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Some Common Fixed Point Theorems of Contractive Mappings in Cone b-metric Spaces

Research Article

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Abstract: In this paper, we prove some common fixed point theorems of contraction mappings in cone b-metric spaces.

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1. Introduction

Fixed point theory plays a basic role in application of many branches of mathematics. Finding a fixed point of contractive mapping becomes the center of strong research activity. There are many works about the fixed point of contractive maps (see, for example, [1, 2]). In [2] Polish Mathematician Banach proved a very important result regarding a contraction mapping, known as the Banach contraction principle, in 1922. In [3], Bakhtin introduced b-metric spaces as a generalized of metric spaces. He proved the contraction mapping principle in b-metric spaces that generalized the famous Banach contraction principle in metric spaces. A lucid survey shows that there are many generalizations of metric spaces. One of them is b-metric space. The concept of b-metric space was introduced by Czerwik [11, 12]. Using this idea, he proved Banach'fixed point theorem in b-metric spaces. Later on, many researchers including Aydi [8], Bota [9], Chug [10], Shi [20], Du [13], Kir [19], Huang and Zhang [4] introduced cone metric spaces as generalized of metric spaces, replacing the real number by an ordered Banach spaces and define cone metric spaces. Moreover, they proved some fixed point theorems for contraction mapping that expanded certain results of fixed point in metric spaces. In [5], Hussain and Shah introduced cone b-metric spaces as a generalized of b-metric spaces and cone metric spaces. We prove some fixed point theorems in cone b-metric spaces on contraction mapping.

Before going to the main results, we define some definition, example and lemma required in sequel.

2. Preliminaries

Let E be real Banach space and P be a subset of E. Then P is called a cone if

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(1). P is closed, nonempty, and satisfies $P \neq \{\theta\}$,

(2). $ax + by \in P$ for all $x, y \in P$ and non-negative real number a, b,

(3). $x \in P$ and $x \in P \Rightarrow x = \theta$, i.e., $P \cap (-P) = \theta$, where θ denote the zero elements of E and by int P the interior of P.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but$x \neq y$, while $x \ll y$ will stand for $y - x \in int\ P$. A cone P is a solid cone if $int\ P \neq \emptyset$. We write $\|.\|$ as the norm on E. The cone P is called normal if there is a number $k > 0$ such that $\forall\ x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| = k\|y\|$. The least positive number k satisfying the above is called the normal constant of P. It is well known that $k \geq 1$.
In the following, we always suppose that E is a Banach space, P is cone in E with $int\ P \neq \emptyset$ and $\leq$ is a partial ordering with respect to P.

Definition 2.1 ([4]). *Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ Satisfies:*

(CM1) $\theta \leq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = \theta$ *if and only if* $x = y$,

(CM2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$,

(CM3) $d(x, y) = d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.2 ([5]). *Let X be a nonempty set and $s \geq 1$ be a real number. Suppose that the mapping $d : X \times X \to E$ Satisfies:*

(CBM1) $\theta \leq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = \theta$ *if and only if* $x = y$,

(CBM2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$,

(CBM3) $d(x, y) = s[d(x, z) + d(z, y)]$ *for all* $x, y, z \in X$.

Then d is called a cone b-metric on X and (X, d) is called a cone b-metric space.

Remark 2.3. *The class of cone b-metric spaces is larger than the class of cone metric spaces since any cone metric space must be a cone b-metric space. Therefore, it is obvious that cone b-metric space generalized b-metric spaces and cone metric spaces.*

Following is an example which shows that a cone b-metric spaces which are not cone metric spaces:

Example 2.4 ([7]). $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subset E$, $X = R$ *and* $d : X \times X \to E$ *such that* $d(x, y) = (|x - y|^p, \alpha|x - y|^p)$ *where* $\alpha \geq 0$ *and* $p > 1$ *are two constant. Then (X, d) is a cone b-metric space but not a cone metric space. In fact, we only need to prove (iii) in Definition 2.2 as follows: Let* $x, y, z \in X$. *Set* $u = x - z$, $v = z - y$, *so* $x - y = u + v$ *from the inequality* $(a + b)^p = (2\max\{a, b\})^p \leq 2^p(a^p + b^p)$ *for all* $a, b \geq 0$, *we have*

$$|x - y|^p = |u + v|^p \leq (|u| + |v|)^P = 2^p(|u|^p + |v|^p) = 2^p(|x - z|^p + |z - y|^p),$$

$$\Rightarrow d(x, y) = s[d(x, z) + d(z, y)]$$

with $s = 2^p > 1$. *But* $|x - y|^p = |x - z|^p + |z - y|^p$ *is impossible for all* $x > z > y$, *indeed, taking account of the inequality.*

$$(a + b)^p > a^p + b^p \ \ for\ all\ a, b > 0,$$

We arrive at

$$|x-y|^p = |u+v|^p = (u+v)^p > u^p + v^p = (x-z)^p + (z-y)^p = |x-z|^p + |z-y|^p,$$

for all $x > z > y$. *Thus, (CM3) in Definition 2.1 is not satisfied, ie., (X, d) is not a cone metric space.*

Example 2.5 ([7]). *Let* $X = l^p$ *with* $0 < p < 1$, *where* $l^p = \{\{x_n\} \subset R: \sum_{n=1}^{\infty} |x_n|^p < \infty\}$. *Let* $d: X \times X \to R^+$,

$$d(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{\frac{1}{p}}, \quad \text{where } x = \{x_n\},\ y = \{y_n\} \in l^p.$$

Than (x, d) is a b-metric space (see [5]). Put $E = l^1$, $P = \{\{x_n\} \in E: \ x_n \geq 0, \ \text{for all } n \geq 1\}$. *Letting the mapping* $d: X \times X \to E$ *be defined by* $d(x,y) = \left\{\frac{d(x,y)}{2^n}\right\}_{n\geq 1}$, *we conclude that (X, d) is a cone b-metric with coefficient* $s = 2^{\frac{1}{p}} > 1$, *but it is not a cone metric space.*

Example 2.6 ([7]). *Let* $X = \{1,2,3,4\}$, $E = R^2$, $P = \{(x,y) \in E: x \geq 0, y \geq 0\}$. *Define* $d: X \times X \to E$ *by*

$$d(x,y) = \begin{cases} \left(|x-y|^{-1}, |x-y|^{-1}\right), & \text{if } x \neq y; \\ \theta, & \text{if } x = y. \end{cases}$$

Then (X, d) is a cone b-metric space with the coefficient $s = \frac{6}{5}$. *But it is not a cone metric space since the triangle inequality is not satisfied. Indeed,* $d(1,2) > d(1,4) + d(4,2)$ *and* $d(3,4) > d(3,1) + d(1,4)$.

Definition 2.7 ([5]). *Let (X, d) be a cone b-metric space,* $x \in X$ *and* $\{x_n\}$ *be a sequence in X. Than*

1. $\{x_n\}$ *converges to* x *whenever, for every* $c \in E$ *with* $\theta \ll c$, *there is a natural number* N *such that* $d(x_n, x) \ll c$ *for all* $n \geq N$. *We denote this by* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x$ $(n \to \infty)$.

2. $\{x_n\}$ *is a Cauchy sequence whenever,for every* $c \in E$ *with* $\theta \ll c$, *there is a natural number* N *such that* $d(x_n, x_m) \ll c$ *for all* $n, m \geq N$.

3. (X, d) *is a complete cone b-metric space if every Cauchy sequence is convergent.*

Lemma 2.8 ([6]). *Let P be a cone and* $\{x_n\}$ *be a sequence in E. If* $c \in int\ P$ *and* $\theta \leq x_n \to \theta\,(as\ n \to \infty)$, *then there exist* N *such that for all* $n > N$, *we have* $x_n \ll c$.

Lemma 2.9 ([6]). *Let* $x, y, z \in E$, *if* $x \leq y$ *and* $y \leq z$, *then* $x \leq z$.

Lemma 2.10 ([5]). *Let P be a cone and* $\theta \leq u \ll c$ *for each* $c \in int\ P$, *then* $u = \theta$.

Lemma 2.11 ([21]). *Let P be a cone. If* $u \in P$ *and* $u \leq ku$ *for some* $0 \leq k < 1$, *then* $u = \theta$.

Lemma 2.12 ([6]). *Let P be a cone and* $a = b + c$ *for each* $c \in int\ P$, *then* $a \leq b$.

3. Main Results

In 2013, Huaping Huang and Shaoyuan Xu [7] proved the following theorem: Let (X,d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose the mapping $T: X \times X \to X$ satisfies the condition $d(Tx, Ty) \leq \alpha d(x,y)$, for all $x, y \in X$. Where $\alpha \in [0,1)$ is a constant. Then T has a unique fixed point in X. Furthermore, the iterative sequence $\{T^n x\}$ converges to the fixed point. Now we expand this theorem in cone b-metric space as.

Theorem 3.1. *Let (X,d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose the mapping $T : X \times X \to X$ satisfies the contractive condition.*

$$d(Tx, Ty) \leq \alpha_1 d(x, y) + \alpha_2 d(x, Tx) + \alpha_3 d(y, Ty) + \alpha_4 d(x, Ty) + \alpha_5 d(y, Tx) \tag{1}$$

where the constant $\alpha_i \in [0, 1)$; $i = 1, 2, 3, 4, 5$ and $\alpha_1 + \alpha_2 + \alpha_3 + s(\alpha_4 + \alpha_5) < \min\{1, \frac{2}{s}\}$. Then T has a unique fixed point in X. Moreover, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Proof. Fix $x_0 \in X$ and set $x_1 = Tx_0$ and $x_{n+1} = Tx_n = T^{n+1}x_0$. Firstly, we see

$$\begin{aligned}
d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\
d(x_{n+1}, x_n) &\leq \alpha_1 d(x_n, x_{n-1}) + \alpha_2 d(x_n, Tx_n) + \alpha_3 d(x_{n-1}, Tx_{n-1}) + \alpha_4 d(x_n, Tx_{n-1}) + \alpha_5 d(x_{n-1}, Tx_n) \\
&\leq \alpha_1 d(x_n, x_{n-1}) + \alpha_2 d(x_n, x_{n+1}) + \alpha_3 d(x_{n-1}, x_n) + \alpha_4 d(x_n, x_n) + \alpha_5 d(x_{n-1}, x_{n+1}) \\
&\leq \alpha_1 d(x_n, x_{n-1}) + \alpha_2 d(x_n, x_{n+1}) + \alpha_3 d(x_{n-1}, x_n) + s\alpha_5 [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\
&\leq \alpha_1 d(x_n, x_{n-1}) + \alpha_2 d(x_n, x_{n+1}) + \alpha_3 d(x_{n-1}, x_n) + s\alpha_5 d(x_{n-1}, x_n) + s\alpha_5 d(x_n, x_{n+1}) \\
&\leq (\alpha_2 + s\alpha_5) d(x_{n+1}, x_n) + (\alpha_1 + \alpha_3 + s\alpha_5) d(x_n, x_{n-1})
\end{aligned}$$

It follows that

$$(1 - \alpha_2 - s\alpha_5) d(x_{n+1}, x_n) \leq (\alpha_1 + \alpha_3 + s\alpha_5) d(x_n, x_{n-1}) \tag{2}$$

Secondly,

$$\begin{aligned}
d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) = d(Tx_{n-1}, Tx_n) \\
d(x_{n+1}, x_n) &\leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 d(x_{n-1}, Tx_{n-1}) + \alpha_3 d(x_n, Tx_n) + \alpha_4 d(x_{n-1}, Tx_n) + \alpha_5 d(x_n, Tx_{n-1}) \\
&\leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 d(x_{n-1}, x_n) + \alpha_3 d(x_n, x_{n+1}) + \alpha_4 d(x_{n-1}, x_{n+1}) + \alpha_5 d(x_n, x_n) \\
&\leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 d(x_{n-1}, x_n) + \alpha_3 d(x_n, x_{n+1}) + s\alpha_4 [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\
&\leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 d(x_{n-1}, x_n) + \alpha_3 d(x_n, x_{n+1}) + s\alpha_4 d(x_{n-1}, x_n) + s\alpha_4 d(x_n, x_{n+1}) \\
&= (\alpha_3 + s\alpha_4) d(x_{n+1}, x_n) + (\alpha_1 + \alpha_2 + s\alpha_4) d(x_n, x_{n-1})
\end{aligned}$$

This establishes that

$$(1 - \alpha_3 - s\alpha_4) d(x_{n+1}, x_n) \leq (\alpha_1 + \alpha_2 + s\alpha_4) d(x_n, x_{n-1}) \tag{3}$$

Adding up (2) and (3)

$$d(x_{n+1}, x_n) \leq \frac{2\alpha_1 + \alpha_2 + \alpha_3 + s(\alpha_4 + \alpha_5)}{2 - \alpha_2 - \alpha_3 - s(\alpha_4 + \alpha_5)} d(x_n, x_{n-1})$$

Put $\alpha = \frac{2\alpha_1 + \alpha_2 + \alpha_3 + s(\alpha_4 + \alpha_5)}{2 - \alpha_2 - \alpha_3 - s(\alpha_4 + \alpha_5)}$ it is easy to see that $0 \leq \alpha < 1$. Now, proceeding in the same manner up to n iterations, we have: $d(x_{n+1}, x_n) \leq \alpha d(x_n, x_{n-1}) \leq \alpha^2 d(x_{n-1}, x_{n-2}) \leq \cdots \leq \alpha^n d(x_1, x_0)$ for $n \geq 1$ and letting $m \geq n$, we have:

$$\begin{aligned}
d(x_{n+m}, x_n) &\leq s[d(x_{n+m}, x_{n+m-1}) + d(x_{n+m-1}, x_n)] \\
&= sd(x_{n+m}, x_{n+m-1}) + sd(x_{n+m-1}, x_n) \\
&\leq sd(x_{n+m}, x_{n+m-1}) + s^2[d(x_{n+m-1}, x_{n+m-2}) + d(x_{n+m-2}, x_n)] \\
&\leq sd(x_{n+m}, x_{n+m-1}) + s^2 d(x_{n+m-1}, x_{n+m-2}) + s^2 d(x_{n+m-2}, x_n) \\
&\leq sd(x_{n+m}, x_{n+m-1}) + s^2 d(x_{n+m-1}, x_{n+m-2}) + s^3[d(x_{n+m-2}, x_{n+m-3}) + d(x_{n+m-3}, x_n)
\end{aligned}$$

$$\begin{aligned}
d(x_{n+m},x_n) &\le sd(x_{n+m},x_{n+m-1}) + s^2 d(x_{n+m-1},x_{n+m-2}) + \cdots + s^{m-1}d(x_{n+2},x_{n+1}) + s^{m-1}d(x_{n+1},x_n)\\
&\le s\alpha^{n+m-1}d(x_1,x_0) + s^2\alpha^{n+m-2}d(x_1,x_0) + s^3\alpha^{n+m-3}d(x_1,x_0) + \cdots + s^{m-1}\alpha^{n+1}d(x_1,x_0) + s^{m-1}\alpha^n d(x_1,x_0)\\
&= \left(s\alpha^{n+m-1} + s^2\alpha^{n+m-2} + \cdots + s^{m-1}\alpha^{n+1}\right)d(x_1,x_0) + s^{m-1}\alpha^n d(x_1,x_0)\\
&= s\alpha^{n+m}\frac{\left[\left(s\alpha^{-1}\right)^{m-1} - 1\right]d(x_1,x_0)}{s-\alpha} + s^{m-1}\alpha^n d(x_1,x_0)\\
&\le \frac{s^m\alpha^{n+1}}{s-\alpha}d(x_1,x_0) + s^{m-1}\alpha^n d(x_1,x_0)
\end{aligned}$$

Let $\theta \ll c$ be given, notice that $\frac{s^m\alpha^{n+1}}{s-\alpha}d(x_1,x_0) + s^{m-1}\alpha^n d(x_1,x_0) \to \theta$ as $m\to\infty$ for any k. Making full use of Lemma 2.8, we find $n_0 \in N$ such that

$$\frac{s^m\alpha^{n+1}}{s-\alpha}d(x_1,x_0) + s^{m-1}\alpha^n d(x_1,x_0) \ll c$$

$d\left(x_{n+m},x_n\right) \le \frac{s^m\alpha^{n+1}}{s-\alpha}d\left(x_1,x_0\right) + s^{m-1}\alpha^n d\left(x_1,x_0\right)\alpha \ll c$ for all $n > n_0$ and any m. So by Lemma 2.9. Therefore $\{x_n\}$ is a Cauchy sequence in (X,d). Since (X,d) is a complete cone-b metric space, there exists $x^* \in X$ such that $x_n \to x^*$ as $n\to\infty$ for some $x^*\in X$. Now we prove that x^* is the unique fixed point of T. Actually, on the one hand.

$$\begin{aligned}
d\left(Tx^*,x^*\right) &\le s\left[d\left(Tx^*,Tx_n\right) + d\left(Tx_n,x^*\right)\right]\\
&\le sd\left(Tx^*,Tx_n\right) + sd\left(Tx_n,x^*\right)\\
&\le sd\left(Tx^*,Tx_n\right) + sd\left(x_{n+1},x^*\right)\\
&\le s\left[\alpha_1 d\left(x^*,x_n\right) + \alpha_2 d\left(x^*,Tx^*\right) + \alpha_3 d\left(x_n,Tx_n\right) + \alpha_4 d\left(x^*,Tx_n\right) + \alpha_5 d\left(x_n,Tx^*\right)\right] + s\left(x_{n+1},x^*\right)\\
&\le s\left[\alpha_1 d\left(x^*,x_n\right) + \alpha_2 d\left(x^*,Tx^*\right) + \alpha_3 d\left(x_n + x_{n+1}\right) + \alpha_4 d\left(x^*,x_{n+1}\right) + \alpha_5 d\left(x_n,Tx^*\right)\right] + sd\left(x_{n+1},x^*\right)\\
&\le s\left[\alpha_1 d\left(x^*,x_n\right) + \alpha_2 d\left(x^*,Tx^*\right) + s\alpha_3 d\left(x_n,x^*\right) + s\alpha_3 d\left(x^*,x_{n+1}\right) + \alpha_4 d\left(x^*,x_{n+1}\right)\right.\\
&\quad \left. + s\alpha_5 d\left(x_n,x^*\right) + s\alpha_5 d\left(x^*,Tx^*\right)\right] + sd\left(x_{n+1},x^*\right)\\
&= s\alpha_1 d\left(x^*,x_n\right) + s\alpha_2 d\left(x^*,Tx^*\right) + s^2\alpha_3 d\left(x_n,x^*\right) + s^2\alpha_3 d\left(x^*,x_{n+1}\right) + s\alpha_4 d\left(x^*,x_{n+1}\right) + s^2\alpha_5 d\left(x_n,x^*\right)\\
&\quad + s^2\alpha_5 d\left(x^*,Tx^*\right) + sd\left(x_{n+1},x^*\right)d\left(Tx^*,x^*\right)\\
&\le \left(s\alpha_2 + s^2\alpha_5\right)d\left(x^*,Tx^*\right) + \left(s\alpha_1 + s^2\alpha_3 + s^2\alpha_5\right)d\left(x_n,x^*\right) + \left(s^2\alpha_3 + s\alpha_4 + s\right)d\left(x^*,x_{n+1}\right)
\end{aligned}$$

Such implies that

$$\left(1 - s\alpha_2 - s^2\alpha_5\right)d\left(x^*,Tx^*\right) \le \left(s\alpha_1 + s^2\alpha_3 + s^2\alpha_5\right)d\left(x_n,x^*\right) + \left(s^2\alpha_3 + s\alpha_4 + s\right)d\left(x^*,x_{n+1}\right) \tag{4}$$

On the other hand

$$\begin{aligned}
d\left(x^*,Tx^*\right) &\le s\left[d\left(x^*,Tx_n\right) + d\left(Tx_n,Tx^*\right)\right]\\
&\le sd\left(x^*,x_{n+1}\right) + sd\left(Tx_n,Tx^*\right)\\
&\le sd\left(x^*,x_{n+1}\right) + s\left[\alpha_1 d\left(x_n,x^*\right) + \alpha_2 d\left(x_n,Tx_n\right) + \alpha_3 d\left(x^*,Tx^*\right) + \alpha_4 d\left(x_n,Tx^*\right) + \alpha_5 d\left(x^*,Tx_n\right)\right]\\
&= sd\left(x^*,x_{n+1}\right) + s\left[\alpha_1 d\left(x_n,x^*\right) + \alpha_2 d\left(x_n,x_{n+1}\right) + \alpha_3 d\left(x^*,Tx^*\right) + \alpha_4 d\left(x_n,Tx^*\right) + \alpha_5 d\left(x^*,x_{n+1}\right)\right]\\
&\le sd\left(x^*,x_{n+1}\right) + s\left[\alpha_1 d\left(x_n,x^*\right) + s\alpha_1 d\left(x_n,x^*\right) + s\alpha_2 d\left(x^*,x_{n+1}\right) + \alpha_3 d\left(x^*,Tx^*\right)\right.\\
&\quad \left. + s\alpha_4 d\left(x_n,x^*\right) + s\alpha_4 d\left(x^*,Tx^*\right) + \alpha_5 d\left(x^*,x_{n+1}\right)\right]\\
d\left(x^*,Tx^*\right) &\le sd\left(x^*,x_{n+1}\right) + s\alpha_1 d\left(x_n,x^*\right) + s^2\alpha_2 d\left(x_n,x^*\right) + s^2\alpha_2 d\left(x^*,x_{n+1}\right) + s\alpha_3 d\left(x^*,Tx^*\right) + s^2\alpha_4 d\left(x_n,x^*\right)\\
&\quad + s^2\alpha_4 d\left(x^*,Tx^*\right) + s\alpha_5 d\left(x^*,x_{n+1}\right)\\
d\left(x^*,Tx^*\right) &\le \left(s\alpha_3 + s^2\alpha_4\right)d\left(x^*,Tx^*\right) + \left(s\alpha_1 + s^2\alpha_2 + s^2\alpha_4\right)d\left(x_n,x^*\right) + \left(s + s^2\alpha_2 + s\alpha_5\right)d\left(x^*,x_{n+1}\right)
\end{aligned}$$

$$\left(1 - s\alpha_3 - \alpha_4 s^2\right) d\left(x^*, Tx^*\right) \leq \left(s\alpha_1 + s^2\alpha_2 + s^2\alpha_4\right) d\left(x_n, x^*\right) + \left(s + s^2\alpha_2 + s\alpha_5\right) d\left(x^*, x_{n+1}\right) \tag{5}$$

Adding (4) and (5), we get

$$d\left(x^*, Tx^*\right) \leq \frac{\left(2s\alpha_1 + s^2\alpha_2 + s^2\alpha_3 + s^2\alpha_4 + s^2\alpha_5\right)}{2 - s\alpha_2 - s\alpha_3 - s^2\alpha_4 - s^2\alpha_5} d\left(x_n, x^*\right) + \frac{\left(2s + s^2\alpha_2 + s^2\alpha_3 + s\alpha_5\right)}{2 - s\alpha_2 - s\alpha_3 - s^2\alpha_4 - s^2\alpha_5} d\left(x^*, x_{n+1}\right)$$

Since $\{x_n\}$ is a Cauchy sequence, there for every $c \in int\, P$, we select an $n_i \in N$ for all $n \geq n_i$. Such that

$$d\left(x^*, x_{n+1}\right) \ll \frac{\left(2s + s^2\alpha_2 + s^2\alpha_3 + s\alpha_5\right) c}{2 - s\alpha_2 - s\alpha_3 - s^2\alpha_4 - s^2\alpha_5}$$
$$d\left(x_n, x^*\right) \ll \frac{\left(2s\alpha_1 + s^2\alpha_2 + s^2\alpha_3 + s^2\alpha_4 + s^2\alpha_5\right) c}{2 - s\alpha_2 - s\alpha_3 - s^2\alpha_4 - s^2\alpha_5}$$

This for any $c \in int\ P$ $d\left(x^*, Tx^*\right) \ll c \quad \forall\ n \geq n_i$. By Lemma 2.10 that $d\left(x^*, Tx^*\right) = \theta$ i.e. x^* is fixed point of T i.e. $x^* = Tx^*$. Finally, we show the uniqueness of the fixed point. Indeed if there is another fixed point y^* then.

$$\begin{aligned}
d\left(x^*, y^*\right) &= d\left(Tx^*, Ty^*\right)\\
&\leq \alpha_1 d\left(x^* y^*\right) + \alpha_2 d\left(x^* Tx^*\right) + \alpha_3 d\left(y^*, Ty^*\right) + \alpha_4 d\left(x^*, Ty^*\right) + \alpha_5 d\left(y^*, Tx^*\right)\\
&\leq \alpha_1 d\left(x^*, y^*\right) + s\alpha_4 d\left(x^*, y^*\right) + s\alpha_4 d\left(y^*, Ty^*\right) + s\alpha_5 d\left(y^*, x^*\right) + s\alpha_5 d\left(x^*, Tx^*\right)\\
&\leq \alpha_1 d\left(x^*, y^*\right) + s\alpha_4 d\left(x^* y^*\right) + s\alpha_5\, d\left(y^*, x^*\right)\\
&\leq \left\{\alpha_1 + s\left(\alpha_4 + \alpha_5\right)\right\} d\left(x^*, y^*\right)
\end{aligned}$$

Owing to $0 \leq \alpha_1 + s\left(\alpha_4 + \alpha_5\right) < 1$. We deduce from Lemma 2.11 that $x^* = y^*$. □

Corollary 3.2. *Let (X, d) be complete cone b- metric space with the coefficient $s \geq 1$. Suppose the mapping $T : X \to X$ satisfies the contractive condition.*

$$d\left(Tx, Ty\right) \leq \alpha_1 d\left(x, y\right) + \alpha_2 d\left(x, Tx\right) + \alpha_3 d\left(y, Ty\right) + \alpha_4 \left[d\left(x, Ty\right) + d\left(y, Ty\right)\right] \quad \textit{for } x, y \in X.$$

Where the constants $\alpha_i \in [0, 1)$ and $\alpha_1 + \alpha_2 + \alpha_2 + 2s\alpha_4 < \min\left\{1, \frac{2}{s}\right\}$, $i = 1, 2, 3, 4$. Then T has a unique fixed point in X.

Corollary 3.3. *Let (X, d) be complete cone b- metric space with the coefficient $s \geq 1$. Suppose the mapping $T : X \to X$ satisfies the contractive condition.*

$$d\left(Tx, Ty\right) \leq \alpha_1 d\left(x, y\right) \quad \textit{for } x, y \in X.$$

Where $\alpha_1 \in [0, 1)$ is a constant. Then T has an unique fixed point.

Proof. Taking $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$ in (1) we get the required result. □

Corollary 3.4. *Let (X, d) be complete cone b- metric space with the coefficient $s \geq 1$. Suppose the mapping $T : X \to X$ satisfies the contractive condition.*

$$d\left(Tx, Ty\right) \leq \alpha_1 d\left(x, y\right) + \alpha_2 d\left(x, Tx\right) + \alpha_3 d\left(y, Ty\right).$$

Where the constants $\alpha_i \in [0, 1)$ and $\alpha_1 + \alpha_2 + \alpha_3 < \min\left(1, \frac{2}{s}\right)$; $i = 1, 2, 3,$. Then T has a unique fixed point.

Proof. Taking $\alpha_4 = \alpha_5 = 0$ in (1) we get the required result. □

Corollary 3.5. *Let (X, d) be complete cone b- metric space with the coefficient $s \geq 1$. Suppose the mapping $T : X \to X$ satisfies the contractive condition.*

$$d(Tx, Ty) \leq \alpha_1 d(x, y) + \alpha_2 [d(x, Tx) + d(y, Ty)].$$

Where the constants $\alpha_i \in [0, 1)$ and $\alpha_1 + 2\alpha_2 < \min\left(1, \frac{2}{s}\right)$; $i = 1, 2,$. Then T has a unique fixed point.

Proof. Taking $\alpha_4 = \alpha_5 = 0$, $\alpha_3 = \alpha_2$ in (1) we get the required result. □

Corollary 3.6. *Let (X, d) be complete cone b- metric space with the coefficient $s \geq 1$. Suppose the mapping $T : X \to X$ satisfies the contractive condition.*

$$d(Tx, Ty) \leq \alpha_4 [d(y, Tx) + d(x, Ty)].$$

Where the constants $\alpha_4 \in \left[0, \frac{1}{2}\right)$. Then T has a unique fixed point.

Proof. Taking $\alpha_1 = \alpha_2 = \alpha_3 = 0$, $\alpha_4 = \alpha_5$ in (1) we get the required result. □

Corollary 3.7. *Let (X, d) be complete cone b- metric space with the coefficient $s \geq 1$. Suppose the mapping $T : X \to X$ satisfies the contractive condition.*

$$d(Tx, Ty) \leq \alpha_2 [d(x, Tx) + d(y, Ty)].$$

Where the constants $\alpha_2 \in \left[0, \frac{1}{2}\right)$. Then T has a unique fixed point.

Proof. Taking $\alpha = \alpha_4 = \alpha_5 = 0$, $\alpha_3 = \alpha_2$ in (1) we get the required result. □

Corollary 3.8. *Let (X, d) be complete cone b-metric space with the coefficient $s \geq 1$. Suppose the mapping $T : X \to X$ satisfies the contractive condition.*

$$d(Tx, Ty) \leq \alpha_1 d(x, y) + \alpha_2 [d(x, Tx) + d(y, Ty)] + \alpha_3 [d(x, Ty) + d(y, Tx)]$$

Where the constants $\alpha_i \in [0, 1)$ and $\alpha_1 + 2(\alpha_2 + \alpha_3 s) < \min\left\{1, \frac{2}{s}\right\}$, $i = 1, 2, 3$. Then T has a unique fixed point.

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