta)$ be the generalized Bruck-Reilly *-extension of the semilattice of cancellative monoid $T = \bigcup_{i=0}^{d-1} M_i$. Let $(x_i,m,n),(y_j,p,q) \in S$. Then*

(1). $(x_i,m,n)\,\mathcal{R}^*\,(y_j,p,q)$ if and only if $m = p$ and $i = j$.

(2). $(x_i,m,n)\,\mathcal{L}^*\,(y_j,p,q)$ if and only if $n = q$ and $i = j$.

*(3). $(x_i,m,n)\,\mathcal{J}^*\,(y_j,p,q)$. That is S is *-simple.*

Proof.

(1). Suppose $(x_i,m,n),(y_j,p,q)$ are elements in S such that $(x_i,m,n)\,\mathcal{R}^*\,(y_j,p,q)$ where $x_i \in M_i$ and $y_j \in M_j$. Then there exists $(e_i,0,0),(e_i,m,m) \in S = GBR^*(T,\theta)$ such that

$$(e_0,0,0)(x_i,m,n) = (e_i,m,m)(x_i,m,n),$$
$$(e_0,0,0)(y_j,p,q) = (e_i,m,m)(y_j,p,q).$$

Consequently, we have that

$$\begin{aligned}(y_j,p,q) &= (e_i,m,m)(y_j,p,q)\\ &= \begin{cases}\left(e_i.y_j\theta^{m-p}, m, m+q-p\right) & \text{if } m \geq p\\ \left(e_i\theta^{p-m}.y_j, m+p-m, q\right) & \text{if } m \leq p\end{cases}\end{aligned}$$

If $m > p$, this gives $(y_j,p,q) = \left(e_i.y_j\theta^{m-p}, m, m+q-p\right)$. If we compare the middle coordinates, then $m = p$, which is a contradiction. Thus $m \leq p$. Similarly it can be shown that $p \leq m$, and from inequality follows $m = p$. Obviously $e_i \in M_i$, $y_j \in M_j$, thus $e_i.y_j \in M_{i,j}$. But $e_i.y_j = y_j$ implies $i \leq j$. Similarly, $e_j.x_i = x_i$ implies that $j \leq i$. As a result $m = p$ and $i = j$.

Conversely, let $m = p$ and $x_i, y_j \in M_i$. Then for any arbitrary elements $(v_i,c,z),(w_j,l,k) \in S = GBR^*(T,\theta)$, where $v_i, w_j \in M_i$,

$$(v_i,c,z)(x_i,m,n) = (w_j,l,k)(x_i,m,n).$$

If $z \geq m$ and $k \geq m$. Then

$$\left(v_i.f^{-1}_{z-m,m}.x_i\theta^{z-m}.f_{z-m,n}, c, z+n-m\right) = \left(w_j.f^{-1}_{k-m,m}.x_i\theta^{k-m}.f_{k-m,n}, l, k+n-m\right).$$

Comparing the first and the third coordinates we have

$$v_i.f^{-1}_{z-m,m}.x_i\theta^{z-m}.f_{z-m,n} = w_j.f^{-1}_{k-m,m}.x_i\theta^{k-m}.f_{k-m,n}, z+n-m = k+n-m$$

respectively. Consequently,

$$v_i.f^{-1}_{z-m,m}.y_j\theta^{z-m}.f_{z-m,q} = w_j.f^{-1}_{k-m,m}.y_j\theta^{k-m}.f_{k-m,q},\ z+n-m = k+n-m.$$

Hence, $(v_i,c,z)(y_j,m,q) = (w_j,l,k)(y_j,m,q)$. A similar argument shows that

$$(v_i,c,z)(y_j,m,q) = (w_j,l,k)(y_j,m,q) \Longrightarrow (v_i,c,z)(x_i,m,n) = (w_j,l,k)(x_i,m,n).$$

Thus $(x_i,m,n)\,\mathcal{R}^*\,(y_j,p,q)$.

(2). The proof is similar to that of (1).

(3). Let $(x_i, m, n), (y_j, p, q) \in S = GBR^*(T, \theta)$ where $x_i \in M_i$ and $y_j \in M_j$. Then

$$(e_j, p, m+1)(x_i, m, n) = (e_j.x_i\theta, p, n+1).$$

Obviously, $e_j.x_i\theta \in M_j$. Then $(e_j.x_i\theta, p, n+1)\,\mathcal{D}^*\,(y_j, p, q)$. In a similar way, we have

$$(e_i, m, p+1)(y_j, p, q) = (e_i.y_j\theta, m, q+1).$$

So $e_i.y_j\theta \in M_i$. Hence $(e_i.y_j\theta, m, q+1)\,\mathcal{D}^*\,(x_j, m, n)$. Thus $(x_i, m, n)\,\mathcal{J}^*\,(y_j, p, q)$. Then by Lemma 1.1, we conclude that S is *-simple. □

Lemma 2.2. *$S = GBR^*(T, \theta)$ is an adequate semigroup if and only if T is adequate.*

Proof. Let $S = GBR^*(T, \theta)$ be adequate. Suppose that $x_i \in T$, $(x_i, 0, 0)\mathcal{L}^*(e_i, m, m) \in S$. Thus, each $\mathcal{L}^*$-class contains an idempotent. Similarly, each $\mathcal{R}^*$-class contains an idempotent. Let e_i, e_j be idempotents in T. Then $(e_i, 0, 0)$ and $(e_j, 0, 0)$ are idempotents in S. Consequently,

$$(e_i, 0, 0)(e_j, 0, 0) = (e_ie_j, 0, 0)$$
$$(e_j, 0, 0)(e_i, 0, 0) = (e_je_i, 0, 0)$$

which implies that

$$\begin{aligned}(e_i, 0, 0)(e_j, 0, 0) = (e_ie_j, 0, 0) &= (e_j, 0, 0)(e_i, 0, 0)\\ &= (e_je_i, 0, 0)\end{aligned}$$

hence $e_ie_j = e_je_i$. Thus idempotents commute showing that T is adequate.

Conversely, let T be adequate, it follows from Lemma 2.1 that each $\mathcal{L}^*$-class and $\mathcal{R}^*$-class of $S = GBR^*(T, \theta)$ contain an idempotent element. Suppose (e_i, m, m) and (e_j, n, n) be any two idempotents in $S = GBR^*(T, \theta)$ where $e_i, e_j \in T$. Let $m > n$. Then we have

$$(e_i, m, m)(e_j, n, n) = (e_j, n, n)(e_i, m, m)$$

since $e_j\theta^{m-n} = e_i$, $f^{-1}_{m-m,n}f_{m-n,n} = e_i$ and $e_ie_j = e_je_i$. □

Lemma 2.3. *$S = GBR^*(T, \theta)$ is a type A semigroup if and only T is a type A semigroup.*

Proof. Let $S = GBR^*(T, \theta)$ be a type A semigroup. It follows from Lemma 2.2 that S is adequate and T is also adequate. Let $(x_i, 0, 0), (e_i, 0, 0) \in S$ where $x_i \in T$ and $e_i \in E(T)$. So we have that

$$(e_i, 0, 0)(x_i, 0, 0) = (e_ix_i, 0, 0),$$
$$(x_i, 0, 0)((e_i, 0, 0)(x_i, 0, 0))^* = \left(x_i(e_ix_i)^*, 0, 0\right).$$

Consequently, $(e_ix_i, 0, 0) = \left(x_i(e_ix_i)^*, 0, 0\right)$. Hence $e_ix_i = x_i(e_ix_i)^*$ which implies that T is right type A. That T is left type A follows similarly. Thus T is a type A semigroup.

Conversely, let T be a type A semigroup. We are to check that for $(x_i, p, q) \in S$, $(e_i, m, m) \in E(S)$,

$$(e_i, m, m)(x_i, p, q) = (x_i, p, q)((e_i, m, m)(x_i, p, q))^* \quad \text{(for right type A)}$$

$$(x_i, p, q)(e_i, m, m) = ((x_i, p, q)(e_i, m, m))^\dagger (x_i, p, q) \quad \text{(for left type A)}$$

Suppose $m \geq p$, we have that

$$(e_i, m, m)(x_i, p, q) = (e_i.f_{m-p,p}^{-1}.x_i?^{m-p}.f_{m-p,q}, m, m+q-p).$$

Consequently,

$$\begin{aligned}(x_i, p, q)((e_i, m, m)(x_i, p, q))^* &= \Big(f_{m+q-p-q,p}^{-1}.x_i\theta^{m+q-p-q}.f_{m+q-p-q,q}.\big(e_i.f_{m-p,p}^{-1}.x_i\theta^{m-p}.f_{m-p,q}\big)^*,\\ &\quad p+m+q-p-q, m+q-p)\\ &= \Big(f_{m-p,p}^{-1}.x_i\theta^{m-p}.f_{m-p,q}\big(e_i.f_{m-p,p}^{-1}.x_i\theta^{m-p}.f_{m-p,q}\big)^*, m, m+q-p\Big)\end{aligned}$$

Since T is type A, we have that

$$f_{m-p,p}^{-1}.x_i\theta^{m-p}.f_{m-p,q}.\big(e_i.f_{m-p,p}^{-1}.x_i\theta^{m-p}.f_{m-p,q}\big)^* = e_i.f_{m-p,p}^{-1}.x_i\theta^{m-p}.f_{m-p,q}.$$

Thus $(e_i, m, m)(x_i, p, q) = (x_i, p, q)((e_i, m, m)(x_i, p, q))^*$. Hence S is a right type A semigroup. That S is a left type A semigroup follows similarly. Therefore S is a type A semigroup. □

Theorem 2.4. *Let $S = GBR^*(T, \theta)$ be the generalized Bruck-Reilly *-extension of the semilattice of cancellative monoids $T = \bigcup_{i=0}^{d-1} M_i$. Then S is a *-simple type A I-semigroup with d $\mathcal{D}^*$-classes.*

Proof. Since $S = GBR^*(T, \theta)$ is a *-simple type A semigroup, we need to show that S is an I-semigroup. Let $(e_i, m, m), (e_j, n, n) \in E(S)$ where $m > n$. Then

$$(e_i, m, m)(e_j, n, n) = (e_j, n, n)(e_i, m, m)$$

because $(e_j\theta\theta m - n)$ is the identity of T. Thus $(e_i, m, n) < (e_j, n, n)$ if and only if $m > n$. On the other hand, if $m = n$ and $i \geq j$, then

$$(e_i, m, m)(e_j, m, m) = (e_ie_j, m, m) = (e_i, m, m).$$

Thus $(e_i, m, m) \leq (e_j, m, m)$ if and only if $e_i \leq e_j \in T$. This shows that $E(S)$ is a chain

$$\begin{aligned}&\cdots\\ &> (e_0, -1, -1) > (e_1, -1, -1) > \cdots > (e_{d-1}, -1, -1)\\ &> (e_0, 0, 0) > (e_1, 0, 0) > \cdots > (e_{d-1}, 0, 0)\\ &> (e_0, 1, 1) > (e_1, 1, 1) > \cdots > (e_{d-1}, 1, 1)\\ &> \cdots\end{aligned}$$

Hence S is a *-simple type A I-semigroup.

Finally, we show that S has d $\mathcal{D}^*$-classes. But $\mathcal{D}^* = \mathcal{L}^* \circ \mathcal{R}^*$. Let $(x_i, m, n)\mathcal{L}^*(z_k, p, q)\mathcal{R}^*(y_j, h, k)$. Then it follows that $n = q$, $p = h$ and $x_i\mathcal{L}^*(T)z_k$, $z_k\mathcal{R}^*(T)y_j$. If $x_i \in M_i$, $y_j \in M_j$ and $z_k \in M_k$, then it is evident that $i = j = k$, which shows that a $\mathcal{D}^*$-class of $S = GBR^*(T, \theta)$ is contained in $M_i \times I \times I$. Also, $(x_i, m, n)\mathcal{D}^*(y_j, p, q)$. Thus each $\mathcal{D}^*$-class of $S = GBR^*(T, \theta)$ equals $M_i \times I \times I$, $0 \leq i \leq d-1$ and the proof of the theorem is completed. □

3. The Structure Theorem

Let S denote a *-simple type A I-semigroup and let C^* be a *-ideal of S consisting of the $\mathcal{H}^*$-classes.

$$S = \bigcup_{(md+i,nd+i)\in B_d^*} H^*_{md+i,nd+i},$$

where d denotes the number of $\mathcal{D}^*$-classes of S. Observe that with respect to *-simple type A ω-semigroups, we have $B_d = \{(m,n) \ : \ \mathbb{N}^0 \times \mathbb{N}^0 \ : m \equiv n \ (\text{mod } d)\}$, the bicyclic semigroup. Let us put

$$B_d^* = \{(md+i, nd+i) \in I \times I \ : \ md+i \equiv nd+i \ (\text{mod } d)\},$$

the extended bicyclic semigroup. Put $T = \bigcup_{i=0}^{d-1} M_i$ where $M_i = H^*_{i,i}$, $i = 0,1,2,\ldots,d-1$. Then T is a finite chain of cancellative monoids, and the idempotents form a chain $e_0 > e_1 > \cdots > e_{d-1}$. Let $H^*_{i,i}$, $H^*_{j,j} \in T$, then we have that $H^*_{i,i}.H^*_{j,j} \subseteq H^*_{i,j}$. Define a map $f_{i,j} : H^*_{i,i} \to H^*_{j,j}$ by the rule $mf_{i,j} = e_j m$ where $i \leq j$ For all $m \in H^*_{i,i}$ and $e_j \in H^*_{j,j}$ we have that $me_j, e_j m \in H^*_{j,j}$ so that $me_j = e_j m$. Since $mn = m\varphi_{i,j}.n\varphi_{i,j}$, it follows that the maps are morphisms and they satisfy the following

(a). $\varphi_{i,i}$ is the identity map.

(b). $\varphi_{i,j}\varphi_{j,k} = \varphi_{i,k}$ for $k \leq j \leq i$.

The following Lemma establishes some important relationship

Lemma 3.1. *Let $a \in H^*_{0,d} \cap C^*$ then $a^{-1} \in H^*_{d,0}$ and $a^k a^{-k} = e_0, a^{-k} a^k = e_{kd}$.*

Proof. Since $(0,d)^{-1} = (d,0) \in B_d^* \cong C^*$, we have that $a^{-1} \in H^*_{d,0}$. Also we have that $a^2 = a.a \in H^*_{0,d} H^*_{0,d} \subseteq H^*_{0,2d}$, and more generally by induction we have that $a^k \in H^*_{0,kd}, a^{-k} \in H^*_{kd,0} \quad (k \in I)$. Now $a^k a^{-k} \in H^*_{0,kd} H^*_{kd,0} \subseteq H^*_{0,0}$ which implies that $a^k a^{-k} = e_0$, since $a^k a^{-k} \in C^*$. Similarly, we have $a^{-k} a^k \in H^*_{kd,0} H^*_{0,kd} \subseteq H^*_{kd,kd}$ which implies that $a^{-k} a^k = e_{kd}$. $\square$

Lemma 3.2. *Every element $z \in S$ can be uniquely written in the form: $z = x_i f^{-1}_{u,v} a^{-m} f_{u,w} a^n$ where $m,n,u,v,w \in I$, $x_i \in H^*_{i,i}$, $f^{-1}_{u,v} f_{u,w} = f_{u,w} f^{-1}_{u,v} = e_i \in T$.*

Proof. First we show that m,n,u,v,w are uniquely determined by the $\mathcal{H}^*$-class of z. By Lemma 3.1, we have that for any $m \in I$, $a^m \in H^*_{0,md}$, $a^{-m} \in H^*_{md,0}$, $a^m a^{-m} \in H^*_{md,md}$. Let $m,n \in I$ and $x_i \in H^*_{i,i}$, then we have

$$x_i f^{-1}_{u,v} a^{-m} f_{u,w} a^n \in H^*_{i,i} H^*_{i,i} H^*_{md,0} H^*_{i,i} H^*_{0,nd} \subseteq H^*_{md+i,nd+i}.$$

So if $z \in H^*_{k,l}$ and $z = x_i f^{-1}_{u,v} a^{-m} f_{u,w} a^n$ then $i \equiv k \equiv l \ (\text{mod } d)$ and $k = md+i$, $l = nd+i$. So $m = \frac{(k-i)}{d}$ and $n = \frac{(l-i)}{d}$. Hence we can define a map $f : H^*_{i,i} \to H^*_{md+i,nd+i}$ by the rule that $x_i f = x_i f^{-1}_{u,v} a^{-m} f_{u,w} a^n$. It is clear that f is injective and surjective. Consequently,

$$\begin{aligned} x_i f^{-1}_{u,v} a^{-m} f_{u,w} a^n &= \left(u f_{u,w} a^m f^{-1}_{u,v} a^{-n} \right) f^{-1}_{u,v} a^{-m} f_{u,w} a^n \\ &= u a^{-m} a^m a^{-n} a^n \\ &= u e_{md} e_{nd} \\ &= u e_{md} e_{md+i} e_{nd+i} e_{nd} \end{aligned}$$

$$= u.$$

Hence φ is a bijection from $H^*_{i,i} \to H^*_{md+i,nd+i}$ with the inverse map given by

$$v \longmapsto va^m a^{-n} \qquad (v \in H^*_{md+i,nd+i})$$

showing that z can be uniquely as $x_i f^{-1}_{u,v} a^{-m} f_{u,w} a^n$ where $x_i \in H^*_{i,i}$. □

Lemma 3.3. *For any $x_i f^{-1}_{u,v} \in T$ there exists a unique $x^*_i f^{-1}_{u,v} \in H^*_{0,0}$ such that $a x_i f^{-1}_{u,v} = x^*_i f^{-1}_{u,v} a$, $x_i f^{-1}_{u,v} a^{-1} = a^{-1} x^*_i f^{-1}_{u,v}$. Let $\theta : T \to H^*_{0,0}$, $x_i f^{-1}_{u,v} \longmapsto ax^*$. Then $f^{-1}_{u,v} a^k x_i = f^{-1}_{u,v} \left(x_i \theta^k\right) a^k$, $x_i f^{-1}_{u,v} a^{-k} = f^{-1}_{u,v} a^{-k} \left(x_i \theta^k\right)$ and θ is a monoid morphism.*

Proof. Suppose $x^*_i = a x_i f^{-1}_{u,v} a^{-1}$. Then we have that

$$x^*_i = a x_i f^{-1}_{u,v} a^{-1} \in H^*_{0,d} H^*_{i,i} H^*_{d,0} \subseteq H^*_{0,0}.$$

Consequently, we have

$$x^*_i f^{-1}_{u,v} a = a x_i f^{-1}_{u,v} a^{-1} a = a x_i f^{-1}_{u,v}.$$

That $x_i f^{-1}_{u,v} a^{-1} = a^{-1} x^*_i f^{-1}_{u,v}$ follows similarly. Now let us define $\theta : T \to H^*_{0,0}$, $x_i f^{-1}_{u,v} \longmapsto a x^*_i$. Then for $x_i, y_i \in T$ we have that

$$\begin{aligned} \left(x_i f^{-1}_{u,v} y_i f^{-1}_{u,v}\right) \theta &= a \left(x_i f^{-1}_{u,v} y_i f^{-1}_{u,v}\right) a^{-1} \qquad \left(x^*_i = \left(x_i f^{-1}_{u,v}\right) \theta = a x_i f^{-1}_{u,v} a^{-1} \subseteq H^*_{0,0}\right) \\ &= a x_i f^{-1}_{u,v} e_d y_i f^{-1}_{u,v} a^{-1} \\ &= \left(x_i f^{-1}_{u,v}\right) \theta \left(y_i f^{-1}_{u,v}\right) \theta. \end{aligned}$$

Thus θ is a morphism. Also it can be easily seen that $f^{-1}_{u,v} \left(x_i \theta^k\right) a^k = f^{-1}_{u,v} a^k x_i$. That $x_i f^{-1}_{u,v} a^{-k} = f^{-1}_{u,v} a^{-k} \left(x_i \theta^k\right)$ follows similarly. □

Lemma 3.4. *Let $u_n \in T$, $m \in \mathbb{N}^0$, $n \in I$ and let $\theta : T \to T$ where $T = \bigcup_{i=0}^{d-1} H^*_{i,i}$. Then for $m > 0$, $f_{m,n} = u_{n+1}\theta^{m-1}.u_{n+2}\theta^{m-2} \ldots u_{n+(m-1)}\theta^{m-(m-1)}.u_{n+m}\theta^{m-m}$, where $f_{0,n} = e_0$ is the identity of T.*

Proof. Now since $u_n \in T$, then $u_{n+i} \in T$. Obviously, $u_n\theta$, $u_{n+i}\theta^{m-i} \in T$ for $m \in \mathbb{N}^0$ and $i = 0, 1, \ldots d-1$ where d is a positive integer. For $i = 0$, we have $u_n\theta^m$ while for $i = 1$, we have $u_{n+1}\theta^{m-1}$ and subsequently $\ldots i = d-1$, we have $u_{n+(d-1)}\theta^{m-(d-1)}$. For $i = m$, we have $u_{n+m}\theta^{m-m} = u_{n+m}$. Now if we let $f_{m,n}$ be the collection of the images of T and $m > 0$, we obtain the desired result. □

We will now prove the structure theorem for *-simple type A I-semigroups.

Theorem 3.5. *Let S be a *-simple type A I-semigroup with d $\mathcal{D}^*$-classes. Then S is isomorphic to a generalized Bruck-Reilly *-extension $S = GBR^*(T, \theta)$ of a monoid T, where $T = \bigcup_{i=0}^{d-1} H^*_{i,i}$ is a finite chain of cancellative monoids M_i and θ is an endomorphism of T with image in M_0.*

Proof. Let S be a *-simple type A I-semigroup. From Lemma 3.2, every element of S has a unique expression in the form $x_i f^{-1}_{u,v} a^{-m} f_{u,w} a^n$ for $x_i \in H^*_{i,i}$, $u, v, w \in I$ and a is a fixed regular element in $H^*_{0,d}$ and $T = \bigcup_{i=0}^{d-1} H^*_{i,i}$. Thus we can define a bijection $\psi : S \to T \times I \times I$ by the rule that

$$\left(x_i f^{-1}_{u,v} a^{-m} f_{u,w} a^n\right) \psi = (x_i, m, n).$$

From Lemma 3.3, we have that for any $x_i \in T$, there exists a unique $x_i^* f_{u,v}^{-1} \in H_{0,0}^*$ such that $ax_i f_{u,v}^{-1} = x_i^* f_{u,v}^{-1} a$, $x_i f_{u,v}^{-1} a^{-1} = a^{-1} x_i^* f_{u,v}^{-1}$. Let us define $\theta : T \to H_{0,0}^*$ as $x_i f_{u,v}^{-1} \longmapsto ax_i^*$. Thus θ is a monoid morphism, and for all $k \in I$ clearly $f_{u,v}^{-1} a^k x_i = f_{u,v}^{-1}(x_i\theta^k)a^k$ and $x_i f_{u,v}^{-1} a^{-k} = f_{u,v}^{-1} a^{-k}(x_i\theta^k)$. Let $x_i f_{u,v}^{-1} a^{-m} f_{u,w} a^n$, $y_j f_{u,v}^{-1} a^{-m} f_{u,w} a^n \in S$ where $x_i, y_j \in T$. We consider the following cases;

Case 1: If $n \geq p$, we have that

$$\begin{aligned}
\left(x_i f_{u,v}^{-1} a^{-m} f_{u,w} a^n\right)\left(y_j f_{u,v}^{-1} a^{-m} f_{u,w} a^n\right) &= x_i f_{u,v}^{-1} a^{n-p} y_j f_{u,w} a^{-m} a^q \\
&= x_i f_{u,v}^{-1}(y_j\theta^{n-p}) a^{n-p} f_{u,w} a^{-m} a^q \quad (\text{since } \ f_{u,v}^{-1} a^k x_i = f_{u,v}^{-1}(x_i\theta^k)a^k) \\
&= x_i f_{u,v}^{-1} y_j \theta^{n-p} f_{u,w} a^{-m} a^{n+q-p} \\
&= x_i f_{n-p,p}^{-1} y_j \theta^{n-p} f_{n-p,q} a^{-m} a^{n+q-p}
\end{aligned}$$

where $u, v = n-p, p$ and $u, w = n-p, q$ (since $n-p, p, q \in I$).

Case 2: If $n \leq p$, we have that

$$\begin{aligned}
\left(x_i f_{u,v}^{-1} a^{-m} f_{u,w} a^n\right)\left(y_j f_{u,v}^{-1} a^{-m} f_{u,w} a^n\right) &= x_i f_{u,v}^{-1} a^{-(p-n)} y_j f_{u,w} a^{-m} a^q \\
&= f_{u,v}^{-1} a^{-(p-n)}(x_i\theta^{p-n}) y_j f_{u,w} a^{-m} a^q \quad (\text{since } \ x_i f_{u,v}^{-1} a^{-k} = f_{u,v}^{-1} a^{-k}(x_i\theta^k)) \\
&= f_{u,v}^{-1} x_i \theta^{p-n} f_{u,w} y_j a^{-(m+p-n)} a^q \\
&= f_{p-n,m}^{-1} x_i \theta^{p-n} f_{p-n,n} y_j a^{-(m+p-n)} a^q
\end{aligned}$$

where $u, v = p-n, m$ and $u, w = p-n, n$ (since $p-n, m, n \in I$). Thus the mapping $\psi : S \to T \times I \times I$ defined by the rule that

$$\left(x_i f_{u,v}^{-1} a^{-m} f_{u,w} a^n\right)\psi = (x_i, m, n)$$

is an isomorphism. This completes the proof. □

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