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An Extended Wright Function

Research Article

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Abstract: In this paper we will extend the classical function Wright $W_{\alpha,\beta}(z) \to W^{\lambda,\xi}_{\alpha,\beta}(z)$ using the relationship between Euler beta function with the symbol Pochhammer $\frac{B(\lambda+n,\xi-\lambda)}{B(\lambda,\xi-\lambda)} = \frac{(\lambda)_n}{(\xi)_n}$. Some basic properties are studied and Laplace transform is evaluate [1, 3]. We will study the Riemann-Liouville fractional integral and fractional derivative arbitrary order v of $W^{\lambda,\xi}_{\alpha,\beta}(z)$.

Keywords: Fractional Calculus, Laplace Transform, Wright Function, Extended Mittag-Leffler function.

1. Introduction and Preliminaries

The simplest Wright function $W_{\alpha,\beta}(z)$ is defined (for $z, \alpha, \beta \in \mathbb{C}$) by the series

$$W_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n + 1)} \frac{z^n}{n!} \tag{1}$$

If $\alpha > 1$, the series in (1) is absolutely convergent for all $z \in \mathbb{C}$, while for $\alpha = 1$ this series is absolutely convergent for $|z| < 1$ and for $|z| = 1$ and $R_e(\beta) > -1$. Moreover, for $\alpha > -1$, $W_{\alpha,\beta}(z)$ is an entire function of z (for more details see [1, 5]). The Wright function along with the Mittag-Leffler function plays a prominent role in the theory of the partial differential equations of the fractional order that are actively used nowadays for modeling of many phenomena physical including the anomalous diffusion (for more details see [3]). Starting recalling some lemmas and definitions elementary that well be used in developing this paper.

Definition 1.1. *Let $f \in L^1_{loc}[a,b]$ $\ -\infty < a \leq x \leq b < \infty,\ v > 0$. Then, the Riemann-Liouville fractional integrals of order v is defined as (see [5])*

$$I^v_z f(z) = \frac{1}{\Gamma(v)} \int_a^b (z-t)^{v-1} f(t) dt \tag{2}$$

Where $\Gamma(z)$ is the Euler Gamma Function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \quad for\ R_e(z) > 0$$

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Definition 1.2. *Let $f \in L^1_{loc}[a,b]$; $-\infty < a \leq x \leq b < \infty$, $v > 0$, $m \in \mathbb{N}$, $m-1 \leq v < m$. Then, the Riemann-Liouville fractional derivatives of order v is defined as [5]*

$$\begin{aligned} D^v_z f(z) &= D^m \left(\frac{1}{\Gamma(m-v)} \int_a^b (z-t)^{m-v-1} f(t) dt \right) \\ &= D^m \left(I_z^{m-v} f(z) \right) \end{aligned} \tag{3}$$

Definition 1.3. *Let $x, y \in \mathbb{C}$ such that $R_e(x) > 0$, $R_e(y) > 0$ Then, the Euler Beta function is defined as [5]:*

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt \tag{4}$$

Lemma 1.4. *Let $x, y \in \mathbb{C}$ such that $R_e(x) > 0$, $R_e(y) > 0$ Then, the Euler Beta function is relations with the Euler Gamma function given by [5]:*

$$B(x,y) = \frac{\Gamma(x) \cdot \Gamma(y)}{\Gamma(x+y)} \tag{5}$$

Definition 1.5. *Let $z \in \mathbb{C}$, $n \in \mathbb{N}$. Then, the Pochhammer symbol is defined as:*

$$(z)_n = z(z+1)(z+2)(z+3)\dots(z+n-1)$$

An alternative definition of Pochhammer symbol is:

$$(z)_n = \begin{cases} 1 & \text{si } n = 0 \\ \frac{\Gamma(z+n)}{\Gamma(z)} & \text{si } n \in \mathbb{N} \end{cases} \tag{6}$$

Lemma 1.6. *Let $\lambda, \xi \in \mathbb{C}$ such that $R_e(\lambda) > 0$, $R_e(\xi - \lambda) > 0$ $n \in \mathbb{N}$. Then*

$$\frac{B(\lambda+n, \xi-\lambda)}{B(\lambda, \xi-\lambda)} = \frac{(\lambda)_n}{(\xi)_n} \tag{7}$$

Definition 1.7. *Let $f : \mathbb{R}^+ \to \mathbb{R}$ an exponential order function and piecewise continuous, then the Laplace transform of f is*

$$\mathcal{L}\left\{f(z)\right\}(s) = \int_0^\infty e^{-st} f(t) dt \quad s \in \mathbb{C} \tag{8}$$

The integral exist for $R_e(s) > 0$

Definition 1.8. *Let $\alpha, \beta, \delta, c \in \mathbb{C}$ such that $R_e(\alpha) > 0$, $R_e(\beta) > 0$ $R_e(\delta) > 0$ and $R_e(\xi - \delta) > 0$ Then, the Extended Mittag-Leffler function is defined as [2]:*

$$E^{(\delta,c)}_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{B(\delta+n, \xi-\delta)}{B(\delta, \xi-\delta)\Gamma(\alpha n+\beta)} (\xi)_n \frac{z^n}{n!} \tag{9}$$

Where $B(x,y)$ is Beta function

Definition 1.9. *Let $\alpha, \beta, \in \mathbb{C}$ such that $R_e(\alpha) > -1$, $R_e(s) > 0$ Then, the Wright function is defined by the series [1]:*

$$W_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{1}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!} \tag{10}$$

Remark 1.10.

(1). If $\alpha > -1$ *the series* (10) *is absolutely convergence for all* $z \in \mathbb{C}$

(2). If $\alpha = 1$ *this series is absolutely convergent for* $|z| < 1$

Lemma 1.11. *Let* $\alpha, \beta \in \mathbb{C}$ *such that* $R_e(\alpha) > 0$, $R_e(\beta) > 0$. *Then, the Laplace transform of the Wright function is expressed in term of the Mittag-Leffler function [1]:*

$$\mathcal{L}\left\{W_{\alpha,\beta}(z)\right\}(s) = \frac{1}{s} E_{\alpha,\beta}\left(\frac{1}{s}\right) \tag{11}$$

Where $E_{\alpha,\beta}(z)$ is the two parameter Mittag-Leffler function and is defined as:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}$$

2. Main Result

From the classical Wright function, we have

$$W_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!} \tag{12}$$

Now, considerer the Wright function following:

$$W^{\lambda}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{(n!)^2} \tag{13}$$

Note that if $\lambda = 1$ (13) is reduced to the classic Wright function (12). If in (13) we multiply and divide by $(\xi)_n$ and using (7), we obtain

$$\begin{aligned} W^{\lambda}_{\alpha,\beta}(z) &= \sum_{n=0}^{\infty} \frac{(\lambda)_n}{\Gamma(\alpha n + \beta)} \frac{(\xi)_n}{(\xi)_n} \frac{z^n}{(n!)^2} \\ &= \sum_{n=0}^{\infty} \frac{B(\lambda + n, \xi - \lambda)}{B(\lambda, \xi - \lambda)\Gamma(\alpha n + \beta)} (\xi)_n \frac{z^n}{(n!)^2} \end{aligned}$$

Thus, we obtain the following

Definition 2.1. *Let* $\alpha, \beta, \lambda, \xi \in \mathbb{C}$ *such that* $R_e(\alpha) > 0$, $R_e(\beta) > 0$, $R_e(\lambda) > 0$ *and* $R_e(\xi - \lambda) > 0$. *Then, the extended Wright function* $W^{\lambda,\xi}_{\alpha\beta}(z)$ *is defined by series*

$$W^{\lambda,\xi}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{B(\lambda + n, \xi - \lambda)}{B(\lambda, \xi - \lambda)\Gamma(\alpha n + \beta)} (\xi)_n \frac{z^n}{(n!)^2} \tag{14}$$

Remark 2.2. *Note that* $W^{\lambda,\xi}_{\alpha,\beta}(z) \rightarrow W_{\alpha,\beta}(z)$ *as* $\xi \rightarrow 1$ *and* $\lambda \rightarrow 1$.

Lemma 2.3. *Let* $\alpha, \beta, \lambda, \xi \in \mathbb{C}$, $R_e(\alpha) > 0$, $R_e(\beta) > 0$ $R_e(\lambda) > 0$ *and* $R_e(\xi - \lambda) > 0$. *Then*

$$\frac{d}{dz} W^{\lambda,\xi}_{\alpha,\beta}(z) = \frac{\xi}{B(\lambda, \xi - \lambda)} \sum \frac{B((\lambda + 1) + n, \xi - \lambda)}{\Gamma(\alpha n + (\alpha + \beta))} (\xi)_n \frac{z^n}{(n+1)(n!)^2} \tag{15}$$

Proof. From definition (14) and using $(\xi)_{n+1} = (\xi)_n$, we have

$$\begin{aligned}
\frac{d}{dz}W_{\alpha,\beta}^{\lambda,\xi}(z) &= \frac{d}{dz}\left(\sum_{n=0}^{\infty}\frac{B(\lambda+n,\xi-\lambda)}{B(\lambda,\xi-\lambda)\Gamma(\alpha n+\beta)}(\xi)_n\frac{z^n}{(n!)^2}\right)\\
&= \sum_{n=0}^{\infty}\frac{B(\lambda+n,\xi-\lambda)}{B(\lambda,\xi-\lambda)\Gamma(\alpha n+\beta)(n!)^2}(\xi)_n\frac{d}{dz}z^n\\
&= \sum_{n=0}^{\infty}\frac{B(\lambda+n,\xi-\lambda)}{B(\lambda,\xi-\lambda)\Gamma(\alpha n+\beta)(n!)^2}(\xi)_n n z^{n-1}\\
&= \sum_{n=1}^{\infty}\frac{B(\lambda+n,\xi-\lambda)}{B(\lambda,\xi-\lambda)\Gamma(\alpha n+\beta)n(n-1)!^2}(\xi)_n z^{n-1}\\
&= \sum_{n=0}^{\infty}\frac{B((\lambda+1)+n,\xi-\lambda)}{B(\lambda,\xi-\lambda)\Gamma(\alpha(n+1)+\beta)(n+1)(n!)^2}(\xi)_{n+1}z^n\\
&= \frac{\xi}{B(\lambda,\xi-\lambda)}\sum_{n=0}^{\infty}\frac{B((\lambda+1)+n,\xi-\lambda)}{\Gamma(\alpha n+(\alpha+\beta))}(\xi)_n\frac{z^n}{(n+1)(n!)^2}
\end{aligned}$$

□

Lemma 2.4. *Let $\alpha,\beta,\lambda,\xi\in\mathbb{C}$, $R_e(\alpha)>0$, $R_e(\beta)>0$ $R_e(\lambda)>0$ and $R_e(\xi-\lambda)>0$. Then*

$$\alpha z\frac{d}{dz}W_{\alpha,\beta}^{\lambda,\xi}(z) = W_{\alpha,\beta-1}^{\lambda,\xi}(z)+(1-\beta)W_{\alpha,\beta}^{\lambda,\xi}(z) \tag{16}$$

Proof. From definition (14), using (15), $\Gamma(x+1)=x\Gamma(x)$ and $(\xi)_{n+1}=\xi(\xi)_n$, we have

$$\begin{aligned}
W_{\alpha,\beta-1}^{\lambda,\xi}(z)+(1-\beta)W_{\alpha,\beta}^{\lambda,\xi}(z) &= \sum_{n=0}^{\infty}\frac{B(\lambda+n,\xi-\lambda)}{B(\lambda,\xi-\lambda)\Gamma(\alpha n+\beta-1)}(\xi)_n\frac{z^n}{(n!)^2}+(1-\beta)\sum_{n=0}^{\infty}\frac{B(\lambda+n,\xi-\lambda)}{B(\lambda,\xi-\lambda)\Gamma(\alpha n+\beta)}(\xi)_n\frac{z^n}{(n!)^2}\\
&= \frac{1}{B(\lambda,\xi-\lambda)}\sum_{n=0}^{\infty}\frac{B(\lambda+n,\xi-\lambda)}{\Gamma(\alpha n+\beta-1)}(\xi)_n\frac{z^n}{(n!)^2}\\
&+ \frac{(1-\beta)}{B(\lambda,\xi-\lambda)}\sum_{n=0}^{\infty}\frac{B(\lambda+n,\xi-\lambda)}{\Gamma(\alpha n+\beta-1)(\alpha n+\beta-1)}(\xi)_n\frac{z^n}{(n!)^2}\\
&= \frac{1}{B(\lambda,\xi-\lambda)}\left(\sum_{n=0}^{\infty}\frac{B(\lambda+n,\xi-\lambda)}{\Gamma(\alpha n+\beta-1)}(\xi)_n\frac{z^n}{(n!)^2}\left(1+\frac{(1-\beta)}{\alpha n+\beta-1}\right)\right)\\
&= \frac{1}{B(\lambda,\xi-\lambda)}\sum_{n=0}^{\infty}\frac{B(\lambda+n,\xi-\lambda)}{\Gamma(\alpha n+\beta-1)}(\xi)_n\frac{z^n}{(n!)^2}\frac{\alpha n}{\alpha n+\beta-1}\\
&= \frac{1}{B(\lambda,\xi-\lambda)}\sum_{n=1}^{\infty}\frac{B(\lambda+n,\xi-\lambda)}{\Gamma(\alpha n+\beta-1)}(\xi)_n\frac{z^n}{n((n-1)!)^2}\frac{\alpha}{\alpha n+\beta-1}\\
&= \frac{\alpha z\xi}{B(\lambda,\xi-\lambda)}\sum_{n=0}^{\infty}\frac{B((\lambda+1)+n,\xi-\lambda)}{\Gamma(\alpha n+(\alpha+\beta))}(\xi)_n\frac{z^n}{(n+1)(n!)^2}\\
&= \alpha z\left(\frac{\xi}{B(\lambda,\xi-\lambda)}\sum_{n=0}^{\infty}\frac{B((\lambda+1)+n,\xi-\lambda)}{\Gamma(\alpha n+(\alpha+\beta))}(\xi)_n\frac{z^n}{(n+1)(n!)^2}\right)\\
&= \alpha z W_{\alpha,\beta}^{\lambda,\xi}(z)
\end{aligned}$$

□

Lemma 2.5. *Let $\alpha,\beta,\lambda,\xi\in\mathbb{C}$, $R_e(\alpha)>0$, $R_e(\beta)>0$ $R_e(\lambda)>0$ $R_e(\xi-\lambda)>0$ and $R_e(s)>0$, $s\neq 0$. Then*

$$\mathcal{L}\left\{W_{\alpha,\beta}^{\lambda,\xi}(z)\right\}(s) = \frac{1}{s}E_{\alpha,\beta}^{\lambda,\xi}\left(\frac{1}{s}\right) \tag{17}$$

Proof. From definition Laplace transform and from (14) we have

$$\begin{aligned}
\mathcal{L}\left\{W_{\alpha,\beta}^{\lambda,\xi}(z)\right\}(s) &= \int_0^{\infty}e^{-sz}\frac{1}{B(\lambda,\xi-\lambda)}\sum_{n=0}^{\infty}\frac{B(\lambda-n,\xi-\lambda)}{\Gamma(\alpha n+\beta)}(\xi)_n\frac{z^n}{(n!)^2}dz\\
&= \sum_{n=0}^{\infty}\frac{B(\lambda-n,\xi-\lambda)}{B(\lambda,\xi-\lambda)\Gamma(\alpha n+\beta)}\frac{(\xi)_n}{(n!)^2}\int_0^{\infty}e^{-sz}z^n dz
\end{aligned} \tag{18}$$

taking into account that the integral in (18) is

$$\int_0^\infty e^{-sz} z^n dz = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}} \tag{19}$$

From (18) and (19) we have

$$\begin{aligned}\mathcal{L}\left\{W_{\alpha,\beta}^{\lambda,\xi}(z)\right\}(s) &= \frac{1}{B(\lambda,\xi-\lambda)}\sum_{n=0}^{\infty}\frac{B(\lambda-n,\xi-\lambda)}{\Gamma(\alpha n+\beta)}\frac{(\xi)_n}{(n!)^2}n!s^{-n-1}\\ &= \frac{1}{s}\frac{1}{B(\lambda,\xi-\lambda)}\sum_{n=0}^{\infty}\frac{B(\lambda-n,\xi-\lambda)}{\Gamma(\alpha n+\beta)}\frac{(\xi)_n}{n!}\left(s^{-1}\right)^n\\ &= \frac{1}{s}E_{\alpha,\beta}^{\lambda,\xi}\left(\frac{1}{s}\right)\end{aligned}$$

□

Note that $\mathcal{L}\left\{W_{\alpha,\beta}^{\lambda,\xi}(z)\right\}(s) \to \quad \mathcal{L}\left\{W_{\alpha,\beta}\right\}(s)$ as $\lambda \to 1$ and $\xi \to 1$

Lemma 2.6 (Integral Representation)**.** *Let* $\alpha,\beta,\lambda,\xi \in \mathbb{C}$, $R_e(\alpha) > 0$, $R_e(\beta) > 0$ $R_e(\lambda) > 0$ *and* $R_e(\xi-\lambda) > 0$. *Then*

$$W_{\alpha,\beta}^{\lambda,\xi}(z) = \int_0^1 u^{\lambda-1}(1-u)^{\xi-\lambda-1}W_{\alpha,\beta}^{\xi}(uz)du \tag{20}$$

Proof. From definition (14), (4) and using the absolutely convergence of the series, we have

$$\begin{aligned}W_{\alpha,\beta}^{\lambda,\xi}(z) &= \frac{1}{B(\lambda,\xi-\lambda)}\sum_{n=0}^{\infty}\frac{B(\lambda+n,\xi-\lambda)}{\Gamma(\alpha n+\beta)}(\xi)_n\frac{z^n}{(n!)^2}\\ &= \frac{1}{B(\lambda,\xi-\lambda)}\sum_{n=0}^{\infty}\int_0^1 u^{\lambda+n-1}(1-u)^{\xi-\lambda-1}\frac{(\xi)_n z^n}{\Gamma(\alpha n+\beta)(n!)^2}du\\ &= \int_0^1 u^{\lambda-1}(1-u)^{\xi-\lambda-1}\frac{1}{B(\lambda,\xi-\lambda)}\sum_{n=0}^{\infty}\frac{(\xi)_n(uz)^n}{\Gamma(\alpha n+\beta)(n!)^2}du\\ &= \int_0^1 u^{\lambda-1}(1-u)^{\xi-\lambda-1}W_{\alpha,\beta}^{\xi}(uz)du\end{aligned}$$

□

Theorem 2.7. *Let* $v,\alpha,\beta,\lambda,\xi \in \mathbb{C}$ *such that* $R_e(\alpha) > 0$, $R_e(\beta) > 0$, $R_e(\lambda) > 0$,$R_e(\xi-\lambda) > 0$ *and* $R_e(v) > 0$. *Then*

$$I_z^v\left(z^{\beta-1}W_{\alpha,\beta}^{\lambda,\xi}(z^\alpha)\right) = z^{\beta+v-1}W_{\alpha,\beta+v}^{\lambda,\beta}(z^\alpha) \tag{21}$$

Proof. From (2) and (14), we have

$$\begin{aligned}I_z^v\left(z^{\beta-1}W_{\alpha,\beta}^{\lambda,\xi}(z^\alpha)\right) &= \frac{1}{\Gamma(v)}\int_0^z (z-t)^{v-1}t^{\beta-1}\sum_{n=0}^{\infty}\frac{B(\lambda+n,\xi-\lambda)}{B(\lambda,\xi-\lambda)\Gamma(\alpha n+\beta)}(\xi)_n \times \frac{z^n}{(n!)^2}dt\\ &= \frac{1}{B(\lambda,\xi-\lambda)\Gamma(v)}\sum_{n=0}^{\infty}\frac{B(\lambda+n,\xi-\lambda)(\xi)_n}{\Gamma(\alpha n+\beta)(n!)^2}\times\int_0^z (z-t)^{v-1}t^{\alpha n+\beta-1}dt\\ &= \frac{1}{B(\lambda,\xi-\lambda)\Gamma(v)}\sum_{n=0}^{\infty}\frac{B(\lambda+n,\xi-\lambda)(\xi)_n}{\Gamma(\alpha n+\beta)(n!)^2}\int_0^z z^{v-1}\left(1-\frac{t}{z}\right)^{v-1}t^{\alpha n+\beta-1}dt\end{aligned} \tag{22}$$

Marking the change of variable $u = \frac{t}{z}$, $dt = zdu$, $t = 0$, $u = 0$ and $t = z$, $u = 1$ and replacing in (22) it result

$$\begin{aligned}I_z^v\left(z^{\beta-1}W_{\alpha,\beta}^{\lambda,\xi}(z^\alpha)\right) &= \frac{z^{\beta+v-1}}{B(\lambda,\xi-\lambda)\Gamma(v)}\sum_{n=0}^{\infty}\frac{B(\lambda+n,\xi-\lambda)(\xi)_n}{\Gamma(\alpha n+\beta)(n!)^2}\times\int_0^1 u^{\alpha n+\beta-1}(1-u)^{v-1}dt\\ &= \frac{z^{\beta+v-1}}{B(\lambda,\xi-\lambda)\Gamma(v)}\sum_{n=0}^{\infty}\frac{B(\lambda+n,\xi-\lambda)(\xi)_n}{\Gamma(\alpha n+\beta)(n!)^2}\times B(\alpha n+\beta,v)\end{aligned} \tag{23}$$

From (5) and replacing in (23), we have

$$I_z^v\left(z^{\beta-1}W_{\alpha,\beta}^{\lambda,\xi}(z^\alpha)\right) = \frac{z^{\beta+v-1}}{B(\lambda,\xi-\lambda)}\sum_{n=0}^{\infty}\frac{B(\lambda+n,\xi-\lambda)(\xi)_n}{\Gamma(\alpha n+(\beta+v))}\frac{(z^\alpha)^n}{(n!)^2}$$

Thus

$$I_z^v\left(z^{\beta-1}W_{\alpha,\beta}^{\lambda,\xi}(z^\alpha)\right) = z^{\beta+v-1}W_{\alpha,\beta+v}^{\lambda,\xi}(z^\alpha)$$

□

Theorem 2.8. *Let $v,\alpha,\beta,\lambda,\xi\in\mathbb{C}$ such that $R_e(\alpha)>0$, $R_e(\beta)>0$, $R_e(\lambda)>0$,$R_e(\xi-\lambda)>0$, $R_e(v)>0$ and $m\in\mathbb{N}$, $m-1\le v<m$. Then*

$$D_z^v\left(z^{\beta-1}W_{\alpha,\beta}^{\lambda,\xi}(z^\alpha)\right) = z^{\beta-v-1}\frac{(\xi)_m}{B(\lambda,\xi-\lambda)}\sum_{n=0}^{\infty}\frac{B((\lambda+m)+n,\xi-\lambda)}{\Gamma(\alpha n+(\beta+v))}\times(\xi)_{m+n}\frac{(z^\alpha)^n}{((n+m)!)^2} \quad (24)$$

Proof. From (3), (21) and the definition (14), we have

$$\begin{aligned}
D_z^v\left(z^{\beta-1}W_{\alpha,\beta}^{\lambda,\xi}(z^\alpha)\right) &= D^m\left(I_z^{m-v}\left(z^{\beta-1}W_{\alpha,\beta}^{\lambda,\xi}(z)^\alpha\right)\right)\\
&= D^m\left(z^{\beta+m-v-1}W_{\alpha,\beta+m-v}^{\lambda,\xi}(z)^\alpha\right)\\
&= D^m\frac{z^{\beta+m-v-1}}{B(\lambda,\xi-\lambda)}\sum_{n=0}^{\infty}\frac{B(\lambda+n,\xi-\lambda)}{\Gamma(\alpha n+\beta+m-v)}\times(\xi)_n\frac{z^{\alpha n}}{(n!)^2}\\
&= \frac{z^{\beta+m-v-1}}{B(\lambda,\xi-\lambda)}\sum_{n=0}^{\infty}\frac{B(\lambda+n,\xi-\lambda)(\xi)_n}{\Gamma(\alpha n+\beta+m-v)}\times\frac{d^m}{dz^m}\frac{z^{\alpha n}}{(n!)^2}\\
&= \frac{1}{B(\lambda,\xi-\lambda)}\sum_{n=m}^{\infty}\frac{B(\lambda+n,\xi-\lambda)(\xi)_n}{\Gamma(\alpha n+\beta-v)(n!)^2}z^{\alpha n+\beta-v-1}\\
&= \frac{z^{\beta-v-1}}{B(\lambda,\xi-\lambda)}\sum_{n=0}^{\infty}\frac{B((\lambda+m)+n,\xi-\lambda)(\xi)_{n+m}}{\Gamma(\alpha(n+m)+\beta-v)((n+m)!)^2}(z^\alpha)^n\\
&= z^{\beta-v-1}\frac{(\xi)_m}{B(\lambda,\xi-\lambda)}\sum_{n=0}^{\infty}\frac{B((\lambda+m)+n,\xi-\lambda)}{\Gamma(\alpha(n+m)+\beta-v)}\times(\xi+m)_n\frac{(z^\alpha)^n}{((n+m)!)^2}
\end{aligned}$$

Thus

$$D_z^v\left(z^{\beta-1}W_{\alpha,\beta}^{\lambda,\xi}(z^\alpha)\right) = z^{\beta-v-1}\frac{(\xi)_m}{B(\lambda,\xi-\lambda)}\sum_{n=0}^{\infty}\frac{B((\lambda+m)+n,\xi-\lambda)}{\Gamma(\alpha n+(\beta+v))}\times(\xi)_{m+n}\frac{(z^\alpha)^n}{((n+m)!)^2}$$

□

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