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On W_9-Curvature Tensor of Generalized Sasakian-Space-Forms

Research Article

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Abstract: The object of the present paper is to study generalized Sasakian-space-forms satisfying certain curvature conditions on W_9- curvature tensor. In this paper, we study W_9- semisymmetric, W_9-flat, $\xi - W_9-$ flat, generalized Sasakian-space-forms satisfying $I(\xi, X).S = 0$, $I(\xi, X).R = 0$, $I(\xi, X).P = 0$ and $I(\xi, X).\widetilde{C} = 0$.

MSC: 53C25, 53D15.

Keywords: Generalized Sasakian-space form, W_9- curvature tensor, Concircular curvature tensor, Ricci tensor, $\eta-$Einstein Manifold, scalar curvature.

1. Introduction

P. Alegre, D. Blair and A. Carriazo [9] introduced and studied generalized Sasakian-space-forms. In 2011, M.M. Tripathi and P. Gupta [7] introduced and studied $\tau-$ curvature tensor in semi -Riemannian manifolds. They studied some properties of $\tau-$ curvature tensor. They defined W_9- curvature tensor of type $(0, 4)$ for $(2n+1)-$dimensional Riemannian manifold, as

$$W_9(X, Y, Z, U) = R(X, Y, Z, U) - \frac{1}{2n}\{S(X, Y)g(Z, U) - g(Y, Z)S(X, U)\} \tag{1}$$

where R and S denote the Riemannian curvature tensor of type $(0, 4)$ defined by ‘$R(X, Y, Z, U) = g(R(X, Y)Z, U)$ and the Ricci tensor of type $(0, 2)$ respectively. The curvature tensor defined by (1) is known as W_9- curvature tensor. A manifold whose W_9- curvature tensor vanishes at every point of the manifold is called W_9- flat manifold. They also define $\tau-$conservative semi-Riemannian manifolds and give necessary and sufficient condition for semi-Riemannian manifolds to be $\tau-$ conservative. Given an almost contact metric manifold $M(\phi, \xi, \eta, g)$, we say that M is generalized Sasakian-space-form if there exist three functions f_1, f_2, f_3 on M such that the curvature tensor R is given by

$$\begin{aligned} R(X, Y)Z = f_1\{g(Y, Z)X - g(X, Z)Y\} + f_2\{g(X, \Phi Z)\Phi Y - g(Y, \Phi Z)\Phi X + 2g(X, \Phi Y)\Phi Z\} \\ + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\} \end{aligned} \tag{2}$$

for any vector fields X, Y, Z on M. In such a case we denote the manifold as $M(f_1, f_2, f_3)$. In [8] the authors cited several examples of generalized Sasakian-space-forms. Alegre et al. [10] have given results on B.Y. Chen's inequality on submanifolds

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of generalized complex space-forms and generalized Sasakian-space-forms. Al. Ghefari et al. analyse the CR submanifolds of generalized Sasakian-space-forms [11, 12]. Sreenivasa. G.T. Venkatesha and Bagewadi C.S. [13] have studied some results on $(LCS)_{2n+1}-$Manifolds. S. K. Yadav, P.K. Dwivedi and D. Suthar [14] studied $(LCS)_{2n+1}-$ Manifolds satisfying certain conditions on the concircular curvature tensor. De and Sarakar [15] have studied generalized Sasakian-space-forms regarding projective curvature tensor. Motivated by the above studies, in the present paper, we study flatness and symmetry property of generalized Sasakian-space-forms regarding W_9-curvature tensor. The present paper is organized as follows:

In this paper, we study the W_9-curvature tensor of generalized Sasakian-space-forms with certain conditions. In section 2, some preliminary results are recalled. In section 3, we study W_9- semisymmetric generalized Sasakian-space-forms. Section 4 deals with $\xi - W_9$ flat generalized Sasakian-space-forms. Generalized Sasakian-space-forms satisfying $I.S = 0$ are studied in section 5. In section 6, W_9- flat generalized Sasakian-space-forms are studied. Section 7 is devoted to study of generalized Sasakian-space-forms satisfying $I.R = 0$. In section 8, generalized Sasakian-space-forms satisfying $I.P = 0$. The last section contains generalized Sasakian-space-forms satisfying $I.\widetilde{C} = 0$.

2. Preliminaries

An odd $-$ dimensional differentiable manifold M^{2n+1} of differentiability class C^{r+1}, there exists a vector valued real linear function Φ, a 1-form η, associated vector field ξ and the Riemannian metric g satisfying

$$\Phi^2(X) = -X + \eta(X)\xi, \Phi(\xi) = 0 \tag{3}$$

$$\eta(\xi) = 1, g(X, \xi) = \eta(X), \eta(\Phi X) = 0 \tag{4}$$

$$g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{5}$$

for arbitary vector fields X and Y, then (M^{2n+1}, g) is said to be an almost contact metric manifod [4], and the structure (Φ, ξ, η, g) is called an almost contact metric structure to M^{2n+1}. In view of (3), (4) and (5), we have

$$g(\Phi X, Y) = -g(X, \Phi Y), g(\Phi X, X) = 0 \tag{6}$$

$$\nabla_X \eta(Y) = g(\nabla_X \xi, Y) \tag{7}$$

Again we know [9] that in a $(2n + 1)-$ dimensional generalized Sasakian-space-form, we have

$$\begin{aligned} R(X, Y)Z = {} & f_1\{g(Y, Z)X - g(X, Z)Y\} + f_2\{g(X, \Phi Z)\Phi Y - g(Y, \Phi Z)\Phi X + 2g(X, \Phi Y)\Phi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\} \end{aligned} \tag{8}$$

for any vector field X, Y, Z on M^{2n+1}, where R denotes the curvature tensor of M^{2n+1} and f_1, f_2, f_3 are smooth functions on the manifold. The Ricci tensor S and the scalar curvature r of the manifold of dimension $(2n+1)$ are respectively, given by

$$S(X, Y) = (2nf_1 + 3f_2 - f_3)g(X, Y) - (3f_2 + (2n - 1)f_3)\eta(X)\eta(Y) \tag{9}$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n - 1)f_3)\eta(X)\xi \tag{10}$$

$$r = 2n(2n + 1)f_1 + 6nf_2 - 4nf_3 \tag{11}$$

Also for a generalized Sasakian-space-forms, we have

$$R(X,Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\} \tag{12}$$

$$R(\xi,X)Y = -R(X,\xi)Y = (f_1 - f_3)\{g(X,Y)\xi - \eta(Y)X\} \tag{13}$$

$$\eta(R(X,Y)Z) = (f_1 - f_3)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} \tag{14}$$

$$S(X,\xi) = 2n(f_1 - f_3)\eta(X) \tag{15}$$

$$Q\xi = 2n(f_1 - f_3)\xi \tag{16}$$

where Q is the Ricci Operator, i.e.

$$g(QX,Y) = S(X,Y) \tag{17}$$

For a $(2n+1)-$ dimensional $(n>1)$ Almost Contact Metric, the W_9- curvature tensor I is given by

$$I(X,Y)Z = R(X,Y)Z - \frac{1}{2n}\{S(X,Y)Z - g(Y,Z)QX\} \tag{18}$$

The W_9- curvature tensor I in a generalized Sasakian-space-form satisfies

$$I(X,Y)\xi = (f_1 - f_3)(\eta(Y)X - \eta(X)Y) - \frac{1}{2n}\{(2nf_1 + 3f_2 - f_3)(g(X,Y)\xi - \eta(Y)X)\} \tag{19}$$

$$I(\xi,Y)\xi = (f_1 - f_3)\{\eta(Y)\xi - Y\}$$

$$I(X,\xi)\xi = \frac{1}{2n}(4nf_1 + 3f_2 - (2n+1)f_3)(X - \eta(X)\xi) \tag{20}$$

$$I(\xi,X)Y = (f_1 - f_3)\{2g(X,Y)\xi - \eta(X)Y - \eta(Y)X\} \tag{21}$$

$$I(\xi,X)\xi = (f_1 - f_3)\{\eta(X)\xi - X\}$$

Given an $(2n+1)-$ dimensional Riemannian manifold (M,g), the Concircular curvature tensor $\widetilde{C}$ is given by

$$\widetilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}\{g(Y,Z)X - g(X,Z)Y\} \tag{22}$$

$$\widetilde{C}(\xi,X)Y = [f_1 - f_3 - \frac{r}{2n(2n+1)}]\{g(X,Y)\xi - \eta(Y)X\} \tag{23}$$

and

$$\eta(\widetilde{C}(X,Y)Z) = [f_1 - f_3 - \frac{r}{2n(2n+1)}]\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} \tag{24}$$

and Projective curvature tensor is given by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y] \tag{25}$$

and related term

$$\eta(P(X,Y)\xi) = 0 \tag{26}$$

$$\eta(P(X,\xi)Z) = \frac{1}{2n}S(X,Z) - (f_1 - f_3)g(X,Z) \tag{27}$$

$$\eta(P(\xi,Y)Z) = (f_1 - f_3)g(Y,Z) - \frac{1}{2n}S(Y,Z) \tag{28}$$

for any vector field X, Y, Z on M.

3. W_9- Semisymmetric Generalized Sasakian-Space-Forms

Definition 3.1. *A $(2n+1)-$ dimensional $(n>1)$ generalized Sasakian-space-form is said to be W_9- semisymmetric if it satisfies $R.I=0$, where R is the Riemannian curvature tensor and I is the W_9- curvature tensor of the space forms.*

Theorem 3.2. *A $(2n+1)-$ dimensional $(n>1)$ generalized Sasakian-space-form is W_9- semisymmetric if and only if $f_1=f_3$.*

Proof. Let us suppose that the generalized Sasakian-space-form $M(f_1,f_2,f_3)$ is W_9- semisymmetric, then we have

$$R(\xi,U)I(X,Y)\xi=0 \tag{29}$$

The above equation can be written as

$$R(\xi,U)I(X,Y)\xi-I(R(\xi,U)X,Y)\xi-I(X,R(\xi,U)Y)\xi-I(X,Y)R(\xi,U)\xi=0 \tag{30}$$

In view of (4), (12) and (13) the above equation reduces to

$$\begin{aligned}(f_1-f_3)\{g(U,I(X,Y)\xi)\xi-\eta(I(X,Y)\xi)U-g(X,U)I(\xi,Y)\xi+\eta(X)I(U,Y)\xi\\ -g(U,Y)I(X,\xi)\xi+I(X,U)\eta(Y)\xi-I(X,Y)\eta(U)\xi+I(X,Y)U\}=0\end{aligned} \tag{31}$$

In view of (18), (19) and (20) and taking the inner product of above equation with ξ, we get

$$\begin{aligned}(f_1-f_3)\{g(U,I(X,Y)\xi)-\frac{1}{2n}(2nf_1+3f_2-f_3)(-g(X,Y)\eta(U)\\ +g(U,Y)\eta(X)+g(X,U)\eta(Y)-g(X,Y)\eta(U)+\eta(I(X,Y)U)\}=0\end{aligned} \tag{32}$$

On solving above equation, we get

$$\frac{1}{2n}(f_1-f_3)\{(3f_2+(2n-1)f_3)(g(Y,U)\eta(X)-\eta(X)\eta(Y)\eta(U))\}=0 \tag{33}$$

From the above equation, we have either $f_1=f_3$ or

$$g(Y,U)\eta(X)-\eta(X)\eta(Y)\eta(U)=0 \tag{34}$$

which is not possible in generalized Sasakian-space-form. Conversely, if $f_1=f_3$, then from (13), $R(\xi,U)=0$.Then obviously $R.I=0$ is satisfied. This completes the proof. □

4. $\xi-W_9-$ Flat Generalized Sasakian-Space-Forms

Definition 4.1. *A $(2n+1)-$ dimensional $(n>1)$ generalized Sasakian-space-form is said to be W_9- flat* [5] *if $I(X,Y)\xi=0$ for all $X,Y\in TM$.*

Theorem 4.2. *A $(2n+1)-$ dimensional $(n>1)$ generalized Sasakian-space-form is $\xi-W_9-$ flat if and only if it is $\eta-$ Einstein Manifold.*

Proof. Let us consider that a generalized Sasakian-space-form is $\xi - W_9-$ flat, i.e. $I(X,Y)\xi = 0$. Then in view of (18), we have

$$R(X,Y)\xi = \frac{1}{2n}\{S(X,Y)\xi - g(Y,\xi)QX\} \tag{35}$$
$$R(X,Y)\xi = \frac{1}{2n}\{S(X,Y)\xi - \eta(Y)QX\} \tag{36}$$

By using (12) and (14) above equation becomes

$$\eta(Y)QX = (2nf_1 + 3f_2 - f_3)g(X,Y)\xi - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y)\xi - 2n(f_1 - f_3)(\eta(Y)X - \eta(X)Y) \tag{37}$$

Putting $Y = \xi$ in above equation, we get

$$QX = 2n(f_1 - f_3)(2\eta(X)\xi - X) \tag{38}$$

Now, taking the inner product of the above equation with U, we get

$$S(X,U) = 2n(f_1 - f_3)\{g(X,U) - 2\eta(X)\eta(U)\} \tag{39}$$

which shows that generalised Sasakian-space-form is an $\eta-$Einstein Manifold. Conversely, suppose that (39) is satisfied. Then by virtue of (35) and (38), we get $I(X,Y)\xi = 0$. □

5. Generalized Sasakian-Space-Form Satisfying $I.S = 0$

Theorem 5.1. *A $(2n+1)-$dimension $(n > 1)$ generalised Sasakian-space-form satisfying $I.S = 0$ is an $\eta-$Einstein Manifold.*

Proof. Let us consider generalised Sasakian-space-form M^{2n+1} satisfying $I(\xi,X).S = 0$. In this case, we can write $S(I(\xi,X)Y,Z) + S(Y,I(\xi,X)Z) = 0$ for any vector fields X,Y,Z on M. Substituting (21) in above equation, we obtain

$$2g(X,Y)S(\xi,Z) - \eta(X)S(Y,Z) - \eta(Y)S(X,Z) + 2S(Y,\xi)g(X,Z) - \eta(X)S(Y,Z) - \eta(Z)S(Y,X) = 0 \tag{40}$$

For $Z = \xi$, the last equation is equivalent to

$$2.2n(f_1 - f_3)g(X,Y) - 2n(f_1 - f_3)\eta(X)\eta(Y) - S(Y,X) = 0 \tag{41}$$

which implies that,

$$S(X,Y) = 2n(f_1 - f_3)\{2g(X,Y) - \eta(X)\eta(Y)\} \tag{42}$$

This proves our assertion. □

6. W_9- flat Generalized Sasakian-space-forms

Theorem 6.1. *A $(2n+1)-$ dimensional $(n > 1)$ generalized Sasakian-space-form is W_9- flat if and only if $f_1 = \frac{3f_2}{(1-2n)} = f_3$.*

Proof. For a $(2n+1)-$ dimensional $(n > 1)$ W_9- flat generalized Sasakian-space-forms, we have from (18)

$$R(X,Y)Z = \frac{1}{2n}\{S(X,Y)Z - g(Y,Z)QX\} \tag{43}$$

In view of (9) and (10), the above equation takes the form

$$R(X,Y)Z = \frac{1}{2n}\{(2nf_1+3f_2-f_3)(g(X,Y)Z-g(Y,Z)X)-(3f_2+(2n-1)f_3)(\eta(X)\eta(Y)Z+g(Y,Z)\eta(X)\xi)\} \tag{44}$$

By virtue of (8) the above equation reduces to

$$\begin{aligned} &f_1\{g(Y,Z)X-g(X,Z)Y\}+f_2\{g(X,\phi Z)\phi Y-g(Y,\phi Z)\phi X+2g(X,\phi Y)\phi Z\} \\ &\quad + f_3\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi\} \\ &\quad = \frac{1}{2n}\{(2nf_1+3f_2-f_3)(g(X,Y)Z-g(Y,Z)X)-(3f_2+(2n-1)f_3)(\eta(X)\eta(Y)Z+g(Y,Z)\eta(X)\xi)\} \end{aligned} \tag{45}$$

Now, replacing Z by ϕZ in the above equation, we obtain

$$\begin{aligned} &f_1\{g(Y,\phi Z)X-g(X,\phi Z)Y\}+f_2\{g(X,\phi^2 Z)\phi Y-g(Y,\phi^2 Z)\phi X+2g(X,\phi Y)\phi^2 Z\}+f_3\{g(X,\phi Z)\eta(Y)\xi-g(Y,\phi Z)\eta(X)\xi\} \\ &= \frac{1}{2n}\{(2nf_1+3f_2-f_3)(g(X,Y)\phi Z-g(Y,\phi Z)X)-(3f_2+(2n-1)f_3)(\eta(X)\eta(Y)\phi Z+g(Y,\phi Z)\eta(X)\xi)\} \end{aligned} \tag{46}$$

Taking inner product of above equation with ξ, we get

$$\begin{aligned} &f_1\{g(Y,\phi Z)\eta(X)-g(X,\phi Z)\eta(Y)\}+f_3\{g(X,\phi Z)\eta(Y)-g(Y,\phi Z)\eta(X)\} \\ &\quad = \frac{1}{2n}\{(2nf_1+3f_2-f_3)(-g(Y,\phi Z)\eta(X))-(3f_2+(2n-1)f_3)g(Y,\phi Z)\eta(X)\} \end{aligned} \tag{47}$$

Putting $X=\xi$ in above equation, we get

$$(4nf_1+6f_2-2f_3)g(Y,\phi Z)=0 \tag{48}$$

Since $g(Y,\phi Z)\neq 0$ in general, we obtain

$$4nf_1+6f_2-2f_3=0 \tag{49}$$

Again replacing X by ϕX in equation (45), we get

$$\begin{aligned} &f_1\{g(Y,Z)\phi X-g(\phi X,Z)Y\}+f_2\{g(\phi X,\phi Z)\phi Y-g(Y,\phi Z)\phi^2 X+2g(\phi X,\phi Y)\phi Z\}+f_3\{-\eta(Y)\eta(Z)\phi X+g(\phi X,Z)\eta(Y)\xi\} \\ &\quad = \frac{1}{2n}\{(2nf_1+3f_2-f_3)(g(\phi X,Y)Z-g(Y,Z)\phi X)\} \end{aligned} \tag{50}$$

Taking inner product with ξ

$$f_1\{-g(\phi X,Z)\eta(Y)\}+f_3 g(\phi X,Z)\eta(Y)=\frac{1}{2n}(2nf_1+3f_2-f_3)g(\phi X,Y)\eta(Z) \tag{51}$$

putting $Y=\xi$, we get

$$(f_1-f_3)g(\Phi X,Z)=0 \tag{52}$$

Since $g(\phi X,Z)\neq 0$ in general, we obtain

$$f_3=f_1 \tag{53}$$

From equation (49) and (53), we get

$$f_1=\frac{3f_2}{1-2n}=f_3 \tag{54}$$

Conversely, suppose that $f_1 = \frac{3f_2}{1-2n} = f_3$ satisfies in generalized Sasakian-space-form and then we have

$$S(X, Y) = 0, \tag{55}$$

$$QX = 0 \tag{56}$$

Also, in view of (18), we have

$$I(X, Y, Z, U) = `R(X, Y, Z, U) \tag{57}$$

where $I(X, Y, Z, U) = g(X, Y, Z, U)$ and $`R(X, Y, Z, U) = g(X, Y, Z, U)$. Putting $Y = Z = e_i$ in above equation and taking summation over $i, 1 \leq i \leq 2n + 1$, we get

$$\sum_{i=1}^{2n+1} I(X, e_i, e_i, U) = \sum_{i=1}^{2n+1} `R(X, e_i, e_i, U) = S(X, U) \tag{58}$$

In view of (8) and (58), we have

$$I(X, Y, Z, U) = f_1\{g(Y, Z)g(X, U) - g(X, Z)g(Y, U)\} + f_2\{g(X, \phi Z)g(\phi Y, U) - g(Y, \phi Z)g(\phi X, U) + 2g(X, \phi Y)g(\phi Z, U)\}$$
$$+ f_3\{\eta(X)\eta(Z)g(Y, U) - \eta(Y)\eta(Z)g(X, U) + g(X, Z)\eta(Y)\eta(U) - g(Y, Z)\eta(X)\eta(U)\} \tag{59}$$

Now, putting $Y = Z = e_i$ in above equation and taking summation over $i, 1 \leq i \leq 2n + 1$, we get

$$\sum_{i=1}^{2n+1} I(X, e_i, e_i, U) = 2nf_1g(X, U) + 3f_2g(\phi X, \phi U) - f_3\{(2n + 1)\eta(X)\eta(U) + g(X, U)\} \tag{60}$$

In view of (55), (56) and (58), we have

$$2nf_1g(X, U) + 3f_2g(\phi X, \phi U) - f_3\{(2n + 1)\eta(X)\eta(U) + g(X, U)\} = 0 \tag{61}$$

Putting $X = U = e_i$ in above equation and taking summation over $i, 1 \leq i \leq 2n + 1$, we get $f_1 = 0$. Then in view of (54), $f_2 = f_3 = 0$. Therefore, we obtain from (8)

$$R(X, Y)Z = 0 \tag{62}$$

Hence in view of (55), (56) and (62), we have $I(X, Y)Z = 0$. This completes the proof. □

7. Generalized Sasakian-space-forms Satisfying $I.R = 0$

Theorem 7.1. *A generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ satisfies the condition $I(\xi, X).R = 0$ if and only if the functions f_1 and f_3 has the sectional curvature $(f_1 - f_3)$.*

Proof. Let generalized Sasakian-space-form satisfying

$$I(\xi, X)R(Y, Z)U = 0 \tag{63}$$

This can be written as

$$I(\xi, X)R(Y, Z)U - R(I(\xi, X)Y, Z)U - R(Y, I(\xi, X)Z)U - R(Y, Z)I(\xi, X)U = 0 \tag{64}$$

for any vector fields X, Y, Z, U on M. In view of (21), we obtain

$$I(\xi, X)R(Y, Z)U = (f_1 - f_3)\{2g(X, R(Y, Z)U)\xi - \eta(X)R(Y, Z)U - \eta(R(Y, Z)U)X\} \tag{65}$$

On the other hand, by direct calculations, we have

$$R(I(\xi, X)Y, Z)U = (f_1 - f_3)\{2g(X, Y)R(\xi, Z)U - \eta(X)R(Y, Z)U - \eta(Y)R(X, Z)U\} \tag{66}$$

$$R(Y, I(\xi, X)Z)U = (f_1 - f_3)\{2g(X, Z)R(Y, \xi)U - \eta(X)R(Y, Z)U - \eta(Z)R(Y, X)U\} \tag{67}$$

$$R(Y, Z)I(\xi, X)U = (f_1 - f_3)\{2g(X, U)R(Y, Z)\xi - \eta(X)R(Y, Z)U - \eta(U)R(Y, Z)X\} \tag{68}$$

Substituting (64), (65), (66) and (67) in (63), we get

$$\begin{aligned}&(f_1 - f_3)\{2g(X, R(Y, Z)U)\xi - \eta(X)R(Y, Z)U - \eta(R(Y, Z)U)X - 2g(X, Y)R(\xi, Z)U + \eta(X)R(Y, Z)U + \eta(Y)R(X, Z)U \\ &- 2g(X, Z)R(Y, \xi)U + \eta(X)R(Y, Z)U + \eta(Z)R(Y, X)U - 2g(X, U)R(Y, Z)\xi + \eta(X)R(Y, Z)U + \eta(U)R(Y, Z)X = 0\end{aligned} \tag{69}$$

Taking inner product with ξ, above equation implies that

$$\begin{aligned}&(f_1 - f_3)\{2g(X, R(Y, Z)U) - \eta(X)\eta(R(Y, Z)U) - 2g(X, Y)\eta(R(\xi, Z)U) + \eta(Y)\eta(R(X, Z)U) - 2g(X, Z)\eta(R(Y, \xi)U) \\ &+ 2\eta(X)\eta(R(Y, Z)U) + \eta(Z)\eta(R(Y, X)U) - 2g(X, U)\eta(R(Y, Z)\xi) + \eta(U)\eta(R(Y, Z)X) = 0\end{aligned} \tag{70}$$

In consequence of (8), (12), (13) and (14) the above equation takes the form

$$2g(X, R(Y, Z)U - 2(f_1 - f_3)(g(X, Y)g(Z, U) - g(X, Z)g(Y, U)) + (f_1 - f_3)(g(X, Y)\eta(Z)\eta(U) - g(X, Z)\eta(Y)\eta(U)\} = 0$$

On solving, we get $2g(X, R(Y, Z)U) - (f_1 - f_3)(g(X, Y)g(Z, U) - g(X, Z)g(Y, U) = 0$, which say us $M^{2n+1}(f_1, f_2, f_3)$ has the sectional curvature $(f_1 - f_3)$. □

8. Generalized Sasakian-space-forms satisfying $I.P = 0$

Theorem 8.1. *A generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ satisfies the condition $I(\xi, X).P = 0$ if and only if $M^{2n+1}(f_1, f_2, f_3)$ has the sectional curvature of the form $(f_1 - f_3)$.*

Proof. The condition $I(\xi, X)P = 0$ implies that

$$(I(\xi, X)P)(Y, Z, U) = I(\xi, X)P(Y, Z)U - P(I(\xi, X)Y, Z)U - P(Y, I(\xi, X)Z)U - P(Y, Z)I(\xi, X)U = 0 \tag{71}$$

for any vector fields X, Y, Z on M. In view of (10), we obtain from (27)

$$\eta(P(X, Y)Z) = 0 \tag{72}$$

Since,

$$I(\xi, X)P(Y, Z)U = (f_1 - f_3)\{2g(X, P(Y, Z)U)\xi - \eta(X)P(Y, Z)U\} \tag{73}$$

$$P(I(\xi, X)Y, Z)U = (f_1 - f_3)\{2g(X, Y)P(\xi, Z)U - \eta(X)P(Y, Z)U) - \eta(Y)P(X, Z)U\} \tag{74}$$

$$P(Y, I(\xi, X)Z)U = (f_1 - f_3)\{2g(X, Z)P(Y, \xi)U - \eta(X)P(Y, Z)U - \eta(Z)P(Y, X)U\} \tag{75}$$

Finally, we conclude that

$$P(Y,Z)I(\xi,X)U = (f_1 - f_3)\{2g(X,U)P(Y,Z)\xi - \eta(X)P(Y,Z)U - \eta(U)P(Y,Z)X\} \tag{76}$$

So, substituting (73), (74), (75) and (76) in (63), we deduce that

$$\begin{aligned}(f_1 - f_3)\{&2g(X,P(Y,Z)U)\xi - \eta(X)P(Y,Z)U - 2g(X,Y)P(\xi,Z)U + \eta(X)P(Y,Z)U) + \eta(Y)P(X,Z)U - 2g(X,Z)P(Y,\xi)U \\ &+ \eta(X)P(Y,Z)U + \eta(Z)P(Y,X)U - 2g(X,U)P(Y,Z)\xi + \eta(X)P(Y,Z)U + \eta(U)P(Y,Z)X\} = 0 \end{aligned} \tag{77}$$

Taking inner product with ξ, we get

$$(f_1 - f_3)\{g(X,R(Y,Z)U) - (f_1 - f_3)(g(X,Y)g(Z,U) - g(X,Z)g(Y,U))\} = 0$$

which say us $M^{2n+1}(f_1, f_2, f_3)$ has the sectional curvature $(f_1 - f_3)$. □

9. Generalized Sasakian-space-forms Satisfying $I.\widetilde{C} = 0$

Theorem 9.1. *A generalized Sasakian-space-forms $M^{2n+1}(f_1, f_2, f_3)$ satisfies the condition $I(\xi, X).\widetilde{C} = 0$ if and only if either the scalar curvature τ of $M^{2n+1}(f_1, f_2, f_3)$ is $\tau = 8n(2n+1)(f_1 - f_3)$ or a real space form with the sectional curvature $(f_1 - f_3)$.*

Proof. The condition $I(\xi, X).\widetilde{C} = 0$ implies that

$$(I(\xi,X)\widetilde{C})(Y,Z,U) = I(\xi,X)\widetilde{C}(Y,Z)U - \widetilde{C}(I(\xi,X)Y,Z)U - \widetilde{C}(Y,I(\xi,X)Z)U - \widetilde{C}(Y,Z)I(\xi,X)U = 0 \tag{78}$$

for any vector fields X, Y, Z and U on M. From (22) and (23), we can easily to see that

$$I(\xi,X)\widetilde{C}(Y,Z)U = (f_1 - f_3)\{2g(X,\widetilde{C}(Y,Z)U)\xi - \eta(X)\widetilde{C}(Y,Z)U - \eta(\widetilde{C}(Y,Z)U)X\} \tag{79}$$

$$\widetilde{C}(I(\xi,X)Y,Z)U = (f_1 - f_3)\{2g(X,Y)\widetilde{C}(\xi,Z)U - \eta(X)\widetilde{C}(Y,Z)U - \eta(Y)\widetilde{C}(X,Z)U\} \tag{80}$$

$$\widetilde{C}(Y,I(\xi,X)Z)U = (f_1 - f_3)\{2g(X,Z)\widetilde{C}(Y,\xi)U - \eta(X)\widetilde{C}(Y,Z)U - \eta(Z)\widetilde{C}(Y,X)U\} \tag{81}$$

and

$$\widetilde{C}(Y,Z)I(\xi,X)U = (f_1 - f_3)\{2g(X,U)\widetilde{C}(Y,Z)\xi - \eta(X)\widetilde{C}(Y,Z)U - \eta(U)\widetilde{C}(Y,Z)X\} \tag{82}$$

Thus, substituting (79), (80), (81) and (82) in (78) and after from necessary abbreviations, (78) takes from

$$\begin{aligned}(f_1 - f_3)\{&2g(X,\widetilde{C}(Y,Z)U)\xi - \eta(X)\widetilde{C}(Y,Z)U - \eta(\widetilde{C}(Y,Z)U)X - 2g(X,Y)\widetilde{C}(\xi,Z)U + \eta(X)\widetilde{C}(Y,Z)U + \eta(Y)\widetilde{C}(X,Z)U \\ &- 2g(X,Z)\widetilde{C}(Y,\xi)U + \eta(X)\widetilde{C}(Y,Z)U + \eta(Z)\widetilde{C}(Y,X)U - 2g(X,U)\widetilde{C}(Y,Z)\xi + \eta(X)\widetilde{C}(Y,Z)U + \eta(U)\widetilde{C}(Y,Z)X\} = 0 \end{aligned} \tag{83}$$

Taking inner product with ξ and solving

$$\begin{aligned}(f_1 - f_3)\{&2g(X,R(Y,Z)U) + (f_1 - f_3)(g(Z,U)g(X,Y) - g(Y,U)g(X,Z)) + \left(f_1 - f_3 - \frac{\tau}{2n(2n+1)}\right)(2g(Z,U)\eta(X)\eta(Y) \\ &- 2g(Y,U)\eta(X)\eta(Z) + g(X,Y)\eta(Z)\eta(U) - g(X,Z)\eta(Y)\eta(U))\} = 0 \end{aligned} \tag{84}$$

Now putting $U = \xi$ in the above equation, we get

$$(f_1 - f_3)(4(f_1 - f_3) - \frac{\tau}{2n(2n+1)})\{g(X,Y)\eta(Z) - g(X,Z)\eta(Y)\} = 0$$

Above equation tells us that $M^{2n+1}(f_1, f_2, f_3)$ has the scalar curvature $\tau = 8n(2n+1)(f_1 - f_3)$.

Conversely, if $M^{2n+1}(f_1, f_2, f_3)$ is either real space form with sectional curvature $(f_1 - f_3)$ or it has the scalar curvature $\tau = 8n(2n+1)(f_1 - f_3)$. This completes the proof. □

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