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Geometric Approach for Banach Space Using Hausdorff Distance

Research Article

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Abstract: In this paper, we study the normed linear spaces which are induced by Hausdorff distance. Barich proved the completeness of Hausdorff metric space [3]. We extend his work for the completeness of the normed linear spaces called Banach spaces, which are induced by Hausdorff distance and we proved convex Hausdorff metric space is Banach space.

Keywords: Convex set, Banach Space, Hausdorff Distance, Hausdorff Metric Space.

1. Introduction

Let's turn the clock ahead to 1922 and given all brief discussion of the contribution of Eduard Helly, Hans Hahn and the great Polish mathematician Banach. While Eduard Helly and Hans Hahn are important players in the story of Functional analysis, making several important contributions to its early development it was Banach who gave the first complete treatment of abstract normed vector space and its the word complete that must be emphasized! in his thesis [6]. Banach discussed several important applications of theory of *functionals* in his own words. Ofcourse, most of us are familiar with the notion of a Banach space, which was introduced in its fully glory that is in Banach thesis. We discuss the normed linear spaces which are induced by Hausdorff distance and some of the basic concepts from Functional analysis. In a nutshell Functional analysis is a study of normed vector spaces and bounded linear operators. Thus it merges the subjects of linear algebra with the points set topology. The topologies that appears in Functional analysis will arise from Hausdorff metric space.

The geometry that follows from these consideration gives a specified approach to Banach space. Considering the above concepts, we have presented a geometric setup that allows us to obtain structure for the existence of an Banach space. Moreover, our geometric frame-work provides that generate a new setup that might be useful to determine conditions that generate the study of functionals for which some interesting results concerning the existence. In this paper, we construct the Hausdorff metric space, is to geometrize the Banach space completely. First Hausdorff distance has been considered, thus by a specific distance, largest length is calculated & properties of this metric function are studied. Finally, convex Hausdorff metric space are considered as complete normed linear space i.e., Banach space.

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2. Preliminaries

The concepts in this section should be familiar to anyone who has taken a course in real analysis. Therefore, we expect the reader to be familiar with the following definitions when applied to the metric space (R, d), where $d(x, y) = |x - y|$. However, with the exclusion of some examples, for the majority of this paper we will be working in a general metric space. Thus our definitions will be given with respect to any metric space (X, d).

Definition 2.1. *Metric space (X, d) consists of a set X and a function $d : X \times X \to R$ that satisfies the following four properties.*

(1). $d(x, y) \geq 0$ for all $x, y \in X$.

(2). $d(x, y) = 0$ if and only if $x = y$.

(3). $d(x, y) = d(y, x)$ for all $x, y \in X$.

(4). $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

The function d, which gives the distance between two points in X, is called a *metric*.

Definition 2.2. *Let $v \in X$ and let $r > 0$. Open ball centered at v with radius r is defined by $B_d(v, r) = \{x \in X : d(x, v) < r\}$.*

Definition 2.3. *A set $E \subseteq X$ is Bounded in (X, d) if there exist $x \in X$ and $M > 0$ such that $E \subseteq B_d(x, M)$.*

Definition 2.4. *A set $K \subseteq X$ is Totally bounded if for each $\epsilon > 0$ there is a finite subset $\{x_i : 1 \leq i \leq n\}$ of K such that $K \subseteq \bigcup_{i=2}^{n} B_d(x_i, \epsilon)$.*

For the following definitions, let $\{x_n\}$ be a sequence in a metric space (X, d).

Definition 2.5. *The sequence $\{x_n\}$ Converges to $x \in X$ if for each $\epsilon > 0$ there exists a positive integer N such that $d(x_n, x) < \epsilon$, for all $n \geq N$. We say $\{x_n\}$ converges if there exists a point $x \in X$ such that $\{x_n\}$ converges to x.*

Definition 2.6. *The sequence $\{x_n\}$ is a Cauchy sequence if for each $\epsilon > 0$ there exists a positive integer N such that $d(x_n, x_m) < \epsilon$ for all $m, n \geq N$.*

Definition 2.7. *A metric space (X, d) is Complete if every Cauchy sequence in (X, d) converges to a point in X.*

Definition 2.8. *A set $K \subseteq X$ is Sequentially compact in (X, d) if each sequence in K has a subsequence that converges to a point in K.*

Definition 2.9. *Norm $\|.\|$ on a linear space X is a mapping X to R satisfying*

(1). $\|x\| \geq 0$ for all $x \in X$.

(2). $\|x\| = 0$ if and only if $x = 0$.

(3). $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in R$ and $x \in X$.

(4). (Triangle inequality) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

A normed linear space $(X, \|.\|)$ is a linear space X equipped with a norm $\|.\|$.

Definition 2.10. *A complete normed linear space is called a Banach space.*

Corollary 2.11 ([5])**.** *Let $\{x_n\}$ and $\{y_n\}$ be sequences in a metric space (X, d). If $\{x_n\}$ converges to x and $\{y_n\}$ converges to y, then $\{d(x_n, y_n)\}$ converges to $d(x, y)$.*

Corollary 2.12 ([5])**.** *If $\{z_k\}$ is a sequence in a metric space (X, d) with the property that $d(z_k, z_{k+1}) < \frac{1}{2^k}$ for all k, then $\{z_k\}$ is a Cauchy sequence.*

Lemma 2.13. *Let (X, d) be a metric space and let A be a closed subset of X. If $\{a_n\}$ converges to x and $a_n \in A$ for all n, then $x \in A$.*

Proof. Suppose $\{a_n\}$ is a sequence that converges to x and $a_n \in A$ for all n. There are two cases to consider. If there exists a positive integer n such that $a_n = x$, then it is clear $x \in A$. If there does not exist a positive integer n such that $a_n = x$, then x is a limit point of A by Theorem 8.49 in [5]. Since A is closed, $x \in A$. □

3. Construction of the Hausdorff Metric

We now define the Hausdorff metric on the set of all nonempty, compact subsets of a metric space. Let (X, d) be a complete metric space and let κ be the collection of all nonempty compact subsets of X. Note that κ is closed under finite union and nonempty intersection. For $x \in X$ and $A, B \in \kappa$, define

$$r(x, B) = inf\{d(x, b) : b \in B\} \qquad and \qquad \rho(A, B) = sup\{r(a, B) : a \in A\}.$$

Note that r is nonnegative and exists by the completeness axiom, since $d(a, b) \geq 0$ by the definition of a metric space. Since r exists and is nonnegative, then both $\rho(A, B)$ and $\rho(B, A)$ exist and are nonnegative. In addition, we define the *Hausdorff distance* between sets A and B in κ as

$$h(A, B) = max\{\rho(A, B), \rho(B, A)\}.$$

Before proving that h defines a metric on the set κ, let us consider a few examples to get a grasp on how these distances work. Consider the following example of closed interval sets in (R, d), where $d(x, y) = |x - y|$.

Example 3.1. *Let $A = [0, 10]$ and let $B = [12, 21]$. We find that $r(x, B)$ is going to be the infimum of the set of distances from each $a \in A$ to the closest point in B. As an example of one of these distances, consider $a = 2$. Then $r(2, B) = inf\{d(2, b) : b \in B\} = d(2, 12) = 10$. We can note that for each $a \in A$, the closest point in B that gives the smallest distance will always be $b = 12$. Therefore, we find that $\rho(A, b) = sup\{d(a, 12) : a \in A\}$. The point $a = 0$ in A maximizes this distance. Therefore $\rho(A, B) = d(0, 12) = |12 - 0| = 12$.*
Similarly, we find that $\rho(B, A) = sup\{d(b, 10) : b \in B\}$, since the point $a = 10$ will give the smallest distance to any point in B. The point $b = 21$ in B maximizes this distance, so we have $\rho(B, A) = d(10, 21) = |10 - 21| = 11$. It follows that $h(A, B) = max\{\rho(A, B), \rho(B, A)\} = 12$.

Now that we have gained a knowledge on how r, ρ, and h work in a few special cases, we refer some basic properties of r and ρ.

Theorem 3.2 ([3])**.** *Let $x \in X$ and let $A, B, C \in \kappa$.*

(1). $r(x, A) = 0$ if and only if $x \in A$.

(2). $\rho(A, B) = 0$ if and only if $A \subseteq B$.

(3). There exists $a_x \in A$ such that $r(x, A) = d(x, a_x)$.

(4). There exists $a^* \in A$ *and* $b^* \in B$ *such that* $\rho(A,B) = d(a^*, b^*)$.

(5). If $A \subseteq B$, *then* $r(x,B) \leq r(x,A)$.

(6). If $B \subseteq C$, *then* $\rho(A,C) \leq \rho(A,B)$.

(7). $\rho(A \cup B, C) = max\{\rho(A,C), \rho(B,C)\}$.

(8). $\rho(A,B) \leq \rho(A,C) + \rho(C,B)$.

4. Hausdorff Metric Space

Normed linear space is a Hausdorff metric space equipped with the metric $d(x,y) = \|x - y\|$. A metric in a linear space defines a norm if it satisfies translational invariant $(d(x-z, y-z) = d(x,y))$ and homogeneity $(d(\lambda x, 0) = \lambda d(x,0))$. Given a complete metric space (X,d), we have now construction of new metric space (κ, h) from the nonempty, compact subsets of X using the Hausdorff distance. The following theorem shows Hausdorff distance defines a metric on κ.

Theorem 4.1 ([3]). *The set* κ *with the Hausdorff distance* h *define a metric space* (κ, h).

Proof. To prove that (κ, h) is a metric space, we need to verify the following four properties.

(1). $h(A,B) \geq 0$ for all $A, B \in \kappa$.

(2). $h(A,B) = 0$ if and only if $A = B$.

(3). $h(A,B) = h(B,A)$ for all $A, B \in \kappa$.

(4). $h(A,B) \leq h(A,C) + h(C,B)$ for all $A, B, C \in \kappa$.

To prove the first property, since $\rho(A,B)$ and $\rho(B,A)$ are nonnegative, it follows that $h(A,B) \geq 0$ for all $A, B \in \kappa$.
For the second property, suppose $A = B$. Therefore $A \subseteq B$ and $B \subseteq A$. By Property (2) of Theorem 2.4 we find that $\rho(A,B)$ and $\rho(B,A) = 0$, and thus $h(A,B) = 0$. Now suppose $h(A,B) = 0$. This implies $\rho(A,B) = \rho(B,A) = 0$. By property (2) of Theorem 3.2, we see that $A \subseteq B$ and $B \subseteq A$ and it follows that $A = B$.
The third property can be proved from the symmetry of the definition since

$$\begin{aligned} h(A,B) &= max\{\rho(A,B), \rho(B,A)\} \\ &= max\{\rho(B,A), \rho(A,B)\} \\ &= h(B,A). \end{aligned}$$

The final property follows from the definition of ρ and h and from property (8) of Theorem 3.2. We find that

$$\rho(A,B) \leq \rho(A,C) + \rho(C,B)$$

Similarly,

$$\rho(B,A) \leq \rho(B,C) + \rho(C,A)$$

Therefore, $h(A,B) = max\{\rho(A,B), \rho(B,A)\} \leq h(A,C) + h(C,B)$. □

Therefore we know that h defines a metric on κ. Hence it defines Hausdorff metric space (κ, h). In the next section, we will look at example of what this metric space might look like, and then one may proceed to prove if the metric space (X, d) is complete, then the metric space (κ, h) which is induced by Hausdorff distance is also complete.

Example 4.2. *Let (R, d_0) be the complete metric space, where d_0 is the discrete metric,*

$$d_0(x,y) = \begin{cases} 0, & when \quad x = y. \\ 1, & when \quad x \neq y. \end{cases}$$

Since κ is the set of all nonempty, compact subsets of (R, d_0), we find that κ is the set of all nonempty finite subsets of R. The infinite sets are not in κ because they are not totally bounded and are thus not compact. Furthermore, we may notice that

$$r(x,B) = inf\{d_0(x,b) : b \in B\} = d_0(x,y) = \begin{cases} 0, & when \quad x \in B. \\ 1, & when \quad x \notin B. \end{cases}$$

Therefore,

$$\rho(A,B) = sup\{r(a,B) : a \in A\} = \begin{cases} 0, & when \quad a \in B. \\ 1, & when \quad a \notin B. \end{cases}$$

So it follows that

$$h(A,B) = \begin{cases} 0, & when \quad A = B. \\ 1, & when \quad A \neq B. \end{cases}$$

Therefore we have a metric space with the set κ of the discrete subsets of R with the Hausdorff metric as the discrete metric. It is easy to verify that our newly created space is not totally bounded. However, we know all discrete metric spaces are complete, so (κ, h) is complete. Therefore, the space (κ, h) of finite sets with the discrete metric is an example of our Hausdorff induced metric space (κ, h).

To illustrate our notion of completeness, now briefly consider a sequence of nonempty compact sets that converges to the unit circle in R^2. This is an example a converging Cauchy sequence in the Hausdorff induced metric space that converges to a set also in the space.

5. Proving that the Hausdorff Metric Space (κ, h) is Complete

As previously stated, to be a complete metric space, every Cauchy sequence in (κ, h) must converge to a point in κ. Therefore, in order to prove that the metric space (κ, h) is complete, we will choose an arbitrary Cauchy sequence $\{A_n\}$ in κ and show that it converges to some $A \in \kappa$. Define A to be the set of all points $x \in X$ such that there is a sequence $\{x_n\}$ that converges to x and satisfies $x_n \in A_n$ for all n. We will eventually show that the set A is an appropriate candidate. However, we must begin with some important theorems regarding A. Given a set $A \in \kappa$ and a positive number ϵ, we define the set $A + \epsilon$ by $\{x \in X : r(x, A) \leq \epsilon\}$. We need to show that this set is closed for all possible choices of A and ϵ. To do this, we will begin by choosing an arbitrary limit point of the set, $A + \epsilon$, and then showing that it is contained in the set.

Proposition 5.1. *$A + \epsilon$ is closed for all possible choices of $A \in \kappa$ and $\epsilon > 0$.*

However, the following theorem gives us an alternative way of proving convergence.

Theorem 5.2 ([3]). *Suppose that $A, B \in \kappa$ and that $\epsilon > 0$. Then $h(A, B) \leq \epsilon$ if and only if $A \subseteq B + \epsilon$ and $B \subseteq A + \epsilon$.*

Extension Lemma: Let $\{A_n\}$ be a Cauchy sequence in κ and let $\{n_k\}$ be an increasing sequence of positive integers. If $\{x_{n_k}\}$ is a Cauchy sequence in X for which $x_{n_k} \in A_{n_k}$ for all k, then there exists a Cauchy sequence $\{y_n\}$ in X such that $y_n \in A_n$ for all n and $y_{n_k} = x_{n_k}$ for all k.

The following lemma makes use of the extension lemma to guarantee that A is closed and nonempty. We will need this fact in proving that A is in κ, since we must show that A is a nonempty, compact subset of κ. This lemma gives us that A is closed and nonempty. Since closed and totally bounded sets are compact, it remains to show that A is totally bounded.

Lemma 5.3 ([5])**.** *Let $\{A_n\}$ be a sequence in κ and let A be the set of all points $x \in X$ such that there is a sequence $\{x_n\}$ that converges to x and satisfies $x_n \in A_n$ for all n. If $\{A_n\}$ is a Cauchy sequence, then the set A is closed and nonempty.*

With the previous lemma, to prove $A \in \kappa$, it only remains to show that A is totally bounded. The following lemma will allow us to do so.

Lemma 5.4 ([5])**.** *Let $\{D_n\}$ be a sequence of totally bounded sets in X and let A be any subset of X. If for each $\epsilon > 0$, there exists a positive integer N such that $A \subseteq D_N + \epsilon$, then A is totally bounded.*

It gives the foundation to prove complete metric space (X, d), we constructed the metric space (κ, h) from the nonempty compact subsets of X using the Hausdorff metric. After examining important theorems and results, we can now state that

Theorem 5.5 ([3])**.** *If (X, d) is complete, then (κ, h) is complete.*

Proof. Let $\{A_n\}$ be a Cauchy sequence in κ, and define A to be the set of all points $x \in X$ such that there is a sequence $\{x_n\}$ that converges to x and satisfies $x_n \in A_n$ for all n. We must prove that $A \in \kappa$ and $\{A_n\}$ converges to A.

By Lemma 5.3, the set A is closed and nonempty. Let $\epsilon > 0$. Since $\{A_n\}$ is Cauchy sequence then there exists a positive integer N such that $h(A_n, A_m) < \epsilon$ for all $m, n \geq N$. $A_m \subseteq A_n + \epsilon$ for all $m > n \geq N$. Let $a \in A$, then we want to show $a \in A_n + \epsilon$. Fix $n \geq N$, by definition of the set A, there exists a sequence $\{x_i\}$ such that $x_i \in A_i$ for all i and $\{x_i\}$ converges to a. By Proposition 5.1 we know that $A_n + \epsilon$ is closed. Since $x_i \in A_n + \epsilon$ for each i, then it follows that $a \in A_n + \epsilon$. This shows that $A \subseteq A_n + \epsilon$. By lemma 5.4, the set A is totally bounded. Additionally, we know A is complete, since it is a closed subset of a complete metric space. Since A is nonempty, complete and totally bounded, then A is compact and thus $A \in \kappa$.

Let $\epsilon > 0$, to show that $\{A_n\}$ converges to $A \in \kappa$, we need to show that there exists a positive integer N such that $h(A_n, A) < \epsilon$ for all $n \geq N$. To do this, we know that $A \subseteq A_n + \epsilon$ and $A_n \subseteq A + \epsilon$. From the first part of our proof, we know there exists N such that $A \subseteq A_n + \epsilon$ for all $n \geq N$.

To prove $A_n \subseteq A + \epsilon$ let $\epsilon > 0$. Since $\{A_n\}$ is a Cauchy sequence, we can choose a positive integer N such that $h(A_m, A_n) < \frac{\epsilon}{2}$ for all $m, n \geq N$. Since $\{A_n\}$ is a Cauchy sequence in κ, there exists a strictly increasing sequence $\{n_i\}$ of positive integers such that $n_1 > N$ and such that $h(A_m, A_n) < \epsilon 2^{-i-1}$ for all $m, n > n_i$. We can use property (3) of Theorem 3.2 to get the following:

$$\begin{aligned} Since \;\; A_n &\subseteq A_{n_1} + \frac{\epsilon}{2}, \;\; \exists\; x_{n_1} \in A_{n_1} \ni d(y, \;\; x_{n_1}) \leq \frac{\epsilon}{2}, \\ since \;\; A_{n_1} &\subseteq A_{n_2} + \frac{\epsilon}{4}, \;\; \exists\; x_{n_2} \in A_{n_2} \ni d(x_{n_1}, x_{n_2}) \leq \frac{\epsilon}{4}, \\ since \;\; A_{n_2} &\subseteq A_{n_3} + \frac{\epsilon}{8}, \;\; \exists\; x_{n_3} \in A_{n_3} \ni d(x_{n_2}, x_{n_3}) \leq \frac{\epsilon}{8}, \cdots, \end{aligned}$$

by continuing this process we are able to obtain a sequence $\{x_{n_i}\}$ such that for all positive integers i then $x_{n_i} \in A_{n_i}$ and $d(x_{n_i}, x_{n_i+1}) \leq \epsilon 2^{-i-1}$. By corollary 2.12, we find x_{n_i} is a Cauchy sequence, so by the extension lemma the limit of the sequence a is in A. Additionally we find that

$$d(y, x_{n_i}) \leq d(y, x_{n_1}) + d(x_{n_1}, x_{n_2}) + d(x_{n_2}, x_{n_3}) + \cdots + d(x_{n_{i-1}}, x_{n_i}) \leq \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{8} + \cdots + \frac{\epsilon}{2^i} < \epsilon.$$

Since $d(y, x_{n_i}) \leq \epsilon$ for all i, it follows that $d(y, a) \leq \epsilon$ and therefore $y \in A + \epsilon$. Thus we know that there exists N such that $A_n \subseteq A + \epsilon$, so it follows that $h(A_n, A) < \epsilon$ for all $n \geq N$ and thus $\{A_n\}$ converges to $A \in \kappa$. Therefore, if (X, d) is complete, then (κ, h) is complete. □

6. Convex Hausdorff Metric Space as a Banach Space

Banach spaces are less special than Hilbert spaces but still sufficiently simple that their fundamental property can be explained readily several standard results which are true in greater generality have simpler and more transparent proofs in this setting. The Banach-Steinhaus uniform boundedness theorem and the Open Mapping Theorem are significantly more substantial than the first result here, since they invoke the Baire Category Theorem. Then Hahn Banach Theorem is non-trivial but does not use completeness.

Finally as made clear in work of Gelfand and Grothendieck and of many others, many subtler sorts of topological vector spaces are expressible as limit of Banach space, making clear that Banach spaces play an even more central role than would be apparent from many conventional elementary function analysis text. But, just to be on the safer side Banach introduced the axioms for vector space X (these were known at the time, but were apparently not considered well-known) and assumes that the spaces X carries a norm. Banach space named after great mathematician of twentieth century Banach and Banach space theory is presented in a broad mathematical context, using tools from such areas as set theory, topology, algebra, probability theory and logic. Equal emphasis is given to both spaces and operators. The standard notations of distance between two Banach spaces is the Banach measure distance and is given by $d(x, y) = \|x - y\|$.

In the year 1972, A.L. Brown has studied on the subspaces of Banach space[1]. Later N.J. Kalton and M.I. Ostorski worked on distance between two Banach space in 1997[4]. Russ Gordon, worked for real analysis[5]. Katie Barich, worked on Hausdorff distance[3]. Here, we considered the Hausdorff distance, for given normed linear space, Hausdorff metric space gives the completeness property which gives Banach structure.

Since, Barich proved the completeness of Hausdorff metric space i.e., X is complete[3]. Consider $\|.\| : X \to R$ be a normed function. In the first instance, let A and B be two nonempty sets in X. By definition, $h(A, B) = max\{\rho(A, B), \rho(B, A)\}$, but $\rho(A, B) = sup\{r(x, B) : \forall\, x \in A\}$; where $r(x, B) = inf\{\|x - b\| : \forall\, b \in B\}$ then, we have $\|x - b\| \geq 0$. Hence, $h(A, B) \geq 0$. In the second instance, we have $\rho(A, B) = sup\{r(x, B) : \forall\, x \in A\}$; where $r(x, B) = inf\{\|x - b\| : \forall\, b \in B\}$ then, we have $\|x - b\| = 0 \Rightarrow x = b$: x and b are the arbitrary elements then every element of A is element of B. $\Rightarrow A \subseteq B$. Similarly, $B \subseteq A$. Then we have $A = B$.

Conversely, If $A = B$, $\|x - b\| = 0\ \forall\, x \in A\ \&\ b \in B$:

We have $r(x, B) = inf\{\|x - b\| : \forall\, b \in B\} = 0 \quad \Rightarrow \quad \rho(A, B) = sup\{r(x, B) : \forall\, x \in A\} = 0$; Then we have $h(A, B) = 0$. Hence $h(A, B) = 0$ iff $A = B$. The third instance we have triangular inequality. By definition, $h(A, B) = max\{\rho(A, B), \rho(B, A)\}$, but $\rho(A, B) = sup\{r(x, B) : \forall x \in A\}$, where

$$
\begin{aligned}
r(x, b) &= inf\{\|x - b\| : \forall\, x \in A, b \in B\} \\
&\leq inf\{\|x - a + a - b\| : \forall\, a \in C,\, b \in B\} \\
&\leq inf\{\|x - a\| : \forall\, x \in A,\, a \in C\} + inf\,\{\|a - b\| : \forall\, a \in C,\, b \in B\,\} \\
&\leq r(x, a) + r(a, b) \\
\Rightarrow \qquad \rho(A, B) &\leq \rho(A, C) + \rho(C, B) \\
Hence, \qquad h(A, B) &\leq h(A, C) + h(C, B).
\end{aligned}
$$

In the last instance, we get $h(\alpha A, \alpha B) = |\alpha| \ inf \{\|x - b\| : \ \forall \ x \in A, \, b \in B\} = |\alpha| h(A, B)$. Hence we state that.

Theorem 6.1. *Let X be convex Hausdorff metric space. If X is normed linear space then it is Banach.*

7. Conclusion

Before proceeding, we would like to have a short digression. During the last several decades, there have been several distance functions that have been proposed purporting to capture the collective/topological properties of systems of many degrees of freedom. One motivation for the formulation of such distance functions, such as the Hausdorff distance is to determine the topological properties(mainly Geometric structure) of systems with long-range interactions. From the Felix Hausdorff viewpoint, this distance clearly gives the images/structures in this category. Assuming that such distance functions may prove to be applicable to a path-integral formulation of aspects of semi-classical or even quantum gravity, the arguments of the present work will still hold without any major modifications. The minor modification needed in case, such distance functions are pertinent, is to use in Banach spaces whose structure has been determined. One could possibly use the Hausdorff distance subject to appropriate structure of space-time, for such a purpose. A second minor, for our purposes, modification may be to substitute some other measure in the place of the often used metric measure. Beyond these points, we expect the above analysis (Theorem 6.1). Hausdorff distance, that defines a metric on the space of all nonempty, compact subsets of the convex metric space. The main object of this paper is to measure greatest distance that we called as Hausdorff distance between complete normed linear spaces i.e., Banach spaces. Only the construction of Banach space has been considered in this paper, but this simple technique can also be tackled straightforwardly by other constructions like Hilbert space. This task is left to the reader.

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