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A Study on the Algorithms for Equilibrium Problems on Hadamard Manifolds

Research Article

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Abstract: The theory of equilibrium problems provides a unified, natural and general framework to study a wide class of problems, which arise in finance, economics, network analysis, transportation and optimization. This theory has applications across all disciplines of pure and applied sciences. Equilibrium problems include variational inequalities and related problems. The aim of this paper is to provide a survey on the algorithms for equilibrium problems that have been studied by many authors on Hadamard manifolds. This paper should be a useful reference for further research in the field of equilibrium problems and Hadamard manifolds.

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1. Introduction

Hadamard manifold named after Jacques Hadamard, sometimes called a Carter-Hadmard manifold after Elie Cartan is a Riemannian Manifold (M, g), that is complete and simply connected, and has everywhere non-positive sectional curvature [16]. Riemannian manifolds constitute a broad and fruitful framework for the development of different fields. In the last decades concepts and techniques which fit in Euclidean spaces have extended to this non-linear framework. Most of the extended methods require the Riemannian manifold to have non-positive sectional curvature. This is an important property which is enjoyed by a large class of Riemannian manifolds and it is strong enough to imply tight togological restrictions and rigidity phenomena [21, 22]. Particularly, Hadamard manifolds, which are complete simply connected finite-dimensional Riemannian manifolds of non-positive sectional curvature, have turned out to be a suitable setting for diverse disciplines. Hadamard manifolds are examples of hyperbolic spaces and geodesic spaces, more precisely, a Busemain non-positive curvature space [7, 15, 23, 25].

In 2012, M.A. Noor et al. [13] gave an iterative method for solving the equilibrium problem on Hadamard Manifolds using the auxiliary principle technique. Recently, much attention has been given to study the variational inequalities, equalities, equilibrium and related optimization problems on the Riemannian manifold and Hadamard manifold. This work is useful for the development of various fields. Nemeth [26], Tang et al. [5], and Colao et al. [24] have considered the variational inequalities and equilibrium problems on Hadamard manifolds. They have studied the existence of solutions of equilibrium problems under some suitable conditions.

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Let M be a simply connected m-dimensional manifold. Given $x \in M$, the tangent space of M at x is denoted by T_xM and the tangent bundle of M by $TM = \bigcup_{x \in M} T_xM$, which is naturally a manifold. A vector field A on M is a mapping of M into TM which associates to each point $x \in M$, a vector $A(x) \in T_xM$. They assumed that M can be endowed with a Riemannian metric to become a Riemannian manifold. They denoted by $(\cdot, \cdot)$ the scalar product on T_xM with the associated norm $|\cdot|_x$, where the subscript x will be omitted. Given a piecewise smooth curve $\gamma : [a, b] \to M$ joining x to y (that is, $\gamma(a) = x$ and $\gamma(b) = y$) by using the metric, we can define the length of γ as $L(\gamma) = \int_a^b ||\gamma'(t)||dt$. Then for any $x, y \in M$, the Riemannian distance $d(x, y)$, which includes the original topology on M, is defined by minimizing this length over the set of all such curves joining x to y.

Let Δ be the Levi-Civita connection with $(M, \langle ., . \rangle)$. Let γ be a smooth curve in M. A vector field A is said to be parallel along γ if $\Delta_{\gamma'} A = 0$. If γ' itself is parallel along γ, he said that γ is a geodesic and in this case $|\gamma|$ is a constant. When $|\gamma'| = 1$, γ is said to be normalized. A geodesic joining x to y in M is said to be minimal if its length equals $d(x, y)$. A Riemannian manifold is complete if for any $x \in M$, all geodesics emanating from x are defined for all $t \in R$. By the Hopf-Rinow theorem, we know that if M is complete, then any pair of points in M can be joined by a minimal geodesic. Moreover, (M, d) is a complete metric space, and bounded closed subsets are compact. Let M be complete. Then the exponential map $\exp_x : T_xM \to M$ at x is defined by $\exp_x v = \gamma_v(1, x)$ for each $v \in T_xM$, where $\gamma(\cdot) = \gamma_v(\cdot, x)$ is the geodesic starting at x with velocity v (i.e., $\gamma(0) = x$ and $\gamma'(0) = v$). Then $\exp_x tv = \gamma_v(t, x)$ for each real number t.

2. Definitions and Notations

Definition 2.1 (Fixed Point)**.** *Let X be a non empty set and $T : X \to X$ be a mapping. The point $x \in X$ is said to be a fixed point of T if x remains invariant under T, i.e. $Tx = x$.*

Example 2.2.

(1). A translation mapping has no fixed point i.e. $Tx = x + 1$ for all $x \in R$.

(2). The mapping $T : R \to R$ defined by $Tx = \frac{x}{3} - 2$, $x = -3$, is unique fixed point.

(3). A mapping $T : R \to R$ defined by $Tx = x^2$ has two fixed points 0 and 1.

(4). A mapping $T : R \to R$ defined by $Tx = x$ has infinitely many fixed points i.e. every point of R is a point of R.

Definition 2.3 (Euclidean Space)**.** *Euclidean space is a finite dimensional real vector space R^n with an inner product (x, y), $x, y \in R^n$, which is a suitable chosen (Cartesian) coordinate system*

$$x = (x_1, x_2, \ldots, x_n)$$

$$y = (y_1, y_2, \ldots, y_n)$$

is given by the formula $(x, y) = \sum_{i=1}^{n} x_i y_i$.

Definition 2.4 (Manifold)**.** *Manifold is a topological space that is locally Euclidean.*

Definition 2.5 (Riemannian Manifold)**.** *Riemannian manifold or Riemannian space (M, g) a real smooth manifold M equipped with an inner product on the tangent space.*

Definition 2.6 (Hadamard Manifold [13])**.** *Hadamard manifold named after Jacques Hadamard sometimes called a Carten-Hadamard manifold after Elie carter is a Riemannian manifold (M, g) that is complete and simply connected, and has everywhere non-positive sectional curvature.*

Example 2.7.

(1). The real line R with its usual metric is a Hadamard manifold with constant sectional curvature equal to 0.

(2). Standard n-dimensional hyperbolic space H^n is a Hadamard manifold with constant sectional curvature equal to -1.

Definition 2.8 (Equilibrium Problem [13]). *For a given bifunction $F(\cdot,\cdot) : K \times K \to R$, the problem of finding $u \in K$ such that*

$$F(u, v) \geq 0 \;\; \forall \;\; v \in K, \tag{1}$$

is called equilibrium problem on Hadamard manifolds.

Definition 2.9 (Firmly non expansive Mapping [24]). *Given a mapping $T : K \to K$ defined on $K \subseteq M$, we say that T is firmly non-expansive if for any $x, y \in K$, the function $\phi : [0, 1] \to [0, \infty]$ defined by*

$$\phi(t) = d(\gamma_1(t), \gamma_2(t))$$

is non-increasing, where γ_1 and γ_2 denote the geodesics joining x to $T(x)$ and y to $T(y)$, respectively. Every firmly non expansive mapping is non expansive, that is, for all $x, y \in K$

$$d(T(x), T(y)) \leq d(x, y).$$

Definition 2.10 (Fejer Monotone Sequence [24]). *Let X be a complete metric space and $C \subseteq X$ be a non empty set. A sequence $\{x_n\} \subset X$ is called Fejer montone w.r.t. C is*

$$d(x_{n+1}, y) \leq d(x_n, y) \quad \text{for all } y \in C \text{ and } n \geq 1.$$

Definition 2.11 (Resolvent of bifunction [24]). *Let $F : K \times K \to R$. For any $\lambda > 0$, the resolvent of F is the set-valued operator $J^F_\lambda : M \to 2^K$ defined by*

$$J^F_\lambda(x) = \left\{z \in K | \lambda F(z, y) - \left\langle \exp_z^{-1} x, \exp_z^{-1} y \right\rangle \geq 0, \forall \; y \in K \right\}, \;\forall \;\; x \in M.$$

Definition 2.12 (Geodesic Convex Function [26]). *A real valued function $f : M \to R$ defined on a geodesic convex set K is said to be geodesic convex if and only if for $0 \leq t \leq 1$*

$$f(\gamma(t)) \leq (1 - t)f(\gamma(0)) + tf(\gamma(1)).$$

Definition 2.13 (KKM Mapping [10]). *Let $K \subset M$ be a non empty closed geodesic convex set and $G : K \to 2^k$ be a set-valued mapping. We say that G is KKM mapping if for any $\{x_1, \ldots, x_m\} \subset K$, we have $C_0(\{x_1, x_2, \ldots, x_m\}) \subset \bigcup\limits_{i=1}^{m} G(x_i)$*

Definition 2.14 (Hemi continuous Function [20]). *A function $F : K \to R$ is said to be hemi-continuous if for every geodesic $\gamma : [0, 1] \to K$, whenever $t \to 0, F(\gamma(t)) \to F(\gamma(0))$.*

Definition 2.15 (Fixed Point Property [26]). *A topological space T is of the fixed point property if every continuous function $f : T \to T$ has a fixed point.*

Definition 2.16 (Variational inequality [26]). *For a given single valued vector field $T : M \to TN$. Consider the problem of finding $u \in K$ such that*

$$\langle Tu, \exp_u^{-1} v\rangle \geq 0 \ \forall \ v \in K,$$

which is called the variational inequality.

Definition 2.17 (Upper semi-continuous [26]). *Given $T : M \to 2^M$ and $x_0 \in M$, the mapping T is said to be*

(1). Upper semi-continuous (USC), at x_0 if for any open set $V \subseteq M$ satisfying $T(x_0) \subseteq V$, there exists an open neighbourhood $U(x_0)$ of x_0 such that $T(x) \subseteq V$ for any $x \in U(x_0)$.

(2). Upper Kuratowski semicontinuous (UKSC), at x_0 if for any sequences $\{x_k\}$, $\{u_k\} \subset M$ with each $u_k \in T(x_k)$, the relation $\lim\limits_{k\to\infty} x_k = x_0$ and $\lim\limits_{k\to\infty} u_k = u_0$ imply $u_0 \in \ T(x_0)$.

3. Equilibrium Problems on Hadamard Manifolds

In this section, we present some algorithm for equilibrium problems on Hadamard manifolds proved by Vittorio Colao et al. [24] , M.A. Noor et al. [13] using the auxiliary principle technique.

3.1. Existence of Equilibrium Points

An equilibrium theory in Euclidean spaces was first introduced by Ky Fan in [8, 9] and then developed by Brezis, Nirenboag and Stampacchia [6] among others. In order to get an existence result for this equilibrium problem they provide following analogues to KKM Lemma [1] in the setting of Hadamard manifolds.

Lemma 3.1. *Let $G : K \to 2^K$ be a mapping such that or each $x \in K$, G(x) is closed suppose that*

(1). there exist $x_0 \in K$ such that $G(x_0)$ is compact

(2). $\forall \ x_1, x_2, \dots, x_m \in K$, $C_0(\{x_1, \dots, x_m\}) \subset \bigcup\limits_{i=1}^{m} G(x_i)$.

Then $\bigcap\limits_{x\in K}^{G} (x) \neq \phi$.

Theorem 3.2. *Let $F : K \times K \to R$ be a bifunction such that*

(1). for any $x \in K$, $F(x, x) \geq 0$;

(2). for every $x \in K$, the set $\{y \in \ K; F(x, y) < 0\}$ is convex

(3). for every $y \in K$, $x \to F(x, y)$ is upper semicontinuous.

(4). there exist a compact set $L \subseteq M$ and a point $y_0 \in L \cap K$ such that $F(x, y_n) < 0, \forall \ x \in K/L$.

Then there exist a point $x_0 \in L \cap K$ satisfying $F(x_0, y) \geq 0, \forall \ y \in K$.

By setting $L = K$ in the previous theorem, the following corollary is obtained.

Corollary 3.3. *Let $K \subseteq M$ be convex and compact and $F : K \times K \to R$ such that*

(1). for any $x \in K$, $F(x, x) \geq 0$;

(2). for every $x \in K$ the set $\{y \in K : F(x, y) < 0\}$ is convex

(3). for every $y \in K$, $x \to F(x, y)$ is upper semicontinuous.

Then there exist a point $x_0 \in K$ satisfying $F(x_0, y) \geq 0, \forall\ y \in K$.

Example 3.4. *Example of an equilibrium problem defined in a Euclidean space whose set K is not convex so it cannot be solved by using the classical results known in vector spaces. However, if we rewrite the problem in a Riemannian manifold then it turns out to satisfy the conditions required in the Corollary 3.3. Let*

$$K = \{(x, y, z) \in R^3 : 0 \leq x \leq 1,\ y^2 - z^2 = -1, z \geq 0\}$$

and $F : K \times K \to R$ the bifunction defined by

$$F(x_1, y_1, z_1, x_2, y_2, z_2) = 4(x_2 - x_1) + (1 - x_1)((y_2^2 + z_2^2) - (y_1^2 + z_1^2))$$

Note that K is indeed not convex in R^3. Given a natural number $m \geq 1$. Let $E^{m.1}$ denote the vector space R^{m+1} endowed with the symmetric bilinear form (which is called the Lorentz metric) defined by

$$(x, y) = \sum_{i=1}^{m} x_i y_i - x_{m+1} y_{m+1}, \forall\ x = (x_i),\ y = (y_i) \in R^{m+1}$$

The hyperbolic m-spcae H^m is defined by

$$\{x = (x_1, \ldots, x_{m+1}) \in E^{m.1} : \langle x, x \rangle = -1, x_{m+1} > 0\},$$

that is upper sheet of the hyperboloid $\{x \in E^{m.1} : (x, x) = -1\}$. Note that $x_{m+1} \geq 1$ for any $x \in H^m$, with equality if and only if $x_i = 0$ for all $i = 1, \ldots, m$. the metric of H^m is induced from the Lorentz metric $(\cdot, \cdot)$ and it will be denoted by the same symbol. Then H^m is a Hadmard manifold sectional curvature -1 (c.f [15]). Furthermore, the normalized godesic $\gamma : R \to H^m$ starting from $x \in H^m$ is given by $\gamma(t) = (cosht)x + (sinht)v, \forall\ t \in R$, where $v \in T_x H^m$ is a unit vector. Considering the set K immersed in the space $M = R \times H^1$ which is a Hadmard manifold for being the product space of Hadmard manifolds (cf. [15]), it is readily seen that K is convex and compact on M. On the other hand, conditions (i) and (iii) in Corollary 3.3 hold, and the fact that F is convex is the second variable. So Corollary 3.3 implies the existence of an equilibrium point for F.

Theorem 3.5. *Let $K \to TM$ be a continuous vector field and $f : K \to R$ a convex lower semicontinuous function. Assume that the following condition holds: (C) There exists a compact set $L \subseteq M$ and a point $y_0 \in L \cap K$ such that*

$$(Ax, \exp_x^{-1} y_0) + f(y_0) - f(x) < 0,\ \forall\ x \in K \backslash L.$$

Then MVIP(A, f) has a solution in $L \cap K$.

Corollary 3.6. *Let $A : K \to TM$ be a continuous vector field and $f : K \to R$ a convex lower semicontinuous function. If either*

(1). K is compact, or

(2). there exists $y_0 \in K$ such that the coercivity condition

$$\frac{\langle Ay_0, \exp_{y_v}^{-1} x \rangle + \langle Ax, \exp_x^{-1} y_0 \rangle}{d(y_0, x)} \to -\infty \text{ as } d(y_0, x) \to \infty$$

holds, then MVIP (A, f) has a solution.

Remark 3.7. *By considering f the function constantly 0, it follows that Corollary 3.6 extended.*

Lemma 3.8. *Let $D, K \subseteq M$ be closed convex sets with D compact. Assume that $\rho : D \times K \to R$ is upper semicontinuous in the first variable and that for any $x \in D$ and $y \in K$, $-\rho(\cdot, \gamma)$ and $\rho(x, \cdot)$ are convex functions. If $\max_{x \in D} \rho(x, y) \geq 0$, $\forall\ y \in K$ then there exists $\hat{x} \in D$ such that $\rho(\hat{x}, y)$ for any $y \in K$.*

Theorem 3.9. *Let $K \subseteq M$ be a compact convex set and $T : K \to 2^K$ and UKSC mapping. Assume that for any $x \in K$, $T(x)$ is closed and convex. Then there exists a fixed point of T.*

Remark 3.10. *The upper semicontinuity implies the upper Kuratowski semicontinuity, so the previous result remains true assume that T is USC instead.*

3.2. Approximation of Equilibrium Points

The approach that followed to approximate a solution of the equilibrium problem (for the bifunction F and the set K find $x \in K$ such that $F(x, y) \geq 0$, $\forall\, y \in K$) involves the resolvent of the bifunction F, which is firmly non-expansive mapping whose fixed point set coincides with the equilibrium point set of F.

Proposition 3.11 ([2]). *A mapping $T : K \to K$ is firmly nonexpansive iff for any $x, y \in K$*

$$\left\langle \exp^{-1}_{T(x)} T(y), \exp^{-1}_{T(x)} x \right\rangle + \left\langle \exp^{-1}_{T(y)} T(x), \exp^{-1}_{T(y)} y \right\rangle \leq 0$$

As in Banach spaces and the Hilbert Ball [20], the class of firmly nonexpansive mappings is characterized by the good asymptotic behaviour of the sequence of Picard iterates $\{T^n(x)\}$. In order to prove the convergence of this sequence, the following definition and results are necessary.

Lemma 3.12 ([4, 17]). *Let X be a complete metric space. If $\{x_n\} \subset X$ is Fejer monotone with respect to a non empty set $C \subseteq X$, then $\{x_n\}$ is bounded. Moreover, if a cluster x of $\{x_n\}$ belong to C, then $\{x_n\}$ converges to x.*

Theorem 3.13. *Let $T : K \to K$ be a firmly non expansive mapping such that its fixed point set $Fix(T) \neq \phi$. Then for each $x \in k$ the sequence of iterates $\{T^n(x)\}$ converges to a fixed point of T.*

3.3. Resolvents of Bifunction

The definition of the resolvent of a bifunction in the setting of a Hilbert space H appears implicitly in [3] and was first given in [18]. In order to distinguish the resolvent of vector fields and the resolvent of bifunctions, denoted latter with an upper index, J^F. Given a bifunction $F : K \times K \to R$, where $K \subseteq H$ is nonempty closed and convex, the resolvent of F is the set-valued operator $J^F : H \to 2^K$ such that for any $x \in H$, $J^F(x) = \{z \in K | (\forall\, y \in K) F(z, y) + \langle z - x, y - z \rangle \geq 0\}$. Under some conditions on the bifunction F, J^F can be proved to be well defined, single-valued and firmly nonexpansive, and its fixed point set turns out to be the equilibrium point set of F.

The following definition extends the previous one to the setting of a Hadamard manifold M.

Definition 3.14. *Let $F : K \times K \to R$. For any $\lambda > 0$, the resolvent of F is the set-valued operator $J^F_\lambda : M \to 2^K$ defined by*

$$J^F_\lambda(x) = \left\{ z \in K | \lambda F(z, y) - \left\langle \exp^{-1}_z x, \exp^{-1}_z y \right\rangle \geq 0, \forall\, y \in K \right\}, \ \forall\, x \in M.$$

Theorem 3.15. *Let $F : K \times K \to R$ be a bifunction satisfying the following conditions:*

(1). F is monotone, that is, for any $(x, y) \in K \times K$, $F(x, y) + F(y, x) \leq 0$;

(2). for each $\lambda > 0$, J_λ^F is properly defined, that is, the domain $D(J_\lambda^F) \neq \emptyset$.

Then for any $\lambda > 0$,

(i). the resolvent J_λ^F is single-valued;

(ii). the resolvent J_λ^F is firmly nonexpansive;

(iii). the fixed point set of J_λ^F is the equilibrium point set of F, $Fix(J_\lambda^F) = EP(F)$;

(iv). if $D(J_\lambda^F)$ is closed and convex, the equilibrium point set $EP(F)$ is closed and convex.

Remark 3.16. *The resolvent could be defined for a set-valued bifunction $F : K \times K \to 2^R$ as the set-valued function $J_\lambda^F : M \to 2^K$ such that*

$$J_\lambda^F(x) = \left\{ z \in K | \lambda u - \left\langle \exp_z^{-1} x, \exp_z^{-1} y \right\rangle \geq 0, \ \ \forall\, y \in K, \forall \ \ u \in F(z,y) \right\},$$

for any $\lambda > 0$ and any $x \in M$. Then, assuming that F monotone means that $u + v \leq 0$ for any $u \in F(x,y)$, $v \in F(y,x)$ and $x, y \in K$, the previous theorem would remain true except for (iii) which needs F to be single-valued.

3.4. Auxiliary Principle Technique

Muhammad Aslam Noor and Khalida Inayat Noor [13] suggested and analyzed an iterative method for solving the equilibrium problems on Hadamard manifolds using the auxiliary principle technique. They considered the convergence analysis of condition.

Definition 3.17 (Tangent Space)**.** *Let M be a simply connected m-dimensional manifold. Given $x \in M$, the tangent space of M at x is denoted by T_xM and can be defined as, In differential geometry, one can attach to every point x of a differentiable manifold a tangent space, a real vector space that intuitively contains the possible "directions" at which one can tangentially pass through x. The elements of the tangent space are called tangent vectors at x.*

Lemma 3.18. *Let $x \in \ M$. Then $\exp_x : T_xM \to M$ is a diffeomorphism, and for any two points $x, y \in \ M$, there exist a unique normalized geodesic joining x to $y, \gamma_{x,y}$, which is minimal.*

Lemma 3.19. *Comparison theorem for triangles. Let $\Delta(x_1, x_2, x_3)$ be a geodesic triangle. Denote for each $i = 1, 2, 3 \ (mod\ 3)$, by $\gamma_1 : [0, l_i] \to M$ the geodesic joining x_i to x_{i+1} and $\alpha_I = L(\gamma_i'(0) - \gamma_l'(i-1)(l_i - 1))$, the angle between the vectors $\gamma_i'(0)$ and $-\gamma_{i-1}'(l_{i-1})$ and $l_i = L(\gamma_i)$. Then*

$$\alpha_1 + \alpha_2 + \alpha_3 \leq \pi, \ \ l_i^2 + l_{i+1}^2 - 2L_i; \ \ l_{i+1} \cos \alpha_{i+1} \leq l_{i-1}^2$$

In terms of the distance and the exponential map, the above inequality can be rewritten as

$$d^2(x_i, x_{i+1}) + d^2(x_{i+1}, x_{i+2}) - 2(\exp_{x_{i+1}}^{x_i}, \exp_{x_{i=1}}^{-1} x_{i+2}) \leq d^2(x_{i-1}, x)$$

Since

$$\left\langle \exp_{x_{i+1}}^{-1} x_i, \, \exp_{x_{i+1}}^{-1} x_{i+2} \right\rangle = d(x_i, x_{i+1}) d(x_{i+1}, x_{i+2}) \cos \alpha_{i+1}$$

Lemma 3.20. *Let $\Delta(x, y, z)$ be a geodesic triangle in a Hadamard manifold M. Then there exists $x', y', z' \in R^2$ such that*

$$d(x, y) = \|x' - y'\|, \;\; d(y, z) = \|y' - z'\|, \;\; d(z, x) = \|z' - x'\|$$

The triangle $\Delta(x', y', z')$ is called the comparison triangle of the geodesic triangle $\Delta(x, y, z)$, which is unique upto isometry of M.

Lemma 3.21. *Let M be a Hadamard manifold and $f : M \to R$ be convex. Then for any $x \in M$, the subdifferential $\partial f(x)$ of f at x is non-empty. That is, $D(\partial f) = M$.*

3.5. Implicit Iterative Method

Noor et al. [14] used the auxiliary principle technique of Glawinski et al. [19] to analyze an implicit iterative method for solving the equilibrium problem. For given $u \in K$ satisfying equilibrium problem (1). Consider the problem of finding $w \in K$ such that

$$\rho F(u, v) + \langle \exp_u^{-1} w, \exp_w^{-1} v\rangle \geq 0, \;\; \forall \;\; v \in K$$

which is called the auxiliary problem on Hadamard manifolds.

Algorithm 3.22. *For a given u_0, compute the approximate solution by the iterative scheme*

$$\rho F(u_n, v) + \left\langle \exp_{u_n}^{-1} u_{n+1}, \exp_{u_{n+1}}^{-1} v\right\rangle \geq 0, \;\; \forall \;\; v \in K$$

is called the explicit iterative method for solving the equilibrium problem on the Hadamard manifold.

Algorithm 3.23. *For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\left\langle \rho T u_n + (\exp_{u_n}^{-1} u_{n+1}), \exp_{u_{n+1}}^{-1} v\right\rangle \geq 0, \forall \;\; v \in K$$

For $M = R^n$, Algorithm 3.23 reduces to

Algorithm 3.24. *For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho T u_n + u_{n+1} - u_n v - u_{n+1}\rangle \geq 0, \;\; \forall \;\; v \in K$$

Theorem 3.25. *Let $F(\cdot, \cdot)$ be a partially relaxed strongly monotone bifunction with a constant $\alpha > 0$. Let u_n be the approximate solution of the equilibrium problem (1) obtained from Algorithm 3.22, then*

$$d^2(u_{n+1}, u) \leq d^2(u_n, u) - (1 - \rho\alpha) d^2(u_{n+1}, u_n),$$

where $u \in K$ is a solution of the equilibrium problem.

Glowinski et al. [19] suggested and analyzed an implicit iterative method for solving the equilibrium problem (1). For a given $u \in K$ satisfying (1), consider the problem of find $w \in K$ such that

$$\rho F(w, v) + \langle \exp_u^{-1} w, \exp_w^{-1} v\rangle \geq 0, \;\forall\, v \in K$$

which is called the auxiliary equibrium problem on Hadamard manifolds. They have shown that the convergence analysis of this method requires only the pseudomonotonicy which is a weaker condition than monotonicity.

Algorithm 3.26. *For a given u_0, compute the approximate solution by the iterative scheme*

$$\rho F(u_{n=1}, v) + \left\langle \exp^{-1}_{u_n} u_{n+1}, \exp^{-1}_{u_{n+1}} v \right\rangle \geq 0, \ \forall\, v \in K$$

is called the implicit (proximal point) iterative method for solving the equilibrium problem on the Hadmard manifold.

Algorithm 3.26 can be written in the following equivalent form.

Algorithm 3.27. *For a given $u_0 \in K$, find the approximate solution u_{n+1} by the iterative scheme*

$$\rho F(u_n, v) + \langle \exp^{-1}_{u_n} y_n, \exp^{-1}_{y} v \rangle \geq 0 \ \forall \ v \in K$$
$$\rho F(y_n, v) + \left\langle \exp^{-1}_{y}, \exp^{-1}_{u_{n+1}} v \right\rangle \geq 0, \ \forall \ v \in K$$

is a two-step iterative method for solving the equilibrium problems on Hadamard manifolds. This method can be viewed as the extragradient method for solving the equilibrium problems.

If K is a convex set in R^n, then Algorithm 3.26 collapses to the following

Algorithm 3.28. *For a given $u_0 \in K$, find the approximate solution u_{n+1} by the iterative scheme :*

$$\rho F(u_{n+1}, u) + (u_{n+1} - u_n, v - u_{n+1}) \geq 0, \ \forall\, v \in K$$

which is known as the implict method for solving the equilibrium problem.

For the convergence analysis of Algorithm 3.27, see [11, 12]. If $F(u, v) = (Tu, \exp^{-1}_u v)$, where T is a single valued vector filed $T : K \to TM$, then Algorithm 3.26 reduces to the following implicit method for solving the variational inequalities.

Algorithm 3.29. *For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\left\langle \rho T u_{n+1} + (\exp^{-1}_{u}, u_{n+1}), \exp^{-1}_{u_{n+1}} v \right\rangle \geq 0, \ \forall \ v \in K$$

Algorithm 3.29 is due according to Tang et al. [5] and M. A. Noor and K. I. Noor [13]. We can also rewrite Algorithm 3.29 in the following equivalent form.

Algorithm 3.30. *For a given $u_0 \in K$, computer the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho T u_n + \exp^{-1}_{u_n} y_n, \exp^{-1}_{y} v \rangle \geq 0 \ \forall\, v \in K,$$
$$\left\langle \rho T y_n + \exp^{-1}_{u_{n+1}} u_n, \exp^{-1}_{u_{n+1}} v \right\rangle \geq 0, \ \ \forall\, v \in K$$

which is the extragradient method for solving the variational inequalities on Hadmard manifolds and appears to be a new one.

In a similar way, one can obtain several iterative methods for solving the variational inequalities on the Hadmard manifold. We now consider the convergence analysis of Algorithm 3.26 and this is the motivation of this next result.

Theorem 3.31. *Let $F(\cdot, \cdot)$ be a pseudomonotone bifunction. Let u_n be the approximate solution of the equilibrium problem obtained from Algorithm 3.26, then*

$$d^2(u_{n+1}, u) + d^2(u_{n+1}, u_n) \leq d^2(u_n, u)$$

where $u \in K$ is a solution of the equilibrium problem (1).

Theorem 3.32. *Let $u \in K$ be solution of (1) and let u_{n+1} be the approximate solution obtained from Algorithm 3.26, then* $\lim_{n\to\infty} u_{n+1} = u$.

In the next section, we study the existence of solutions of mixed equilibrium problems on Hadamard manifolds. S. Jena et al. [20] introduced the implicit and explicit algorithms to solve these problems. They showed that the sequence generated by both implicit and explicit algorithms converges to a solution of mixed equilibrium problems, whenever it exists, under reasonable assumptions.

3.6. Existence of Solutions of Mixed Equilibrium Problems

Colao et al. [24] studied existence of solutions of equilibrium problems under monotonicity assumptions on Hadamard manifolds.

Definition 3.33 (Mixed Equilibrium Problem)**.** *Let $\psi : K \to R$ be a mapping and $F : K \times K \to R$ be a bifunction satisfying the property $F(x, x) = 0$ for all $x \in K$. Then the problem is to find $\bar{x} \in K$ such that*

$$F(\bar{x}, y) + \psi(y) - \psi(\bar{x}) \geq 0 \ \ \forall \ \ y \in K \tag{2}$$

is called mixed equilibrium problem. Calao et al. [4] called a bifunction F to be monotone on K if for any $x, y \in K$, we have $F(x, y) + F(y, x) \leq 0$. A bifunction F is said to be pseudomonotone with respect to the function ψ if

$$\begin{aligned} &F(x, y) + \psi(y) - \psi(x) \geq 0 \\ \Rightarrow\ &F(y, x) + \psi(x) - \psi(y) \leq 0 \end{aligned}$$

Lemma 3.34. *Let $F : K \times K \to R$ be hemicontinuous in the first argument and for fixed $x \in K$ the mapping $z \to F(x, z)$ be geodesic convex. Also assume that the map $\psi : K \to R$ is geodesic convex and the bifunction F is pseudomonotone with respect to ψ. Then $\bar{x} \in K$ is a solution of the mixed equilibrium problem (2) if and only if $F(y, \bar{x}) + \psi(\bar{x}) - \psi(y) \leq 0$ for all $y \in K$.*

Theorem 3.35. *Let K be a bounded subset of M and $F : K \times K \to R$ be hemicontinuous in the first argument. Suppose for fixed $x \in K$, the mapping $z \to F(x, z)$ and $\psi : K \to R$ are geodesic convex, lower semicontinuous. Also assume that the bifunction F is pseudomonotone with respect to ψ. Then the mixed equilibrium problem (2) has a solution.*

Theorem 3.36. *Let K be an unbounded subset of M and $F : K \times K \to R$ be hemicontinuous in the first argument. Suppose for fixed $x \in K$, the mapping $z \to F(x, z)$ and $\psi : K \to R$ are geodesic convex, lower semicontinuous. Also assume that the bifunction F is pseudomonotone with respect to ψ. If there exists a point $x_0 \in K$, such that $F(x, x_0) + \psi(x_0) - \psi(x) < 0$, whenever $d(0, x) \to +\infty$, $x \in K$ holds, then the mixed equilibrium problem (2) has a solution.*

Now we present some Implicit methods for solving mixed equilibrium problem:

Algorithm 3.37. *At stage n, given $x_n \in K$, $\rho > 0$, compute $x_{n+1} \in K$ as a solution of the following iterative scheme.*

$$F(x_{n+i,y}) + \frac{1}{\rho}\left\langle \exp^{-1}_{x_n} x_{n+1}, \exp^{-1}_{x_{n+1}} y \right\rangle + \psi(y) - \psi(x_{n+1}) \geq 0 \ \ \forall \ \ y \in K \tag{3}$$

(i). When $\psi \equiv 0$, the Algorithm 3.37 reduces to the following implicit iterative algorithm.

Algorithm 3.38. *At stage n, given $x_n \in K$, $\rho > 0$, compute $x_{n+1} \in K$, such that*

$$F(x_{n+1}, y) + \frac{1}{\rho}\left\langle \exp^{-1}_{x_n} x_{n+1}, \exp^{-1}_{x_{n+1}} y \right\rangle \geq 0 \;\; \forall \;\; y \in K$$

This is the implicit algorithm for the equilibrium problems introduced by Noor et al. [11].

(ii). If K is convex set in R^n, then Algorithm 3.37 reduces into the following algorithm [11, 12].

Algorithm 3.39. *At stage n, given $x_n \in K$, $\rho > 0$, compute $x_{n+1} \in K$, as a solution of the iterative scheme*

$$F(x_{n+1}, y) + \frac{1}{\rho}\langle x_{n+1} - x_n, y - x_{n+1}\rangle + \psi(y) - \psi(n+1) \geq 0, \;\; \forall \;\; y \in K$$

(iii). If we take $F(x, y) = \langle V_x, \exp^{-1}_x y\rangle$, then Algorithm 3.37 reduces to the following.

Algorithm 3.40. *At stage n, given $x_n \in K$, $\rho > 0$, compute $x_{n+1} \in K$, as a solution of the iterative scheme*

$$\left\langle \rho V x_{n+1} + \exp^{-1}_{x_n} x_{n+1}, \exp^{-1}_{x_{n+1}} y \right\rangle + \psi(y) - \psi(x_{n+1}) \geq 0, \;\; \forall \;\; y \in K,$$

which is an algorithm for solving mixed variational inequalities studied by Noor et al. [13].

(iv). When $\psi \equiv 0$, the Algorithm 3.40 reduces to the following implicit iterative algorithm for solving variational inequalities.

Algorithm 3.41. *At stage n, given $x_n \in K$, $\rho > 0$, compute $x_{n+1} \in K$, as a solution of the iterative scheme*

$$\langle \rho V x_{n+1} + \exp^{-1}_{x_n} x_{n+1}, \exp^{-1}_{x_n} y\rangle \geq 0, \;\; \forall \;\; y \in K$$

Theorem 3.42. *Let $F : K \times K \to R$ be pseudomonotone with respect to the function ψ and continuous in the first argument and $SOL(MEP) \neq \phi$. Suppose that the sequence $\{x_n\}$ generated by (3) is well defined and $\psi : K \to R$ is continuous. Then $\{x_n\}$ converges to a solution of the mixed equilibrium problem (2).*

Some Explicit methods for solving mixed equilibrium problem:

Definition 3.43. *The bifunction F is said to be partially relaxed pseudomonotone with respect to the function ψ if there exist $\alpha > 0$ such that $\forall \;\; x, y, z \in K$.*

$$F(x, y) + \psi(y) - \psi(x) \geq 0 \Rightarrow F(z, x) + \psi(x) - \psi(z) \leq \alpha d^2(y, z)$$

If we take $z = y$, then F reduces to a pseudomonotone function.

Algorithm 3.44. *At stage n, given $x_n \in K$, $\rho > 0$, compute $x_{n+1} \in K$, as a solution of the iterative scheme*

$$F(x_n, y) + \frac{1}{\rho}\left\langle \exp^{-1}_{x_n} x_{n+1}, \exp^{-1}_{x_{n+1}} y \right\rangle + \psi(y) - \psi(x_n) \geq 0 \;\; \forall \;\; y \in K \tag{4}$$

Some particular cases of Algorithm 3.44 are given as follows:

(i). When $\psi \equiv 0$, the Algorithm 3.44 reduces to the following explicit iterative algorithm for equilibrium problems.

Algorithm 3.45. *At stage n, given $x_n \in K$, $\rho > 0$, compute $x_{n+1} \in K$, such that*

$$F(x_n, y) + \frac{1}{\rho}\left\langle \exp^{-1}_{x_{n+1}}, \exp^{-1}_{x_{n+1}} y \right\rangle \geq 0 \;\; \forall \;\; y \in K.$$

Algorithm 3.46. *At stage n, given $x_n \in\ K$, $\rho > 0$, compute $x_{n+1} \in\ K$, as a solution of the iterative scheme*

$$F(x_n, y) + \frac{1}{\rho} < x_{n+1} - x_n, y - x_{n+1} > +\psi(y) - \psi(x_{n+1}) \geq 0 \ \ \forall \ \ y \in K.$$

Algorithm 3.47. *At stage n, given $x_n \in k$, $\rho > 0$ compute $x_{n+1} \in K$ as a solution of the iterative scheme*

$$\left\langle \rho V x_n + \exp_{x_n}^{-1}, \exp_{x_{n+1}}^{-1} y \right\rangle + \psi(y) - \psi(x_{n+1}) \geq 0, \ \ \forall \ \ y \in \ \ K.$$

Algorithm 3.48. *At stage n, given $x_n \in\ K$, $\rho\ > 0$, compute $x_{n+1} \in\ K$, as a solution of the iterative scheme*

$$\left\langle \rho V x_n + \exp_{x_{n+1}}^{-1}, \exp_{x_{n+1}}^{-1} y \right\rangle \geq 0, \ \ \forall \ \ y \in \ \ K.$$

Algorithm 3.49. *Let $F : K \times K \to R$ be a partially relaxed pseudomonotone bifunction with respect to the function ψ with a constant $\alpha > 0$, and continuous in the first argument. Suppose that the sequence $\{x_n\}$ generated by (4) is well defined, $\psi : K \to R$ is continuous and $SOL(MEP) \neq \phi$. Then*

$$d^2(x_{n+1}, x) \leq d^2(x_n, x) - (1 - 2\rho\alpha)d^2(x_{n+1}, x_n)$$

If in addition $\rho < \frac{1}{2a}$, then $\{x_n\}$ converges to a solution of the mixed equilibrium problem (2).

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