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Time Derivative of Vector and Mathematical Objects in Tensor Calculus and its Application to Newtonian Equation of Motion

Research Article

Shibaji Halder[1]*

1 Department of Mathematics, Vidyasagar College, Kolkata, West Bengal, India.

Abstract: We represent a vector as the contravariant components with respect to covariant basis and we derivative it with respect to time. Now if we put it in the left side of Newtonian equation of motion for acceleration (time derivative of velocity), we get equation of motion in tensorial form. Now finding the acceleration components in spherical polar coordinate system and get the contravariants components of acceleration and changing it to the physical components. Again if we consider two different coordinate system and using the covariant basis formula we get it tensorial rank does not change this type of differentiation.

Keywords: Time rate of change of contravariant vector, Newtonian Mechanics.

1. Introduction

The main theme of tensor calculus hidden in the vector space and its dual spaces. The components of any vector in the dual vector space completely depend upon the corresponding basis in the vector space and in the similar manner components of any vector in the vector space completely depend upon the corresponding basis in the dual vector space, i.e. if a basis of a vector spaces changes according to the covariant rule, then the components of a vector in the corresponding dual basis also changes according to the basis of the vector space and vice-versa [1]. Now covariant derivative of a vector in the vector space be written as the linear combination basis vectors [1, 2]. This paper deals with the effective force of the Newtonian law in any coordinate system, firstly in the tensorial form and then in physical form in spherical polar coordinate system. At last using the changing law of basis vectors, it can be easily shown that covariant derivative of contravariant components changes according to the contravariant changing rule.

2. General Discussion

Any vector $\mathbf{F}(q^1(t), q^2(t), \ldots, q^n(t))$ in S_n can be represented with respect to covariant basis as

$$\mathbf{F}(q^1(t), q^2(t), \ldots, q^n(t)) = F^k(q^1(t), q^2(t), \ldots, q^n(t))r_k,$$

* *E-mail: shibajihalderrrrm@gmail.com*

where $\mathbf{r}_k$ is the tangent vector to the $q^k(t)$ curve. That is

$$\mathbf{r}_k = \frac{\partial r}{\partial q^k} = \frac{\partial r}{\partial \overline{q}^j}\frac{\partial \overline{q}^j}{\partial q^k} = \overline{\mathbf{r}}_j \frac{\partial \overline{q}^j}{\partial q^k}$$

$\mathbf{r}_k$'s are transform as covariant rule. it can be shown F^k's are contravariant components of $\mathbf{F}$ with respect to the basis $\mathbf{r}^k$ which is dual to the basis $\mathbf{r}_k$. That is $\mathbf{r}_k.\mathbf{r}^j = \delta^j_k$. Now I differentiate $\mathbf{F}$ with respect to time,

$$\frac{d}{dt}\mathbf{F}(q^1(t), q^2(t), \ldots, q^n(t)) = \frac{dF^k}{dt}\mathbf{r}_k + F^k\frac{d r_k}{dt}$$

Now

$$\frac{dr_k}{dt} = \frac{d}{dt}\left(\frac{\partial r}{\partial q^k}\right) = \frac{\partial}{\partial q^i}\left(\frac{\partial r}{\partial q^k}\right)\frac{dq^i}{dt} = \Gamma^j_{ik}\mathbf{r}_j\frac{dq^i}{dt}$$

Here, using the property that differentiation of a basis vector $\mathbf{r}_k(q^1(t), \ldots, q^n(t))$ with respect to a same variable q^i be written as the linear combination in the same basis, where the coefficients are function of q^i's and known as Christofell symbols of second kind and denoted as Γ^j_{ik}.

$$\begin{aligned}\frac{d}{dt}\mathbf{F} &= \frac{dF^k}{dt}\mathbf{r}_k + F^k\Gamma^j_{ik}\mathbf{r}_j\frac{dq^i}{dt} \\ &= \frac{dF^k}{dt}\mathbf{r}_k + F^j\Gamma^k_{ij}\mathbf{r}_k\frac{dq^i}{dt}\end{aligned} \tag{1}$$

here interchange the the dummy suffix j and k.

$$= \left(\frac{dF^k}{dt} + F^j\Gamma^k_{ij}\frac{dq^i}{dt}\right)\mathbf{r}_k \tag{2}$$

Therefore, now we apply on the Newtonian mechanics to get easily general form of the equation of motion. Now the equation of motion can be written as

$$m\frac{d\overrightarrow{v}}{dt} = m\left(\frac{dv^k}{dt} + v^j\Gamma^k_{ij}\frac{dq^i}{dt}\right)\mathbf{r}_k = \overrightarrow{F} = F^k\mathbf{r}_k$$

Therefore, equating the components, the tensor form of equation of motion becomes

$$m\left(\frac{dv^k}{dt} + v^j\Gamma^k_{ij}\frac{dq^i}{dt}\right) = F^k$$

Applying to the Spherical polar coordinate system, Here $\mathbf{r} = \mathbf{i}r\sin\theta\cos\phi + \mathbf{j}r\sin\theta\sin\phi + \mathbf{j}r\cos\theta$. Therefore $\mathbf{r}_1 = \frac{\partial}{\partial r}\mathbf{r}(r,\theta,\phi)$, $\hat{r} = \frac{r_1}{|r_1|} = \mathbf{r}_1$, similarly $\mathbf{r}_2 = r$ and $\mathbf{r}_3 = r\sin\theta$. Here $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ are tangent vectors and $\{\hat{r}, \hat{\theta}, \hat{\phi}\}$ are unit tangent vectors to the coordinate curves at any point $P(r,\theta,\phi)$. Now the velocity,

$$\begin{aligned}\mathbf{v} &= \frac{d}{dt}\mathbf{r}(r,\theta,\phi) = \frac{d}{dt}r\hat{r} = \frac{d}{dt}r\mathbf{r}_1 \\ \mathbf{v} &= \frac{dr}{dt}\mathbf{r}_1 + r\Gamma^j_{i1}\mathbf{r}_j\frac{dq^i}{dt}, \text{ using (1), here } q^1 = r, q^2 = \theta, q^3 = \phi \\ &= \left(\frac{dr}{dt} + r\Gamma^1_{i1}\frac{dq^i}{dt}\right)\mathbf{r}_1 + r\Gamma^2_{i1}\frac{dq^i}{dt}\mathbf{r}_2 + r\Gamma^3_{i1}\frac{dq^i}{dt}\mathbf{r}_3\end{aligned}$$

Non zero Christoffel coefficients of second kind for spherical polar coordinates are $\Gamma^1_{22} = -r$, $\Gamma^1_{33} = -r\sin^2\theta$, $\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}$, $\Gamma^2_{33} = -\sin\theta\cos\theta$, $\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}$, $\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta$. $v^1 =$ coefficient of $\mathbf{r}_1 = (\frac{dr}{dt} + r\Gamma^1_{i1}\frac{dq^i}{dt}) = \frac{dr}{dt}$, similarly,

$$v^2 = r\Gamma^2_{i1}\frac{dq^i}{dt} = \frac{d\theta}{dt} \quad and \quad v^3 = r\Gamma^3_{i1}\frac{dq^i}{dt} = \frac{dr}{dt} + r\cot\theta\frac{d\theta}{dt}$$

Therefore, velocity

$$\mathbf{v} = v^1\mathbf{r}_1 + v^2\mathbf{r}_2 + v^3\mathbf{r}_3 = \frac{dr}{dt}\mathbf{r}_1 + \frac{d\theta}{dt}\mathbf{r}_2 + \left(\frac{dr}{dt} + r\cot\theta\frac{d\theta}{dt}\right)\mathbf{r}_3$$

In terms of physical components,

$$\mathbf{v} = \frac{dr}{dt}\hat{r} + \frac{d\theta}{dt}r\hat{\theta} + \left(\frac{dr}{dt} + r\cot\theta\frac{d\theta}{dt}\right)r\sin\theta\hat{\phi}$$

Now acceleration,

$$\mathbf{f} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}v^k\mathbf{r}_k = \left(\frac{dv^k}{dt} + v^j\Gamma^k_{ij}\frac{dq^i}{dt}\right)\mathbf{r}_k, \quad \text{using (2)}$$

Therefore

$$\begin{aligned}
f^1 &= \text{coefficient of } \mathbf{r}_1 = \frac{dv^1}{dt} + v^j\Gamma^1_{ij}\frac{dq^i}{dt}, \quad \text{here } q^1 = r,\ q^2 = \theta,\ q^3 = \phi \\
&= \frac{dv^1}{dt} + v^2\left(-r\frac{d\theta}{dt}\right) + v^3\left(-r\sin^2\theta\frac{d\phi}{dt}\right), \quad \text{putting values of } \Gamma^1_{ij}\text{'s} \\
&= \frac{d}{dt}\left(\frac{dr}{dt}\right) + \frac{d\theta}{dt}\left(-r\frac{d\theta}{dt}\right) + \left(\frac{dr}{dt} + r\cot\theta\frac{d\theta}{dt}\right)\left(-r\sin^2\theta\frac{d\phi}{dt}\right) \\
f^1 &= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - r\sin^2\theta\left(\frac{dr}{dt} + r\cot\theta\frac{d\theta}{dt}\right)\frac{d\phi}{dt}, \quad \text{as } \mathbf{r}_1 = \hat{r}, \text{ therefore it is also physical component.}
\end{aligned}$$

Similarly,

$$\begin{aligned}
f^2 &= \text{coefficient of } \mathbf{r}_2 = \frac{dv^2}{dt} + v^j\Gamma^2_{ij}\frac{dq^i}{dt} \\
&= \frac{dv^2}{dt} + v^1\left(\frac{1}{r}\frac{d\theta}{dt}\right) + v^2\left(\frac{1}{r}\frac{dr}{dt}\right) + v^3\left(-\sin\theta\cos\theta\frac{d\phi}{dt}\right), \quad \text{putting values of } \Gamma^2_{ij}\text{'s.} \\
&= \frac{d}{dt}\left(\frac{d\theta}{dt}\right) + \frac{dr}{dt}\left(\frac{1}{r}\frac{d\theta}{dt}\right) + \frac{d\theta}{dt}\left(\frac{1}{r}\frac{dr}{dt}\right) + \left(\frac{dr}{dt} + r\cot\theta\frac{d\theta}{dt}\right)\left(-\sin\theta\cos\theta\frac{d\phi}{dt}\right) \\
&= \frac{1}{r^2}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) - \sin\theta\cos\theta\left(\frac{dr}{dt} + r\cot\theta\frac{d\theta}{dt}\right)\frac{d\phi}{dt}
\end{aligned}$$

$\mathbf{r}_2 = r\hat{\theta}$, physical component in this direction will be

$$f_\theta = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) - r\sin\theta\cos\theta\left(\frac{dr}{dt} + r\cot\theta\frac{d\theta}{dt}\right)\frac{d\phi}{dt}$$

and

$$\begin{aligned}
f^3 &= \text{coefficient of } \mathbf{r}_3 = \frac{dv^3}{dt} + v^j\Gamma^3_{ij}\frac{dq^i}{dt}, \quad \text{now putting values of } \Gamma^3_{ij}\text{'s we get} \\
&= \frac{dv^3}{dt} + v^1\Gamma^3_{31}\frac{d\phi}{dt} + v^2\Gamma^3_{32}\frac{d\phi}{dt} + v^3\left(\Gamma^3_{13}\frac{dr}{dt} + \Gamma^3_{23}\frac{d\theta}{dt}\right) \\
&= \frac{d}{dt}\left(\frac{dr}{dt} + r\cot\theta\frac{d\theta}{dt}\right) + \frac{dr}{dt}\frac{1}{r}\frac{d\phi}{dt} + \frac{d\theta}{dt}\cot\theta\frac{d\phi}{dt} + \left(\frac{dr}{dt} + r\cot\theta\frac{d\theta}{dt}\right)\left(\frac{1}{r}\frac{dr}{dt} + \cot\theta\frac{d\theta}{dt}\right) \\
&= \frac{d}{dt}\left(\frac{dr}{dt} + r\cot\theta\frac{d\theta}{dt}\right) + \frac{1}{r}\frac{d\phi}{dt}\left(\frac{dr}{dt} + r\cot\theta\frac{d\theta}{dt}\right) + \frac{1}{r}\left(\frac{dr}{dt} + r\cot\theta\frac{d\theta}{dt}\right)^2
\end{aligned}$$

as $\mathbf{r}_3 = r\sin\theta\hat{\phi}$, physical component in this direction will be $f^3 r\sin\theta$. Similarly Cylidrical polar coordinate system I find $f^1 = \frac{d^2\rho}{dt^2} - \rho\left(\frac{d\phi}{dt}\right)^2$, $f^2 = \frac{1}{\rho^2}\frac{d}{dt}\left(\rho^2\frac{d\phi}{dt}\right)$, $f^3 = \frac{d^2z}{dt^2}$, thats are expected values. Now how the component of derivative of a vector transforms. Let

$$\begin{aligned}
\mathbf{F} &= F^k\left(q^1(t), q^2(t), \ldots, q^n(t)\right)\mathbf{r}_k \\
&= \overline{F}^m\left(\overline{q}^1(t), \overline{q}^2(t), \ldots, \overline{q}^n(t)\right)\overline{\mathbf{r}}_m,
\end{aligned}$$

are two representation of $\mathbf{F}$ in two coordinate systems $\{q^i\}$ and $\{\overline{q}^j\}$ and two covariant bases $\{\mathbf{r}_k\}$ and $\{\overline{\mathbf{r}}_j\}$ respectively. As

$$\mathbf{r}_k = \frac{\partial \mathbf{r}}{\partial q^k} = \frac{\partial \mathbf{r}}{\partial \overline{q}^j}\frac{\partial \overline{q}^j}{\partial q^k} = \overline{\mathbf{r}}_m \frac{\partial \overline{q}^m}{\partial q^k}$$

Now,

$$\frac{d}{dt}\mathbf{F} = \left(\frac{dF^k}{dt} + F^j\Gamma^k_{ij}\frac{dq^i}{dt}\right)\mathbf{r}_k = \left(\frac{dF^k}{dt} + F^j\Gamma^k_{ij}\frac{dq^i}{dt}\right)\overline{\mathbf{r}}_m\frac{\partial \overline{q}^m}{\partial q^k} = \left(\frac{d\overline{F}^m}{dt} + \overline{F}^n\overline{\Gamma}^m_{\text{ln}}\frac{d\overline{q}^l}{dt}\right)\overline{\mathbf{r}}_m$$

now equating the coefficients we get

$$\left(\frac{d\overline{F}^m}{dt} + \overline{F}^n\overline{\Gamma}^m_{\text{ln}}\frac{d\overline{q}^l}{dt}\right) = \left(\frac{dF^k}{dt} + F^j\Gamma^k_{ij}\frac{dq^i}{dt}\right)\frac{\partial \overline{q}^m}{\partial q^k}$$

i.e change as contravariant rule.

References

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