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A Comparative Study of Jacobi Method and Givens Method for Finding Eigenvalues and Eigenvectors of a Real Symmetric Matrices

Research Article

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Abstract: The aim of this paper is to compare the Jacobi method and the Givens method for finding the eigenvalues and the corresponding eigenvectors of a real symmetric matrices. Finally, we have seen that with examples Givens method is non iterative and more efficient than Jacobi method, although it requires the given symmetric matrix into a tridiagonal matrix having the same eigenvalues.

Keywords: Eigenvalues and eigenvectors, Jacobi method, Givens method, Symmetric matrix, Bisection method.

1. Introduction

In literature [1–3], there exists several methods to compute the eigenvalues of a given real symmetric matrix. The eigenvalues plays important role in computer engineering and control engineering. The Jacobi method [1, 4] uses plane rotations in each step to compute the eigenvalues of a given real symmetric matrix. The rotation is applied till the off-diagonal elements zero. The principal diagonal elements are the eigenvalues of the matrix. In Givens method [1, 4] we tridiagonalise the given real symmetric matrix A by employing the orthogonal matrices. Tridiagonalise is that form in which the only non-zero elements are on the principal diagonal and the two diagonals just above and below of principal diagonal. Solving tridiagonal linear systems [5, 6] is one of the most important problems in scientific computing. It is involved in the solution of differential equations and in various areas of science and engineering applications such as control system and computer science. There are various numerical techniques available in the literature [7–9] which are useful for determining eigenvalues of a real symmetric matrices. In most of these methods, the given real symmetric matrix is converted into tridiagonal form. In this method, Sturm sequence and bisection method is used to determine the eigenvalues of a given real symmetric matrix. One of the leading methods for computing the eigenvalues of a real symmetric matrix is Givens method. In that method, after transforming the matrix into tridiagonal form say, 'S', the leading principal minors of $|S-\lambda I|$ form a Sturm sequence. Then, using bisection approach, change of sign in various Sturm sequence is observed. Further, based on this, eigenvalue can be determined by repeatedly using bisection method. In order to show the comparative result, we have considered the example for the illustration of Jacobi method and Givens method.

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2. Preliminaries

In this section, we recall some basic concepts which would be used in the sequel.

Definition 2.1. *A square matrix A is said to be symmetric if $A^T = A$.*

Definition 2.2. *A square matrix A of order n is said to be orthogonal if $AA^T = I_n = A^T A$.*

Definition 2.3. *Let $A = [a_{ij}]_{n\times n}$ be a given square matrix. If there exists a number λ and a non zero vector X such that*

$$AX = \lambda X \tag{1}$$

Then λ is termed as eigenvalue or latent root or characteristic value and X is termed as the corresponding eigenvector or characteristic vector of the matrix A.

Equation (1) can be written as

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix} X = 0 \tag{2}$$

This is a homogeneous system of n linear equations. It will have a non-trivial solution if and only if $|A-\lambda I|$ vanishes, i.e., if

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix} = 0$$

On expanding the determinant we can get an algebraic equation of degree n in λ, i.e., we get

$$\lambda^n - (a_{11} + a_{22} + \cdots + a_{nn})\lambda^{n-1} + \cdots + (-1)^n |A| = 0 \tag{3}$$

This equation is termed as characteristic equation. It will have n roots (n value of λ) say $\lambda_1, \lambda_2, \ldots, \lambda_n$. These are the values of λ for which the system (2) has non-trivial solution. These are known as eigenvalues or latent roots or characteristic values of the matrix A. The corresponding values of vector X say $X_1, X_2, \ldots, X_n$ such that $AX_1 = \lambda_1 X_1, AX_2 = \lambda_2 X_2, \ldots, AX_n = \lambda_n X_n$ are called the eigenvectors. Out of the n eigenvalues, some or all of them may coincide. From here it is obvious that problem of determining the eigenvalues is merely a problem of solving the algebraic equation (3), which is a polynomial equation of degree n. But this method is not suitable for matrices of higher orders. In this paper, we have discussed Jacobi method and givens method for determining the eigenvalues of a real symmetric matrices. From equation (3), we observe that

$$\sum_{i=1}^{n} \lambda_i = a_{11} + a_{22} + \cdots + a_{nn} \tag{4}$$

i.e., Sum of the eigenvalues = Sum of the diagonal values and

$$\prod_{i=1}^{n} \lambda_i = \lambda_1 \lambda_2 \ldots \lambda_n = |A| \tag{5}$$

These properties can be used to find the remaining latent root of a matrix whose all except one latent roots are known.

2.1. Bisection Method (Bolzano Method)

If a function $f(x)$ is continuous between a and b and $f(a)$ and $f(b)$ are of opposite signs, then there exists at least one root between a and b. Let $f(a)$ be negative and $f(b)$ be positive so that the approximate value of the root between them is $x_0 = \frac{a+b}{2}$. If $f(x_0) = 0$, then it asserts that x_0 is the correct root of $f(x) = 0$. On the other hand, if $f(x_0) \neq 0$, then the root either lies in between $\left(a, \frac{a+b}{2}\right)$ or $\left(\frac{\mathrm{a+b}}{2}, b\right)$ depending on whether $f(x_0)$ is negative or positive. We again bisect the interval and repeat the process until the root is obtained to desired accuracy.

3. Jacobi Method for Symmetric Matrices

The Jacobi method is suitable for finding the eigenvalues of a real symmetric matrices. A real symmetric matrix is systematically reduced to a diagonal matrix by Jacobi method. This method use the similarity transformed matrix which is simpler but has the same eigenvalues as the given matrix. The transformation matrices which are used are orthogonal matrices. The advantage of using orthogonal matrices is that it minimizes errors in the process. Jacobi method can be used to find all eigenvalues simultaneously of any real symmetric matrix A. We know from matrix theory that, the eigenvalues of a real symmetric matrix A are real. This method reduces the given matrix to a diagonal form, where the diagonal elements are the eigenvalues of the given matrix. In this method, the given matrix A is transformed to a new matrix A_1 by the scheme

$$A_1 = P_1^{-1} A P_1 \tag{6}$$

Where P_1 is an orthogonal matrix. Therefore, $P_1^{-1} = P_1^T$. This transformation introduces a zero at a non-diagonal position of A. Then another matrix A_2 is produced by the equation

$$A_2 = P_2^{-1} A_1 P_2 = P_2^{-1} P_1^{-1} A P_1 P_2 \qquad \text{[by (6)]}$$

in which a new non-diagonal element is reduced to zero. Continuing this process of reducing the non-diagonal elements to zero one by one, we finally obtain a matrix

$$A_k = P_k^{-1} P_{k-1}^{-1} \dots P_1^{-1} A P_1 P_2 \dots P_{k-1} P_k \tag{7}$$

Which is a diagonal matrix. The eigenvalues are the diagonal elements of A_k. The non-diagonal element need not be reduced exactly to zero but must be less than a specified small quantity. The orthogonal matrices P_i used above are extensions of a rotation matrix in a two-dimensional system. P_i's are chosen as follows. Suppose a non-diagonal element, say a_{ij}, has to be reduced to zero. If A is an $n \times n$ matrix, then P is also an $n \times n$ matrix, where the sub matrix

$$\begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix}$$

consisting of the ith and jth rows and columns is replaced by

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

All the other diagonal elements of P are equal to unity. The other non-diagonal elements are taken as zero. For example, if A is a 4×4 matrix and a non-diagonal element, say a_{23}, has to be reduced to zero. Then, we take

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

Note the second and third rows and columns in (8). Now, let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix} \tag{9}$$

is a given symmetric matrix. The transformation $P_1^T A P_1$ gives

$$A_1 = P_1^T A P_1 \tag{10}$$

The element equated to zero in the $(2,3)$ position of A_1 gives the equation

$$-a_{22}\sin\theta\cos\theta + a_{23}\cos^2\theta - a_{23}\sin^2\theta + a_{33}\sin\theta\cos\theta = 0$$

This equation yields,

$$\tan 2\theta = \frac{2a_{23}}{a_{22} - a_{33}} \Rightarrow \theta = \frac{1}{2}\tan^{-1}\left[\frac{2a_{23}}{a_{22} - a_{33}}\right]$$

Solving this trigonometric equation we get four values of θ. If θ has to be small, we take $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$. Substituting for θ in equations (8) and (10), we get the values of P_1 and A_1 respectively. Next, we work with A_1 to annihilate some other non-diagonal element to zero. The process is truncated when all the non-diagonal elements are numerically less than the desired accuracy. The eigenvectors are obtained as the corresponding columns of

$$P = P_1 P_2 \ldots P_k \tag{11}$$

Each step of reduction in the above method is called a rotation. The pair (i,j) is called the plane of rotation and θ is the angle of rotation. The sequence in which the elements are reduced to zero is $a_{12}, a_{13}, \ldots, a_{1n}$; $a_{23}, a_{24}, \ldots, a_{2n}$ and so on. If $a_{ij} (i \neq j)$ is reduced to zero, the element a_{ji} also gets reduced to zero automatically by symmetry. In Jacobi method, the number of iterations increase if the matrix is large. If A is an $n \times n$ matrix, the minimum number of rotations required to reduce A into a diagonal form may be $\frac{n(n-1)}{2}$. For example, if A is a matrix of order 10, then the minimum number of operations may be 45. The Jacobi method is illustrated in Examples 3.1 and 3.2.

Example 3.1. *Let us now consider the real symmetric matrix*

$$A = \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$$

to find the eigenvalues and the corresponding eigenvectors by Jacobi method.

Solution.

The given matrix is real and symmetric. The largest off-diagonal element is $a_{13} = a_{31} = 2$. The other two elements in this 2×2 sub matrix are $a_{11} = 1$ and $a_{33} = 1$. Now, we compute $\tan 2\theta = \frac{2a_{ij}}{a_{ii}-a_{jj}}$, where $|a_{ij}|$ be numerically the largest off-diagonal element of A. Therfore

$$\tan 2\theta = \frac{2a_{13}}{a_{11} - a_{33}} = \frac{2 \times 2}{1-1} = \infty \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$$

Therefore

$$\begin{aligned} S_1 &= \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos\frac{\pi}{4} & 0 & -\sin\frac{\pi}{4} \\ 0 & 1 & 0 \\ \sin\frac{\pi}{4} & 0 & \cos\frac{\pi}{4} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned}$$

The first rotation gives,

$$\begin{aligned} D_1 &= S_1^{-1} A S_1 = S_1^T A S_1 \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

We may observe that the elements d_{13} and d_{31} got annihilated. To make sure that our calculations are correct up to this step, we may also observe that the sum of the diagonal elements of D_1 is same as the sum of the diagonal elements of the original matrix A. As a second step, we choose the largest off-diagonal element of D_1 and is found to be $d_{12} = d_{21} = 2$. The other elements are $d_{11} = 3, d_{22} = 3$. Now, we compute

$$\tan 2\theta = \frac{2d_{12}}{d_{11} - d_{22}} = \frac{2 \times 2}{3-3} = \frac{4}{0} = \infty \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$$

Thus, we construct the second rotation matrix as

$$\begin{aligned} S_2 &= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & 0 \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

At the end of second rotation, we get

$$\begin{aligned} D_2 &= S_2^{-1} D_1 S_2 = S_2^T D_1 S_2 \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned} \tag{12}$$

Which turned out to be a diagonal matrix and therefore we stop the computation. From equation (12), we notice that the eigenvalues of the given matrix are $5, 1$ and -1. The eigenvectors are the column vectors of $S = S_1 S_2$. Therefore,

$$\begin{aligned} S = S_1 S_2 &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned}$$

Hence the eigenvectors corresponding to $5, 1$ and -1 are respectively $\left[\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right]^T$, $\left[-\frac{1}{2}, \frac{1}{\sqrt{2}}, -\frac{1}{2}\right]^T$ and $\left[-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right]^T$. □

Example 3.2. *Let us consider the real symmetric matrix*

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

to find the eigenvalues and the corresponding eigenvectors by Jacobi method.

Solution.

If we find that all the off-diagonal elements are of the same order of magnitude. Then, we can choose any one of them. In this example, we shall first reduce the largest off-diagonal element $a_{12} = 3$ to zero. For this, we take

$$S_1 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let us compute

$$\begin{aligned} A_1 &= S_1^{-1} A S_1 = S_1^T A S_1 \\ &= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 3 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 + 6\sin\theta\cos\theta & 3\cos 2\theta & \cos\theta + 2\sin\theta \\ 3\cos 2\theta & 2 - 6\sin\theta\cos\theta & -\sin\theta + 2\cos\theta \\ \cos\theta + 2\sin\theta & -\sin\theta + 2\cos\theta & 1 \end{bmatrix} \end{aligned}$$

Equating the element in the $(1,2)$ position to zero, we get $\cos 2\theta = 0$, which gives $\theta = \frac{\pi}{4}$. With this value of θ, A_1 and S_1 become

$$A_1 = \begin{bmatrix} 5 & 0 & \frac{3}{\sqrt{2}} \\ 0 & -1 & \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \end{bmatrix} \quad \text{and} \quad S_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Starting with A_1, we reduce the largest off-diagonal element $a_{13} = \frac{3}{\sqrt{2}}$ to zero, we set

$$S_2 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

Computing $A_2 = S_2^{-1} A_1 S_2 = S_2^T A_1 S_2$, we get

$$A_2 = \begin{bmatrix} 3 + 2\cos 2\theta + \frac{3}{\sqrt{2}}\sin 2\theta & \frac{1}{\sqrt{2}}\sin\theta & -2\sin 2\theta + \frac{3}{\sqrt{2}}\cos 2\theta \\ \frac{1}{\sqrt{2}}\sin\theta & -1 & \frac{1}{\sqrt{2}}\cos\theta \\ -2\sin 2\theta + \frac{3}{\sqrt{2}}\cos 2\theta & \frac{1}{\sqrt{2}}\cos\theta & -\frac{3}{\sqrt{2}}\sin 2\theta + 3 - 2\cos 2\theta \end{bmatrix}$$

Equating the element in the $(1,3)$ position to zero, i.e.,

$$-2\sin 2\theta + \frac{3}{\sqrt{2}}\cos 2\theta = 0 \Rightarrow \tan 2\theta = \frac{3}{2\sqrt{2}}$$

Therefore $\sin 2\theta = \frac{3}{\sqrt{17}} = 0.7276$ and $\cos 2\theta = \frac{2\sqrt{2}}{\sqrt{17}} = 0.6860$. Hence, $\sin\theta = 0.3963$ and $\cos\theta = 0.9182$ (calculated to four decimal places using a calculator). Substituting the values of $\sin\theta$ and $\cos\theta$ in A_2 and S_2, we get

$$A_2 = \begin{bmatrix} 5.9155 & 0.2802 & 0 \\ 0.2802 & -1 & 0.6493 \\ 0 & 0.6493 & 0.0845 \end{bmatrix} \quad \text{and} \quad S_2 = \begin{bmatrix} 0.9182 & 0 & -0.3963 \\ 0 & 1 & 0 \\ 0.3963 & 0 & 0.9182 \end{bmatrix}$$

From above two transformations, we have seen that the element in the $(1,2)$ position of A_1, which was reduced to zero, is replaced by an element $(\neq 0)$ in A_2. But, this element, i.e., 0.2802 is definitely less than the corresponding element in A. Finally, all the off-diagonal elements will be gradually reduced to zero (almost zero). Continuing the above process with A_2, we get the following matrices successively:

$$A_3 = \begin{bmatrix} 5.9155 & 0.2538 & 0.1187 \\ 0.2538 & -1.3036 & 0 \\ 0.1187 & 0 & 0.3882 \end{bmatrix} \quad \text{and} \quad S_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.9058 & -0.4237 \\ 0 & 0.4237 & 0.9058 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 5.9246 & 0 & 0.1186 \\ 0 & -1.2993 & -0.005 \\ 0.1186 & -0.005 & 0.3882 \end{bmatrix} \quad \text{and} \quad S_4 = \begin{bmatrix} 0.9994 & -0.0351 & 0 \\ 0.0351 & 0.9994 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 5.9273 & -0.00011 & 0 \\ -0.00011 & -1.2923 & -0.005 \\ 0 & -0.005 & 0.3857 \end{bmatrix} \quad \text{and} \quad S_5 = \begin{bmatrix} 0.9988 & 0 & -0.0214 \\ 0 & 1 & 0 \\ 0.0214 & 0 & 0.9988 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} 5.9273 & -0.00011 & 7.46 \times 10^{-7} \\ -0.00011 & -1.2993 & 0 \\ 7.46 \times 10^{-7} & 0 & 0.3857 \end{bmatrix} \quad \text{and} \quad S_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.9999 & -0.003 \\ 0 & 0.003 & 0.9999 \end{bmatrix}$$

Since the off-diagonal elements are very nearly equal to zero (at least up to the 4^{th} decimal place), A_6 can be considered as a diagonal matrix. However, iterations can be continued further to get greater accuracy, if needed. Hence the eigenvalues of A are the diagonal elements of A_6. That is, $\lambda_1 = 5.9273, \lambda_2 = -1.2993$ and $\lambda_3 = 0.3857$. Here we have,

$$\begin{aligned} A_6 &= S_6^{-1}A_5S_6 = S_6^{-1}S_5^{-1}A_4S_5S_6 \\ &= S_6^{-1}S_5^{-1}S_4^{-1}A_3S_4S_5S_6 \\ &= S_6^{-1}S_5^{-1}S_4^{-1}S_3^{-1}A_2S_3S_4S_5S_6 \\ &= S_6^{-1}S_5^{-1}S_4^{-1}S_3^{-1}S_2^{-1}A_1S_2S_3S_4S_5S_6 \\ &= S_6^{-1}S_5^{-1}S_4^{-1}S_3^{-1}S_2^{-1}S_1^{-1}AS_1S_2S_3S_4S_5S_6 \end{aligned}$$

$\Rightarrow A_6 = S^{-1}AS$, where $S = S_1S_2S_3S_4S_5S_6$. To obtain the eigenvectors, we find $S = S_1S_2S_3S_4S_5S_6$. We get,

$$S = \begin{bmatrix} 0.6145 & -0.5504 & -0.5648 \\ 0.6814 & 0.7313 & 0.0287 \\ 0.3972 & -0.3710 & 0.8246 \end{bmatrix}$$

Thus the eigenvectors corresponding to λ_1, λ_2 and λ_3 are respectively the first, second and third columns of S. □

4. Givens Method For Symmetric Matrices

The Givens method leads to a tridiagonal matrix. The eigenvalues and eigenvectors of the original matrix are to determined from those of the tridiagonal matrix. Let A be a real symmetric matrix. The Givens method consists of the following steps: Step 1. Reduce A to a tridiagonal symmetric matrix using plane rotations. The reduction to a tridiagonal form is achieved by using the orthogonal transformations as in the Jacobi method. However, in this case we start with the subspace containing the elements $a_{22},\ a_{23},\ a_{32},\ a_{33}$. Perform the plane rotation $S_1^{-1}AS_1$ using the orthogonal matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Now, let us consider the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \tag{13}$$

and let the orthogonal rotation matrix S_1 in the plane $(2, 3)$ be

$$S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \therefore S_1^{-1}AS_1 = S_1^TAS_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12}\cos\theta + a_{13}\sin\theta & -a_{12}\sin\theta + a_{13}\cos\theta \\ a_{12}\cos\theta + a_{13}\sin\theta & a_{23}\sin 2\theta + a_{22}\cos^2\theta + a_{33}\sin^2\theta & a_{23}\cos 2\theta - a_{22}\sin\theta\cos\theta + a_{33}\sin\theta\cos\theta \\ -a_{12}\sin\theta + a_{13}\cos\theta & a_{23}\cos 2\theta - a_{22}\sin\theta\cos\theta + a_{33}\sin\theta\cos\theta & -a_{23}\sin 2\theta + a_{22}\sin^2\theta + a_{33}\cos^2\theta \end{bmatrix}$$

Then in the resulting matrix, equating the element in the $(1,3)$ position to zero for reducing $S_1^{-1}AS_1$ to tridiagonal matrix, we get

$$-a_{12}\sin\theta + a_{13}\cos\theta = 0 \Rightarrow \tan\theta = \frac{a_{13}}{a_{12}} \Rightarrow \theta = \tan^{-1}\left(\frac{a_{13}}{a_{12}}\right) \tag{14}$$

By this value of θ, the above transformation gives zeros in $(1,3)$ and $(3,1)$ positions. Let us further perform rotation in the plane $(2,4)$ and put the resulting element $(1,4) = 0$. This would not affect the zeros obtained earlier. Then the transformations are applied to the matrix in turn so as to annihilate the elements $(1,3),(1,4),(1,5),\ldots,(1,n)$; $(2,4),(2,5),\ldots,(2,n)$ and finally we arrive at the tridiagonal matrix

$$P = \begin{bmatrix} p_1 & q_1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ q_1 & p_2 & q_2 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & q_2 & p_3 & q_3 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & q_{n-2} & p_{n-1} & q_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & q_{n-1} & p_n \end{bmatrix}$$

<u>Step 2</u>. To obtain the eigenvalues of the tridiagonal matrix. Let the resulting tridiagonal matrix after first transformation be obtained as

$$S_1^{-1}AS_1 = S_1^TAS_1 = B = \begin{bmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{12} & \alpha_{22} & \alpha_{23} \\ 0 & \alpha_{23} & \alpha_{33} \end{bmatrix} \tag{15}$$

Now, the eigenvalues of (13) and (15) are the same. To find the eigenvalues of (15), we consider

$$\det(B-\lambda I) = 0 \Rightarrow \begin{vmatrix} \alpha_{11}-\lambda & \alpha_{12} & 0 \\ \alpha_{12} & \alpha_{22}-\lambda & \alpha_{23} \\ 0 & \alpha_{23} & \alpha_{33}-\lambda \end{vmatrix} = 0$$

Say $f_3(\lambda) = 0$. Then we have,

$$f_0(\lambda) = 1,\ f_1(\lambda) = \alpha_{11} - \lambda = \alpha_{11} - \lambda f_0(\lambda)$$

and

$$f_2(\lambda) = \begin{vmatrix} \alpha_{11}-\lambda & \alpha_{12} \\ \alpha_{12} & \alpha_{22}-\lambda \end{vmatrix} = (\alpha_{22}-\lambda)f_1(\lambda) - \alpha_{12}^2 f_0(\lambda)$$

Now expanding $f_3(\lambda)$ in terms of the third row, we immediately obtain

$$f_3(\lambda) = (\alpha_{33}-\lambda)\begin{vmatrix} \alpha_{11}-\lambda & \alpha_{12} \\ \alpha_{12} & \alpha_{22}-\lambda \end{vmatrix} - \alpha_{23}\begin{vmatrix} \alpha_{11}-\lambda & 0 \\ \alpha_{12} & \alpha_{23} \end{vmatrix}$$

$$\Rightarrow f_3(\lambda) = (\alpha_{33}-\lambda)f_2(\lambda) - \alpha_{23}^2 f_1(\lambda)$$

The recurrence formula in general is,

$$f_k(\lambda) = (\alpha_{kk}-\lambda)f_{k-1}(\lambda) - (\alpha_{(k-1)k})^2 f_{k-2}(\lambda),\ 2 \le k \le n \tag{16}$$

Above is the characteristic equation which can be solved by any standard method. Thus the roots of (16) will be the eigenvalues of the given real symmetric matrix. If none of the $\alpha_{ij}\quad (i \neq j)$ vanish then this equation generate a sequence

$\{f_k(\lambda) : k = 0, 1, \ldots, n\}$, which is called the Sturm sequence. A table of the sequence for various λ is prepared and the number of changes in sign of the Sturm sequence is noted, the difference between the number of changes of sign for consecutive values of λ gives an approximate location of the eigenvalues. Knowing the location of the eigenvalues, their exact values can be obtained by any iterative method. That is, if $V(x)$ denotes the number of changes in sign in the sequence for a given number x, then the number of zeros of f_n in (a, b) is $|V(a) - V(b)|$ provided a or b is not a zero of f_n. In this way, we can approximately compute the eigenvalues and by repeated bisections, we can improve these estimates.

Step 3. To obtain the eigenvectors of the tridiagonal matrix. Let Y be the eigenvector of the tridiagonal matrix B and let $S_1, S_2, \ldots, S_j$ be the orthogonal matrices employed in reducing the given real symmetric matrix A to the tridiagonal form B, then the corresponding eigenvector X of A is given by $X = SY$, where $S = S_1 S_2 \ldots S_j$ is the product of the orthogonal matrices used in the plane rotations. The number of rotations needed for Givens method are equivalent to the number of non-tridiagonal elements of the matrix. For a 3×3 matrix, only one rotation is required; whereas for a 4×4 matrix, three rotations are required etc. That is, the total number of plane rotations required to bring a matrix of order n to its tridiagonal form is $\frac{(n-1)(n-2)}{2}$.

The Givens method is illustrated in examples 4.1 and 4.2.

Example 4.1. *Let us now consider the real symmetric matrix*

$$A = \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$$

to find the eigenvalues and the corresponding eigenvectors by Givens method.

Solution.

There is only one non-tridiagonal element $a_{13} = 2$. This is to be reduced to zero, hence one rotation is required. Now, to annihilate a_{13}, we define the orthogonal matrix in the plane $(2, 3)$ as:

$$O = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

where θ is obtained by $\tan\theta = \frac{a_{13}}{a_{12}} = \frac{2}{\sqrt{2}} \Rightarrow \sin\theta = \sqrt{\frac{2}{3}}$ and $\cos\theta = \frac{1}{\sqrt{3}}$. Therefore

$$O = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

THerefore

$$A_1 = O^{-1}AO = O^T AO = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{6} & \frac{5}{\sqrt{3}} & 2\sqrt{\frac{2}{3}} \\ 0 & -\sqrt{6} + \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 1 & \sqrt{6} & 0 \\ \sqrt{6} & 3 & -\sqrt{2} \\ 0 & -\sqrt{2} & 1 \end{bmatrix},$$

which is a tridiagonal matrix. Now, to find the eigenvalues of A_1, we proceed as follows:

The characteristic equation of A_1 is

$$\begin{bmatrix} 1-\lambda & \sqrt{6} & 0 \\ \sqrt{6} & 3-\lambda & -\sqrt{2} \\ 0 & -\sqrt{2} & 1-\lambda \end{bmatrix} = 0$$

The Sturm sequence, i.e., the leading minors of order $0,1,2,3$ are given by $f_0(\lambda) = 1$, $f_1(\lambda) = 1-\lambda$, $f_2(\lambda) = (3-\lambda)f_1(\lambda) - 6f_0(\lambda)$ and $f_3(\lambda) = (1-\lambda)f_2(\lambda) - 2f_1(\lambda)$. Let us now consider the changes of sign in the Sturm sequence as

λ	$f_0(\lambda)$	$f_1(\lambda)$	$f_2(\lambda)$	$f_3(\lambda)$	$N(\lambda)$
-2	1	3	9	21	0
0	1	1	-3	-5	1
2	1	-1	-7	9	2
3	1	-2	-6	16	2
4	1	-3	-3	15	2
6	1	-5	9	-35	3

Above table shows that there is an eigenvalue in the intervals $(-2,0)$, $(0,2)$ and $(4,6)$. We now find better estimates of the eigenvalues by repeated bisections. First, we shall find the eigenvalue in the interval $(-2,0)$ by bisecting it at -1.

λ	$f_0(\lambda)$	$f_1(\lambda)$	$f_2(\lambda)$	$f_3(\lambda)$	$N(\lambda)$
-2	1	3	9	21	0
-1	1	2	2	0	...

Note that $f_3(-1) = 0$, so that $\lambda = -1$ is an eigenvalue. Now, we shall find the eigenvalue in the interval $(0,2)$ by bisecting it at 1.

λ	$f_0(\lambda)$	$f_1(\lambda)$	$f_2(\lambda)$	$f_3(\lambda)$	$N(\lambda)$
0	1	1	-3	-5	1
1	1	0	-6	0	...

Since $f_3(1) = 0$, so $\lambda = 1$ is an eigenvalue. Next, we shall find the eigenvalue in the interval $(4,6)$ by bisecting it at 5.

λ	$f_0(\lambda)$	$f_1(\lambda)$	$f_2(\lambda)$	$f_3(\lambda)$	$N(\lambda)$
5	1	-4	2	0	...
6	1	-5	9	-35	3

Again, since $f_3(5) = 0$, so $\lambda = 5$ is an eigenvalue. Therefore, the eigenvalues of A_1 are $5, 1$ and -1 and hence the eigenvalues of A are also $5, 1$ and -1. Now, to find the eigenvectors of A_1 for each of the eigenvalues, we proceed as follows:

For $\lambda = 5$, let the eigenvector of A_1 be $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Then we have,

$$A_1 Y = \lambda Y \Rightarrow \begin{bmatrix} 1 & \sqrt{6} & 0 \\ \sqrt{6} & 3 & -\sqrt{2} \\ 0 & -\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 5 \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Which gives the equations,

$$y_1 + \sqrt{6}y_2 = 5y_1 \tag{17}$$

$$\sqrt{6}y_1 + 3y_2 - \sqrt{2}y_3 = 5y_2$$

$$\text{and} \quad -\sqrt{2}y_2 + y_3 = 5y_3 \tag{18}$$

Equation (17) gives,

$$4y_1 = \sqrt{6}y_2 \Rightarrow \frac{y_1}{\sqrt{6}} = \frac{y_2}{4} \Rightarrow \frac{y_1}{\frac{1}{2}} = \frac{y_2}{\sqrt{\frac{2}{3}}}$$

Equation (18) gives,

$$-\sqrt{2}y_2 = 4y_3 \Rightarrow \frac{y_2}{4} = \frac{y_3}{-\sqrt{2}} \Rightarrow \frac{y_2}{\sqrt{\frac{2}{3}}} = \frac{y_3}{-\frac{1}{2\sqrt{3}}}$$

Therefore, the eigenvector of A_1 for $\lambda = 5$ is $Y = \left[\frac{1}{2}, \sqrt{\frac{2}{3}}, -\frac{1}{2\sqrt{3}}\right]^T$. Therefore, the eigenvector X of A for $\lambda = 5$ is given by

$$\begin{aligned} X = OY &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \sqrt{\frac{2}{3}} \\ -\frac{1}{2\sqrt{3}} \end{bmatrix} \\ &= \left[\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right]^T \end{aligned}$$

where O is the orthogonal matrix used in the plane rotation. For $\lambda = -1$, let the eigenvector of A_1 be $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Then we have,

$$A_1 Y = \lambda Y \Rightarrow \begin{bmatrix} 1 & \sqrt{6} & 0 \\ \sqrt{6} & 3 & -\sqrt{2} \\ 0 & -\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = -1 \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Which gives the equations,

$$y_1 + \sqrt{6}y_2 = -y_1 \tag{19}$$

$$\sqrt{6}y_1 + 3y_2 - \sqrt{2}y_3 = -y_2$$

$$\text{and} \quad -\sqrt{2}y_2 + y_3 = -y_3 \tag{20}$$

Equation (19) gives,

$$2y_1 = -\sqrt{6}y_2 \Rightarrow \frac{y_1}{-\sqrt{6}} = \frac{y_2}{2} \Rightarrow \frac{y_1}{-\frac{1}{\sqrt{2}}} = \frac{y_2}{\frac{1}{\sqrt{3}}}$$

Equation (20) gives,

$$\sqrt{2}y_2 = 2y_3 \Rightarrow \frac{y_2}{2} = \frac{y_3}{\sqrt{2}} \Rightarrow \frac{y_2}{\frac{1}{\sqrt{3}}} = \frac{y_3}{\frac{1}{\sqrt{6}}}$$

Therefore, the eigenvector of A_1 for $\lambda = -1$ is $Y = \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}\right]^T$. Therefore, the eigenvector X of A for $\lambda = -1$ is given by

$$\begin{aligned} X = OY &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} \\ &= \left[-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right]^T \end{aligned}$$

Similarly, the eigenvector of A_1 for $\lambda = 1$ is $Y = \left[-\frac{1}{2}, 0, -\frac{\sqrt{3}}{2}\right]^T$. Therefore, the eigenvector X of A for $\lambda = 1$ is given by

$$X = OY = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ 0 \\ -\frac{\sqrt{3}}{2} \end{bmatrix}$$
$$= \left[-\frac{1}{2}, \frac{1}{\sqrt{2}}, -\frac{1}{2}\right]^T$$

□

Example 4.2. *Let us now consider the real symmetric matrix*

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

to find the eigenvalues by Givens method.

Solution.

To annihilate, i.e., to make zero a_{13}, we take the orthogonal matrix P_1 as

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Computing $P_1^{-1}AP_1$, we get

$$A_1 = P_1^{-1}AP_1 = P_1^TAP_1$$
$$= \begin{bmatrix} 1 & \cos\theta+\sin\theta & -\sin\theta+\cos\theta & 1 \\ \cos\theta+\sin\theta & 2\cos^2\theta+6\sin^2\theta+3\sin 2\theta & 4\cos\theta\sin\theta+3\cos^2\theta-3\sin^2\theta & 4\cos\theta+10\sin\theta \\ -\sin\theta+\cos\theta & 4\cos\theta\sin\theta+3\cos^2\theta-3\sin^2\theta & 2\sin^2\theta+6\cos^2\theta-6\cos\theta\sin\theta & -4\sin\theta+10\cos\theta \\ 1 & 4\cos\theta+10\sin\theta & -4\sin\theta+10\cos\theta & 20 \end{bmatrix}$$

Equating the element in the $(1,3)$ position to zero, we get $\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$. Substituting $\theta = \frac{\pi}{4}$ in A_1, we get

$$A_1 = \begin{bmatrix} 1 & \sqrt{2} & 0 & 1 \\ \sqrt{2} & 7 & 2 & 7\sqrt{2} \\ 0 & 2 & 1 & 3\sqrt{2} \\ 1 & 7\sqrt{2} & 3\sqrt{2} & 20 \end{bmatrix}$$

As the matrix is symmetric, so the $(3,1)$ element is also reduced to zero. Now, to reduce the element in the $(1,4)$ position to zero, we take P_2 as

$$P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{bmatrix}$$

Computing $A_2 = P_2^{-1} A_1 P_2 = P_2^T A_1 P_2$ and equating the element in the $(1,4)$ position to zero, we get (using a calculator) $\sin\theta = 0.5774$, $\cos\theta = 0.8165$. Substituting these values in A_2, we get

$$A_2 = \begin{bmatrix} 1 & 1.7321 & 0 & 0 \\ 1.7321 & 20.6687 & 4.0827 & 9.4281 \\ 0 & 4.0827 & 1 & 2.3093 \\ 0 & 9.4281 & 2.3093 & 6.333 \end{bmatrix}$$

To annihilate the element in the $(2,4)$ position of A_2, we take

$$P_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix}$$

As before, we again compute $A_3 = P_3^{-1} A_2 P_3 = P_3^T A_2 P_3$ and equating the element in the $(2,4)$ position to zero, we get $\sin\theta = 0.9177$, $\cos\theta = 0.3974$. Substituting these values in A_3, we get

$$A_3 = \begin{bmatrix} 1 & 1.7321 & 0 & 0 \\ 1.7321 & 20.6687 & 10.2746 & 0 \\ 0 & 10.2746 & 7.1758 & 0.3648 \\ 0 & 0 & 0.3648 & 0.1580 \end{bmatrix}$$

This is the final tridiagonal form of the given matrix A. The matrices A and A_3 have the same eigenvalues. To find the eigenvalues of A_3, we proceed as follows:

The characteristic equation of A_3 is

$$\begin{vmatrix} 1-\lambda & 1.7321 & 0 & 0 \\ 1.7321 & 20.6687-\lambda & 10.2746 & 0 \\ 0 & 10.2746 & 7.1758-\lambda & 0.3648 \\ 0 & 0 & 0.3648 & 0.1580-\lambda \end{vmatrix} = 0$$

The Sturm sequence, i.e., the leading minors of order $0, 1, 2, 3, 4$ are given by $f_0(\lambda) = 1$, $f_1(\lambda) = 1 - \lambda$, $f_2(\lambda) = (20.6687 - \lambda) f_1(\lambda) - (1.7321)^2$, $f_3(\lambda) = (7.1758 - \lambda) f_2(\lambda) - (10.2746)^2 f_1(\lambda)$ and $f_4(\lambda) = (0.1580 - \lambda) f_3(\lambda) - (0.3648)^2 f_2(\lambda)$. Let us now consider the changes of sign in the Sturm sequence as

λ	$f_0(\lambda)$	$f_1(\lambda)$	$f_2(\lambda)$	$f_3(\lambda)$	$f_4(\lambda)$	$N(\lambda)$
0	+	+	+	+	+	0
0.1	+	+	+	+	-	1
0.5	+	+	+	-	+	2
2	+	-	-	-	+	2
3	+	-	-	+	-	3

From the above table, it is clear that there is an eigenvalue in the intervals $(0, 0.1)$, $(0.1, 0.5)$ and $(2, 3)$. First, we shall find the eigenvalue in the interval $(0, 0.1)$ by bisecting it at 0.05.

λ	$f_0(\lambda)$	$f_1(\lambda)$	$f_2(\lambda)$	$f_3(\lambda)$	$f_4(\lambda)$	$N(\lambda)$
0	+	+	+	+	+	0
0.05	+	+	+	+	−	1

Therefore, the eigenvalue lies in the interval $(0, 0.05)$. Proceeding in this way, using the method of successive bisection, we get an eigenvalue $\lambda_1 = 0.0379309$. Similarly, the eigenvalues in $(0.1, 0.5)$ and $(2, 3)$ are $\lambda_2 = 0.453835$ and $\lambda_3 = 2.20363$. But A_3 is a 4×4 matrix and hence has four eigenvalues. The fourth eigenvalue is obtained from the equation

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = a_{11} + a_{22} + a_{33} + a_{44} \; (of \; A_3) \Rightarrow 2.6953959 + \lambda_4 = 29.0025 \Rightarrow \lambda_4 = 26.3071$$

Since λ_1, λ_2 and λ_3 are approximate eigenvalues, so λ_4 is also the approximate eigenvalue, i.e., $\lambda_4 \approx 26.3071$. □

5. Conclusion

In this paper, we have studied Jacobi method and Givens method for finding the eigenvalues and the corresponding eigenvectors of a real symmetric matrices. In Jacobi method, we have seen that with examples that the elements that are reduced to zero by a transformation may not necessarily remain zero during subsequent transformations. But in Givens method, we have seen that with examples that this method preserves the zeros in the off-diagonal elements, once they are created, i.e., Givens method does not disturb zeros already obtained. In Jacobi method, the minimum number of rotation required to transform the given $n \times n$ real symmetric matrix A in to diagonal form is $\frac{n(n-1)}{2}$ but in Givens method, the total number of plane rotations required to bring a real symmetric matrix of order n to its tridiagonal form is $\frac{(n-1)(n-2)}{2}$. So, Givens method takes less number of rotation as compared to Jacobi method. Although, Jacobi method leads to a diagonal matrix but Givens method leads to a tridiagonal matrix. However, the advantage of Jacobi method is that it gives all the eigenvalues and eigenvectors of a real symmetric matrix at a time where as Givens method finds the eigenvalues one at a time and the method is very well suited for arbitrary real symmetric matrices. Finally, by analyzing all the things we can conclude that Givens method is more efficient than Jacobi method.

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