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Cone $\mathcal{C}$-class Function on Some Common Fixed Point Theorems for Contractive Type Conditions in Random Cone Metric Space

Research Article

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Abstract: In paper we discuss the some common fixed point theorems for contractive type conditions in random cone metric space via Cone $\mathcal{C}$-class function.

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Keywords: Common fixed point, random cone metric space, C-class function.

1. Introduction and preliminaries

In 2007, Huang and Zhang [2] introduced the concept of cone metric spaces and fixed point theorems of contraction mappings; Any mapping T of a complete cone metric space X into itself that satisfies, for some $0 \leq k < 1$, the inequality $d(Tx, Ty) \leq kd(x, y)$, $\forall\, x, y \in X$ has a unique fixed point. In 2011, Smriti Mehta et. al [12] introduce the concept of random cone metric space and proved an existence of random fixed point under weak contraction condition in the setting of random cone metric space. In this paper, we discuss about self maps for altering distance functions and ultra altering distance functions of some common fixed point theorems for contractive type conditions in random cone metric space via cone $\mathcal{C}$-class function.

Definition 1.1 ([12]). *Let (E, τ) be a topological vector space. A subset P of E is called a cone if and only if:*

(1). P is closed, non-empty and $P \neq \{0\}$

(2). $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b

(3). If $x \in P$ or $-x \in P$ implies $x = 0$

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We will write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P.

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Example 1.2. *Let $K > 1$ be given. Consider the real vector space with $E = \{ax + b : a, b \in R; x \in [1 - \frac{1}{k}, 1]\}$ with supremum norm and the cone $P = \{ax + b : a \geq 0, b \leq 0\}$ in E. The cone P is regular and so normal.*

Definition 1.3. *Let X be a non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies*

(b_1) $d(x, y) = 0$ if and only if $x = y$,

(b_2) $d(x, y) = d(y, x)$,

(b_3) $d(x, z) \leq d(x, y) + d(y, z)$.

Then d is called cone metric [2] or K-Metric [17] on X and (X, d) is called a cone metric space [2] (CMS).

Example 1.4. *Let $E = R^2$*

$$P = \{(x, y) : x, y \geq 0\}$$

$X = R$ and $d : X \times X \to E$ such that $d(x, y) = (|x, y|, \alpha|x, y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 1.5. *Let (X, d) be a CMS, $x \in X$ and $\{x_n\}_{n\geq 0}$ be a sequence in X. Then $\{x_n\}_{n\geq 0}$ converges to x whenever for every $c \in E$ with $0 \ll E$, there is a natural number $N \in N$ such that $d(x_n, x) \ll c$ for all $n \geq N$. It is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$*

Definition 1.6. *Let (X, d) be a CMS, $x \in X$ and $\{x_n\}_{n\geq 0}$ be a sequence in X. $\{x_n\}_{n\geq 0}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$, there is a natural number $N \in N$, such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$*

Definition 1.7. *Let (X, d) be a CMS, $x \in X$ and $\{x_n\}_{n\geq 0}$ be a sequence in X. (X, d) is a complete cone normed space if every Cauchy sequence is convergent. Complete cone normed spaces will be called complete cone metric spaces.*

Definition 1.8 (Measurable Function)**.** *Let (Ω, Σ) be a measurable space with Σ-a sigma algebra of subsets of and M be a nonempty subset of a metric space $X = (X, d)$. Let 2^M be the family of nonempty subsets of M and $C(M)$ the family of all nonempty closed subsets of M. A mapping $G : \Omega \to 2^M$ is called measurable if for each open subset U of M, $G^{-1}(U) \in \Sigma$, where $G^{-1}(U) = \{\omega \in \Omega : G(\omega) \cap U \neq \emptyset\}$.*

Definition 1.9 (Measurable Selector)**.** *A mapping $\xi : \Omega \to M$ is called a measurable selector of a measurable mapping $G : \Omega \to 2^M$ if ξ is measurable and $\xi(\omega) \in G(\omega)$ for each $\omega \in \Omega$.*

Definition 1.10 (Random Operator)**.** *The mapping $T : \Omega \times M \to X$ is said to be a random operator if and only if for each fixed $x \in M$, the mapping $T(., x) : \Omega \to X$ is measurable.*

Definition 1.11 (Continuous Random Operator)**.** *A random operator $T : \Omega \times M \to X$ is said to be continuous random operator if for each fixed $x \in M$, and $\omega \in \Omega$, the mapping $T(\omega, .) : X \to X$ is continuous.*

Definition 1.12 (Random fixed point)**.** *A measurable mapping $\xi : \Omega \to M$ is a random fixed point of a random operator $T : \Omega \times M \to X$ if and only if $T(\omega, \xi(\omega)) = \xi(\omega)$ for each $\omega \in \Omega$.*

Definition 1.13 ([12])**.** *Let M be a nonempty set and the mapping $d : \Omega \times M \to P$, where P is a cone, $\omega \in \Omega$ be a selector, satisfy the following conditions:*

(1). $d(x(\omega), y(\omega)) \geq 0$ and $d(x(\omega), y(\omega)) = 0$ if and only if $x(\omega) = y(\omega)$ for all $x(\omega), y(\omega) \in \Omega \times M$,

(2). $d(x(\omega), y(\omega)) = d(y(\omega), x(\omega))$ for all $x, y \in M, \omega \in \Omega$ and $x(\omega), y(\omega) \in \Omega \times M$,

(3). $d(x(\omega), y(\omega)) \leq d(x(\omega), z(\omega)) + d(z(\omega), y(\omega))$ *for all* $x, y \in M$ *and* $\omega \in \Omega$ *be a selector,*

(4). *for any* $x, y \in X, \omega \in \Omega, d(x(\omega), y(\omega))$ *is non-increasing and left continuous.*

Then d *is called random cone metric on* M *and* (M, d) *is called a random cone metric space (RCMS).*

Example 1.14. *Let* $M = R$ *and* $P = \{x \in M : x \geq 0\}$*, also* $\Omega = [0, 1]$ *and* Σ *be the sigma algebra of Lebegsue's measurable subset of* $[0, 1]$*. Let* $X = [0, \infty)$ *and define a mapping* $d : (\Omega \times X) \times (\Omega \times X) \to P$ *by* $d(x(\omega), y(\omega)) = |x(\omega) - y(\omega)|$*. Then* (X, d) *is a random cone metric space.*

Definition 1.15. *A function* $\psi : P \to P$ *is called an altering distance function if the following properties are satisfied:*

(1). ψ *is non-decreasing and continuous,*

(2). $\psi(t) = 0$ *if and only if* $t = 0$*.*

Definition 1.16. *An ultra altering distance function is a continuous, nondecreasing mapping* $\varphi : P \to P$ *such that* $\varphi(t) > 0$*,* $t > 0$ *and* $\varphi(0) \geq 0$*.*

We denote this set with Φ_u.

Definition 1.17 ([9])**.** *A mapping* $F : P^2 \to P$ *is called cone* C*-class function if it is continuous and satisfies following axioms:*

(1). $F(s, t) \leq s$*;*

(2). $F(s, t) = s$ *implies that either* $s = 0$ *or* $t = 0$*; for all* $s, t \in P$*.*

We denote cone C-class functions as $\mathcal{C}$.

Example 1.18 ([9])**.** *The following functions* $F : P^2 \to P$ *are elements of* $\mathcal{C}$*, for all* $s, t \in [0, \infty)$*:*

(1). $F(s, t) = s - t$*,*

(2). $F(s, t) = ks$*, where* $0 < k < 1$*,*

(3). $F(s, t) = s\beta(s)$*, where* $\beta : [0, \infty) \to [0, 1)$*,*

(4). $F(s, t) = \Psi(s)$*, where* $\Psi : P \to P$*,* $\Psi(0) = 0$*,* $\Psi(s) > 0$ *for all* $s \in P$ *with* $s \neq 0$ *and* $\Psi(S) \leq s$ *for all* $s \in P$*.,*

(5). $F(s, t) = s - \varphi(s)$*, where* $\varphi : [0, \infty) \to [0, \infty)$ *is a continuous function such that* $\varphi(t) = 0 \Leftrightarrow t = 0$*;*

(6). $F(s, t) = s - h(s, t)$*, where* $h : [0, \infty) \times [0, \infty) \to [0, \infty)$ *is a continuous function such that* $h(s, t) = 0 \Leftrightarrow t = 0$ *for all* $t, s > 0$*.*

(7). $F(s, t) = \varphi(s), F(s, t) = s \Rightarrow s = 0$*, here* $\varphi : [0, \infty) \to [0, \infty)$ *is a upper semi continuous function such that* $\varphi(0) = 0$ *and* $\varphi(t) < t$ *for* $t > 0$*.*

Lemma 1.19. *Let* ψ *and* φ *are altering distance and ultra altering distance functions respectively ,* $F \in \mathcal{C}$ *and* $\{s_n\}$ *a decreasing sequence in* P *such that* $\psi(s_{n+1}) \leq F(\psi(s_n), \varphi(s_n))$ *for all* $n \geq 1$*. Then* $\lim_{n\to\infty} s_n = 0$*.*

Theorem 1.20 ([16]). *Let (X,d) be a complete random cone metric space. with respect to a cone P and let M be a nonempty separable closed subset of X. Let S and T be two continuous random operators defined on M such that for $\omega \in \Omega, S(\omega.,), T(\omega.,): \Omega \times M \to M$ satisfying the condition:*

$$\begin{aligned} d(S(x(\omega)), T(y(\omega))) \leq a(\omega)[d(x(\omega), Sx(\omega)) + d(y(\omega), Ty(\omega))] + b(\omega)d(x(\omega), y(\omega)) \\ + c(\omega)\max\{d(x(\omega), T(y(\omega))), d(y(\omega), S(x(\omega)))\} \end{aligned} \tag{1}$$

for all $x, y \in M$, $2a(\omega) + b(\omega) + 2c(\omega) < 1$, where $a(\omega), b(\omega), c(\omega) > 0$ and $\omega \in \Omega$. Then S and T have a unique common random fixed point in X.

2. Main Results

Theorem 2.1. *Let (X,d) be a complete random cone metric space. with respect to a cone P and let M be a nonempty separable closed subset of X. Let ψ and φ be altering distance and ultra altering distance functions respectively, $\psi : P \to P$ is an altering distance function, $\varphi \in \Phi_u$ and $F \in \mathcal{C}$, Let S and T be two continuous random operators defined on M such that for $\omega \in \Omega, S(\omega.,), T(\omega.,): \Omega \times M \to M$ satisfying the condition:*

$$\begin{aligned} \psi(d(S(x(\omega)), T(y(\omega)))) \leq F(\psi(a(\omega)[d(x(\omega), Sx(\omega)) + d(y(\omega), Ty(\omega))] + b(\omega)d(x(\omega), y(\omega)) \\ + c(\omega)\max\{d(x(\omega), T(y(\omega))), d(y(\omega), S(x(\omega)))\}), \\ \varphi(a(\omega)[d(x(\omega), Sx(\omega)) + d(y(\omega), Ty(\omega))] + b(\omega)d(x(\omega), y(\omega)) \\ + c(\omega)\max\{d(x(\omega), T(y(\omega))), d(y(\omega), S(x(\omega)))\},)) \end{aligned} \tag{2}$$

for all $x, y \in M$, $2a(\omega) + b(\omega) + 2c(\omega) < 1$, where $a(\omega), b(\omega), c(\omega) > 0$ and $\omega \in \Omega$. Then S and T have a unique common random fixed point in X.

Proof. For each $x_0(\omega) \in \Omega \times M$ and $n = 0, 1, 2, \cdots$, we choose $x_2(\omega), x_2(\omega) \in \Omega \times M$ such that $x_1(\omega) = S(x_0(\omega))$ and $x_2(\omega) = T(x_1(\omega))$. In general we de ne sequence of elements of M such that $x_{2n+1}(\omega) = S(x_{2n}(\omega))$ and $x_{2n+2}(\omega) = T(x_{2n+1}(\omega))$. Then from (2), we have

$$\begin{aligned} \psi(d(x_{2n+1}(\omega), x_{2n}(\omega))) &= \psi(d(S(x_{2n}(\omega)), T(x_{2n-1}(\omega)))) \\ &\leq F(\psi(a(\omega)[d(x_{2n}(\omega), S(x_{2n}(\omega))) + d(x_{2n-1}(\omega), T(x_{2n-1}(\omega))] + b(\omega)d(x_{2n}(\omega), x_{2n-1}(\omega)) \\ &+ c(\omega)\max\{d(x_{2n}(\omega), T(x_{2n-1}(\omega))), d(x_{2n-1}(\omega), S(x_{2n}(\omega)))\}), \\ &\varphi(a(\omega)[d(x_{2n}(\omega), S(x_{2n}(\omega))) + d(x_{2n-1}(\omega), T(x_{2n-1}(\omega))] + b(\omega)d(x_{2n}(\omega), x_{2n-1}(\omega)) \\ &+ c(\omega)\max\{d(x_{2n}(\omega), T(x_{2n-1}(\omega))), d(x_{2n-1}(\omega), S(x_{2n}(\omega)))\}) \\ &= F(\psi(a(\omega)[d(x_{2n}(\omega), x_{2n+1}(\omega)) + d(x_{2n-1}(\omega), x_{2n}(\omega))] + b(\omega)d(x_{2n}(\omega), x_{2n-1}(\omega)) \\ &+ c(\omega)\max\{d(x_{2n}(\omega), x_{2n}(\omega)), d(x_{2n-1}(\omega), x_{2n+1}(\omega))\}), \\ &\varphi(a(\omega)[d(x_{2n}(\omega), x_{2n+1}(\omega)) + d(x_{2n-1}(\omega), x_{2n}(\omega))] + b(\omega)d(x_{2n}(\omega), x_{2n-1}(\omega)) \\ &+ c(\omega)\max\{d(x_{2n}(\omega), x_{2n}(\omega)), d(x_{2n-1}(\omega), x_{2n+1}(\omega))\}) \\ &= F(\psi(a(\omega)[d(x_{2n}(\omega), x_{2n+1}(\omega)) + d(x_{2n-1}(\omega), x_{2n}(\omega))] + b(\omega)d(x_{2n}(\omega), x_{2n-1}(\omega)) \\ &+ c(\omega)\max\{0, d(x_{2n-1}(\omega), x_{2n+1}(\omega))\}), \\ &\varphi(a(\omega)[d(x_{2n}(\omega), x_{2n+1}(\omega)) + d(x_{2n-1}(\omega), x_{2n}(\omega))] + b(\omega)d(x_{2n}(\omega), x_{2n-1}(\omega)) \\ &+ c(\omega)\max\{0, d(x_{2n-1}(\omega), x_{2n+1}(\omega))\}) \end{aligned} \tag{3}$$

Since for non-negative real numbers a and b, we have

$$\max\{a, b\} \leq a + b. \tag{4}$$

Using (4) in (3), we have

$$\begin{aligned}
\psi(d(x_{2n+1}(\omega), x_{2n}(\omega))) &\leq F(\psi(a(\omega)[d(x_{2n}(\omega), x_{2n+1}(\omega)) + d(x_{2n-1}(\omega), x_{2n}(\omega))] \\
&+ b(\omega)d(x_{2n}(\omega), x_{2n-1}(\omega)) + c(\omega)d(x_{2n-1}(\omega), x_{2n+1}(\omega))), \\
&\varphi(a(\omega)[d(x_{2n}(\omega), x_{2n+1}(\omega)) + d(x_{2n-1}(\omega), x_{2n}(\omega))] \\
&+ b(\omega)d(x_{2n}(\omega), x_{2n-1}(\omega)) + c(\omega)d(x_{2n-1}(\omega), x_{2n+1}(\omega)))) \\
&\leq F(\psi(a(\omega)[d(x_{2n}(\omega), x_{2n+1}(\omega)) + d(x_{2n-1}(\omega), x_{2n}(\omega))] + b(\omega)d(x_{2n}(\omega), x_{2n-1}(\omega)) \\
&+ c(\omega)[d(x_{2n-1}(\omega), x_{2n+1}(\omega)) + d(x_{2n}(\omega), x_{2n+1}(\omega))]), \\
&\varphi(a(\omega)[d(x_{2n}(\omega), x_{2n+1}(\omega)) + d(x_{2n-1}(\omega), x_{2n}(\omega))] + b(\omega)d(x_{2n}(\omega), x_{2n-1}(\omega)) \\
&+ c(\omega)[d(x_{2n-1}(\omega), x_{2n+1}(\omega)) + d(x_{2n}(\omega), x_{2n+1}(\omega))])) \\
&= \psi((a(\omega) + b(\omega) + c(\omega))d(x_{2n}(\omega), x_{2n-1}(\omega)) + (a(\omega) + c(\omega))d(x_{2n+1}(\omega), x_{2n}(\omega))).
\end{aligned} \tag{5}$$

The above inequality (5) gives,

$$\begin{aligned}
d(x_{2n+1}(\omega), x_{2n}(\omega)) &\leq \left(\frac{a(\omega) + b(\omega) + (\omega)}{1 - a(\omega) - c(\omega)}\right) d(x_{2n+1}(\omega), x_{2n}(\omega)) \\
&= k(\omega)d(x_{2n+1}(\omega), x_{2n}(\omega)),
\end{aligned} \tag{6}$$

Where

$$k(\omega) = \left(\frac{a(\omega) + b(\omega) + (\omega)}{1 - a(\omega) - c(\omega)}\right).$$

Where By the assumption of the theorem $2a(\omega) + b(\omega) + 2c(\omega) < 1 \Rightarrow a(\omega) + b(\omega) + c(\omega) < 1 - a(\omega) - c(\omega) \Rightarrow k(\omega) = \left(\frac{a(\omega)+b(\omega)+(\omega)}{1-a(\omega)-c(\omega)}\right) < 1$. Similarly, we have

$$d(x_{2n}(\omega), x_{2n-1}(\omega)) \leq k(\omega)d(x_{2n-1}(\omega), x_{2n-2}(\omega)).$$

Hence

$$d(x_{2n+1}(\omega), x_{2n}(\omega)) \leq k(\omega)^2 d(x_{2n-1}(\omega), x_{2n-2}(\omega)).$$

On continuing in this process, we get

$$d(x_{2n+1}(\omega), x_{2n}(\omega)) \leq k(\omega)^{2n} d(x_1(\omega), x_0(\omega)).$$

Also for $n > m$, we have

$$\begin{aligned}
d(x_n(\omega), x_m(\omega)) &\leq d(x_n(\omega), x_{n-1}(\omega)) + d(x_{n-1}(\omega), x_{n-2}(\omega)) + +d(x_{m+1}(\omega), x_m(\omega)) \\
&\leq (k(\omega)^{n-1} + k(\omega)^{n-2} + \cdots + k(\omega)^m)d(x_1(\omega), x_0(\omega)) \\
&\leq (\frac{k(\omega)^m}{1 - k(\omega)})d(x_1(\omega), x_0(\omega)).
\end{aligned}$$

Let $0 \ll \epsilon$ be given. Choose a natural number N such that $(\frac{k(\omega)^m}{1-k(\omega)})d(x_1(\omega), x_0(\omega)) \ll \epsilon$ for every $m \geq N$. Thus $d(x_n(\omega), x_m(\omega)) \leq (\frac{k(\omega)^m}{1-k(\omega)})d(x_1(\omega), x_0(\omega)) \ll \epsilon$, for every $n > m \geq N$. This shows that the sequence $x_n(\omega)$ is a Cauchy sequence in $\Omega \times M$. Since (X, d) is complete, there exists $z(\omega) \in \Omega \times X$ such that $x_n(\omega) \to z(\omega)$ as $n \to \infty$. Choose a natural number N_1 such that

$$d(z(\omega), x_{2n+2}(\omega)) \ll \frac{\epsilon(1 - a(\omega) - c(\omega))}{2(1 + b(\omega) + 2c(\omega))}. \tag{7}$$

and

$$d(x_{2n+1}(\omega), x_{2n+2}(\omega)) \ll \frac{\epsilon(1 - a(\omega) - c(\omega))}{2(a(\omega) + b(\omega) + c(\omega))}, \tag{8}$$

for every $n \geq N_1$. Hence for $n \geq N_1$, we have

$$\begin{aligned}
\psi(d(z(\omega), S(z(\omega)))) &\leq \psi(d(z(\omega), x_{2n+2}(\omega)) + d(x_{2n+2}(\omega), S(z(\omega)))\\
&= \psi(d(z(\omega), x_{2n+2}(\omega))) + \psi(d(S(z(\omega), T(x_{2n+1}(\omega)))\\
&\leq \psi(d(z(\omega), x_{2n+2}(\omega))) + F(\psi(a(\omega)[d(z(\omega), S(z(\omega)))\\
&+ d(x_{2n+1}(\omega), T(x_{2n+1}(\omega))] + b(\omega)d(z(\omega), x_{2n+1}(\omega))\\
&+ c(\omega)\max\{d(z(\omega), T(x_{2n+1}(\omega))) + d(x_{2n+1}(\omega), S(z(\omega)))\}),\\
&\varphi(a(\omega)[d(z(\omega), S(z(\omega))) + d(x_{2n+1}(\omega), T(x_{2n+1}(\omega))] + b(\omega)d(z(\omega), x_{2n+1}(\omega))\\
&+ c(\omega)\max\{d(z(\omega), T(x_{2n+1}(\omega))), d(x_{2n+1}(\omega), S(z(\omega)))\}))\\
&= \psi(d(z(\omega), x_{2n+2}(\omega))) + \psi(a(\omega)[d(z(\omega), S(z(\omega))) + d(x_{2n+1}(\omega), x_{2n+2}(\omega))]\\
&+ b(\omega)d(z(\omega), x_{2n+1}(\omega)) + c(\omega)\max\{d(z(\omega), x_{2n+2}(\omega)), d(x_{2n+1}(\omega), S(z(\omega)))\}).
\end{aligned}$$

Using (4) in the above inequality, we get

$$\begin{aligned}
\psi(d(z(\omega), S(z(\omega)))) &\leq \psi(d(z(\omega), x_{2n+2}(\omega)))\\
&+ F(\psi(+a(\omega)[d(z(\omega), S(z(\omega))) + d(x_{2n+1}(\omega), x_{2n+2}(\omega))]\\
&+ b(\omega)d(z(\omega), x_{2n+1}(\omega)) + c(\omega)[d(z(\omega), x_{2n+2}(\omega)) + d(x_{2n+1}(\omega), S(z(\omega)))]),\\
&\varphi(+a(\omega)[d(z(\omega), S(z(\omega))) + d(x_{2n+1}(\omega), x_{2n+2}(\omega))]\\
&+ b(\omega)d(z(\omega), x_{2n+1}(\omega)) + c(\omega)[d(z(\omega), x_{2n+2}(\omega)) + d(x_{2n+1}(\omega), S(z(\omega)))])\\
&\leq \psi(d(z(\omega), x_{2n+2}(\omega)))
\end{aligned} \tag{9}$$

$$\begin{aligned}
&+ F(\psi(+a(\omega)[d(z(\omega), S(z(\omega))) + d(x_{2n+1}(\omega), x_{2n+2}(\omega))]\\
&+ b(\omega)[d(z(\omega), x_{2n+1}(\omega)) + d(x_{2n+2}(\omega), x_{2n+1}(\omega))]\\
&+ c(\omega)[d(z(\omega), x_{2n+2}(\omega)) + d(x_{2n+1}(\omega), x_{2n+2}(\omega))\\
&+ d(x_{2n+2}(\omega), z(\omega)) + d(z(\omega), S(z(\omega)))]),\\
&\varphi(+a(\omega)[d(z(\omega), S(z(\omega))) + d(x_{2n+1}(\omega), x_{2n+2}(\omega))]\\
&+ b(\omega)[d(z(\omega), x_{2n+1}(\omega)) + d(x_{2n+2}(\omega), x_{2n+1}(\omega))]\\
&+ c(\omega)[d(z(\omega), x_{2n+2}(\omega)) + d(x_{2n+1}(\omega), x_{2n+2}(\omega))\\
&+ d(x_{2n+2}(\omega), z(\omega)) + d(z(\omega), S(z(\omega)))]))\\
&= \psi((1 + b(\omega) + 2c(\omega))d(z(\omega), x_{2n+2}(\omega))\\
&+ (a(\omega) + b(\omega) + c(\omega))d(x_{2n+1}(\omega), x_{2n+2}(\omega)) + (a(\omega) + c(\omega))d(z(\omega), S(z(\omega)))).
\end{aligned} \tag{10}$$

The above inequality gives

$$d(z(\omega),S(z(\omega)))\le\left(\frac{1+b(\omega)+2c(\omega)}{1-a(\omega)-c(\omega)}\right)d(z(\omega),x_{2n+2}(\omega))+\left(\frac{a(\omega)+b(\omega)+c(\omega)}{1-a(\omega)-c(\omega)}\right)d(x_{2n+1}(\omega),x_{2n+2}(\omega)). \tag{11}$$

Using (7) and (8) in (11), we get $d(z(\omega),S(z(\omega)))\ll\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$. Thus $d(z(\omega),S(z(\omega)))\ll\frac{\epsilon}{m}\quad\forall\ m\ge1$. So $\frac{\epsilon}{m}-d(z(\omega),S(z(\omega)))\in P$ for all $m\ge1$. Since $\frac{\epsilon}{m}\to0$ as $m\to\infty$ and P is closed, we obtain $-d(z(\omega),S(z(\omega)))\in P$. But $d(z(\omega),S(z(\omega)))\in P$. Therefore by Definition 2.1 (c_3), $d(z(\omega),S(z(\omega))=0$ and so $S(z(\omega))=z(\omega)$. In an exactly the similar way we can prove that for all $\omega\in\Omega,T(z(\omega))=z(\omega)$. Hence $S(z(\omega))=T(z(\omega))=z(\omega)$. This shows that $z(\omega)$ is a common random fixed point of S and T.

Uniqueness: Let $v(\omega)$ be another random fixed point common to S and T, that is, for $\omega\in\Omega,S(v(\omega))=T(v(\omega))=v(\omega)$ such that $z(\omega)\neq v(\omega)$. Then for $\omega\in\Omega$, we have

$$\begin{aligned}\psi(d(z(\omega),v(\omega)))&=\psi(d(S(z(\omega)),T(v(\omega))))\\&\le F(\psi(a(\omega)[d(z(\omega),S(z(\omega)))+d(v(\omega),T(v(\omega)))]+b(\omega)d(z(\omega),v(\omega))\\&\quad+c(\omega)\max\{d(z(\omega),T(v(\omega))),d(v(\omega),S(z(\omega)))\}),\\&\quad\varphi(a(\omega)[d(z(\omega),S(z(\omega)))+d(v(\omega),T(v(\omega)))]+b(\omega)d(z(\omega),v(\omega))\\&\quad+c(\omega)\max\{d(z(\omega),T(v(\omega))),d(v(\omega),S(z(\omega)))\})\\&\le F(\psi((b(\omega)+c(\omega))d(z(\omega),v(\omega))),\varphi((b(\omega)+c(\omega))d(z(\omega),v(\omega)))\\&<\psi(d(z(\omega),v(\omega))),\quad\text{since}\ \ 0<b(\omega)+c(\omega)<1,\end{aligned}$$

a contradiction. Hence $z(\omega)=v(\omega)$ and so $z(\omega)$ is a unique common random fixed point of S and T. This completes the proof. □

Corollary 2.2. *Let (X,d) be a complete random cone metric space. with respect to a cone P and let M be a nonempty separable closed subset of X. Let ψ and φ be altering distance and ultra altering distance functions respectively, $\psi:P\to P$ is an altering distance function, $\varphi\in\Phi_u$ and $F\in\mathcal{C}$, Let T be a continuous random operator defined on M such that for $\omega\in\Omega,T(\omega,.):\Omega\times M\to M$ satisfying the condition:*

$$\begin{aligned}\psi(d(T(x(\omega)),T(y(\omega))))\le F(&\psi(a(\omega)[d(x(\omega),Tx(\omega))+d(y(\omega),Ty(\omega))]\\&+b(\omega)d(x(\omega),y(\omega))+c(\omega)\max\{d(x(\omega),T(y(\omega))),\\&d(y(\omega),T(x(\omega)))\}),\varphi(a(\omega)[d(x(\omega),Tx(\omega))+d(y(\omega),Ty(\omega))]\\&+b(\omega)d(x(\omega),y(\omega))+c(\omega)\max\{d(x(\omega),T(y(\omega))),d(y(\omega),T(x(\omega)))\}))\end{aligned}$$

for all $x,y\in M$, $2a(\omega)+b(\omega)+2c(\omega)<1$, where $a(\omega),b(\omega),c(\omega)>0$ and $\omega\in\Omega$. Then S and T have a unique common random fixed point in X.

Proof. The proof of the corollary immediately follows by putting $S=T$ in Theorem (2.1). This completes the proof. □

If we take $S=T$ and $a(\omega)=c(\omega)=0$ in Theorem (2.1), then we obtain the following result as corollary.

Corollary 2.3. *Let (X,d) be a complete random cone metric space. with respect to a cone P and let M be a nonempty separable closed subset of X. Let ψ and φ be altering distance and ultra altering distance functions respectively, $\psi:P\to P$*

is an altering distance function, $\varphi \in \Phi_u$ and $F \in \mathcal{C}$, Let T be a random operator defined on M such that for $\omega \in \Omega, T(\omega,.) : \Omega \times M \to M$ satisfying the condition:

$$\psi(d(T(x(\omega)), T(y(\omega)))) \leq F(\psi(b(\omega)d(x(\omega), y(\omega))), \varphi(b(\omega)d(x(\omega), y(\omega)))),$$

for all $x, y \in M, b(\omega) \in (0,1)$ and $\omega \in \Omega$. Then T has a unique random fixed point in X. If we take $S = T$ and $b(\omega) = c(\omega) = 0$ in Theorem 2.1, then we obtain the following result as corollary.

Corollary 2.4. *Let (X, d) be a complete cone random met-ric space with respect to a cone P and let M be a nonempty separable closed subset of X. Let ψ and φ be altering distance and ultra altering distance functions respectively, $\psi : P \to P$ is an altering distance function, $\varphi \in \Phi_u$ and $F \in \mathcal{C}$. Let T be a continuous random operator defined on M such that for $\omega \in \Omega, T(\omega,.) : \Omega \times M \to M$ satisfying the condition:*

$$\begin{aligned}\psi(d(T(x(\omega)), T(y(\omega)))) \leq F(&\psi(a(\omega)[d(x(\omega), T(x(\omega))) + d(y(\omega), T(y(\omega)))]),\\ &\varphi(a(\omega)[d(x(\omega), T(x(\omega))) + d(y(\omega), T(y(\omega)))]))\end{aligned} \tag{12}$$

for all $x, y \in M, a(\omega) \in (0, \frac{1}{2})$ and $\omega \in \Omega$. Then T has a unique random fixed point in X. If we take $S = T$ and $a(\omega) = b(\omega) = 0$ in Theorem (2.1), then we obtain the following result as corollary.

Corollary 2.5. *Let (X, d) be a complete random cone metric space. with respect to a cone P and let M be a nonempty separable closed subset of X. Let ψ and φ be altering distance and ultra altering distance functions respectively, $\psi : P \to P$ is an altering distance function, $\varphi \in \Phi_u$ and $F \in \mathcal{C}$. Let T be a continuous random operator defined on M such that for $\omega \in \Omega, T(\omega,.) : \Omega \times M \to M$ satisfying the condition:*

$$\begin{aligned}\psi(d(T(x(\omega)), T(y(\omega)))) \leq F(&\psi(c(\omega)\max\{d(x(\omega), T(y(\omega))), d(y(\omega), T(x(\omega)))\}),\\ &\varphi(c(\omega)\max\{d(x(\omega), T(y(\omega))), d(y(\omega), T(x(\omega)))\}))\end{aligned} \tag{13}$$

for all $x, y \in M, c(\omega) \in (0, \frac{1}{2})$ and $\omega \in \Omega$. Then T has a unique random fixed point in X.

Theorem 2.6. *Let (X, d) be a complete random cone metric space. with respect to a cone P and let M be a nonempty separable closed subset of X. Let ψ and φ be altering distance and ultra altering distance functions respectively, $\psi : P \to P$ is an altering distance function, $\varphi \in \Phi_u$ and $F \in \mathcal{C}$. Let S and T be two continuous random operators defined on M such that for $\omega \in \Omega, T(\omega,.) : \Omega \times M \to M$ satisfying the condition:*

$$\begin{aligned}\psi(d(S(x(\omega)), T(y(\omega)))) \leq F(&\psi(a(\omega)d(x(\omega), y(\omega)) + b(\omega)\max\{d(x(\omega), S(x(\omega))), d(y(\omega), T(y(\omega)))\}\\ &+ c(\omega)\max\{d(x(\omega), T(y(\omega))), d(y(\omega), S(x(\omega)))\}), \varphi(a(\omega)d(x(\omega), y(\omega))\\ &+ b(\omega)\max\{d(x(\omega), S(x(\omega))), d(y(\omega), T(y(\omega)))\}\\ &+ c(\omega)\max\{d(x(\omega), T(y(\omega))), d(y(\omega), S(x(\omega)))\}))\end{aligned} \tag{14}$$

for all $x, y \in M, a(\omega) + b(\omega) + 2c(\omega) < 1$, where $a(\omega), b(\omega), c(\omega) > 0$ and $\omega \in \Omega$. Then S and T have a unique common random fixed point in X.

Proof. For each $x_0(\omega) \in \Omega \times M$ and $n = 0; 1, 2, ?$, we choose $x_2(\omega), x_2(\omega) \in \Omega \times M$ such that $x_1(\omega) = S(x_0(\omega))$ and $x_2(\omega) = T(x_1(\omega))$. In general we de ne sequence of elements of M such that $x_{2n+1}(\omega) = S(x_{2n}(\omega))$ and $x_{2n+2}(\omega) = T(x_{2n+1}(\omega))$.

Then from (14), we have

$$\begin{aligned}\psi((S(x(\omega)),T(y(\omega)))) &= \psi((S(z(\omega)),T(x_{2n-1}(\omega))))\\ &\le F(\psi(a(\omega)d(x_{2n}(\omega),x_{2n-1}(\omega)) + b(\omega)\max\{d(x_{2n}(\omega),S(x_{2n}(\omega))),d(x_{2n-1}(\omega),T(x_{2n-1}(\omega)))\}\\ &+ c(\omega)\max\{d(x_{2n}(\omega),T(x_{2n-1}(\omega))),d(x_{2n-1}(\omega),S(x_{2n}(\omega)))\}),\varphi(a(\omega)d(x_{2n}(\omega),x_{2n-1}(\omega))\\ &+ b(\omega)\max\{d(x_{2n}(\omega),S(x_{2n}(\omega))),d(x_{2n-1}(\omega),T(x_{2n-1}(\omega)))\}\\ &+ c(\omega)\max\{d(x_{2n}(\omega),T(x_{2n-1}(\omega))),d(x_{2n-1}(\omega),S(x_{2n}(\omega)))\})\\ &\le F(\psi(a(\omega)d(x_{2n}(\omega),x_{2n-1}(\omega)) + b(\omega)\max\{d(x_{2n}(\omega),x_{2n+1}(\omega)),d(x_{2n-1}(\omega),x_{2n}(\omega))\}\\ &+ c(\omega)\max\{d(x_{2n}(\omega),x_{2n}(\omega)),d(x_{2n-1}(\omega),,x_{2n+1}(\omega))\}\\ &\le a(\omega)d(x_{2n}(\omega),x_{2n-1}(\omega))),\varphi(a(\omega)d(x_{2n}(\omega),x_{2n-1}(\omega))\\ &+ b(\omega)\max\{d(x_{2n}(\omega),x_{2n+1}(\omega)),d(x_{2n-1}(\omega),x_{2n}(\omega))\}\\ &+ c(\omega)\max\{d(x_{2n}(\omega),x_{2n}(\omega)),d(x_{2n-1}(\omega),,x_{2n+1}(\omega))\}\\ &\le F(\psi(a(\omega)d(x_{2n}(\omega),x_{2n-1}(\omega))) + b(\omega)\max\{d(x_{2n}(\omega),x_{2n+1}(\omega)),d(x_{2n-1}(\omega),x_{2n}(\omega))\},\\ &+ c(\omega)\max\{0,d(x_{2n-1}(\omega),x_{2n+1}(\omega))\}),\varphi(a(\omega)d(x_{2n}(\omega),x_{2n-1}(\omega)))\\ &+ b(\omega)\max\{d(x_{2n}(\omega),x_{2n+1}(\omega)),d(x_{2n-1}(\omega),x_{2n}(\omega))\},\\ &+ c(\omega)\max\{0,d(x_{2n-1}(\omega),x_{2n+1}(\omega))\})\end{aligned} \tag{15}$$

Since for non-negative real numbers a and b, we have

$$\max\{a,b\} \le a+b. \tag{16}$$

and taking

$$\max\{d(x_{2n}(\omega),d(x_{2n+1}(\omega)),d(x_{2n-1}(\omega),d(x_{2n}(\omega))\} = d(x_{2n-1}(\omega),d(x_{2n}(\omega)) \tag{17}$$

Using (16 (17) in (15), we have

$$\begin{aligned}d(x_{2n+1}(\omega),x_{2n}(\omega)) &\le a(\omega)d(x_{2n}(\omega),x_{2n-1}(\omega)) + b(\omega)d(x_{2n-1}(\omega),x_{2n}(\omega)) + c(\omega)d(x_{2n-1}(\omega),x_{2n+1}(\omega))\\ &\le a(\omega)d(x_{2n}(\omega),x_{2n-1}(\omega)) + b(\omega)d(x_{2n-1}(\omega),x_{2n}(\omega))\\ &+ c(\omega)[d(x_{2n-1}(\omega),x_{2n}(\omega)) + d(x_{2n}(\omega),x_{2n+1}(\omega))]\end{aligned} \tag{18}$$

The above inequality (18) gives,

$$\begin{aligned}d(x_{2n+1}(\omega),x_{2n}(\omega)) &\le (\frac{a(\omega)+b(\omega)+(\omega)}{1-c(\omega)})d(x_{2n+1}(\omega),x_{2n}(\omega))\\ &= h(\omega)d(x_{2n+1}(\omega),x_{2n}(\omega)),\end{aligned} \tag{19}$$

Where

$$h(\omega) = (\frac{a(\omega)+b(\omega)+c(\omega)}{1-c(\omega)}).$$

Where By the assumption of the theorem

$$a(\omega)+b(\omega)+2c(\omega) < 1 \Rightarrow a(\omega)+b(\omega)+c(\omega) < 1-c(\omega) \Rightarrow h(\omega) = \left(\frac{a(\omega)+b(\omega)+(\omega)}{1-c(\omega)}\right) < 1.$$

If we take

$$\max\{d(x_{2n}(\omega), x_{2n+1}(\omega)), d(x_{2n-1}(\omega)), x_{2n}(\omega)\} = d(x_{2n}(\omega), x_{2n+1}(\omega)). \tag{20}$$

in equation (15), then we also get the same result as (19). The rest of the proof is same as that of Theorem (2.1) . This completes the proof. □

Remark 2.7. *If we take $S = T$ and $b(\omega) = c(\omega) = 0$ and $a(\omega) = b(\omega)$ in Theorem (2),then we obtain Corollary (2.3) of this paper.*

Corollary 2.8. *Let (X, d) be a complete random cone metric space. with respect to a cone P and let M be a nonempty separable closed subset of X. Let ψ and φ be altering distance and ultra altering distance functions respectively, $\psi : P \to P$ is an altering distance function, $\varphi \in \Phi_u$ and $F \in \mathcal{C}$. Let T be a continuous random operator defined on M such that for $\omega \in \Omega, T(\omega, .) : \Omega \times M \to M$ satisfying the condition:*

$$\begin{aligned}\psi(d(T(x(\omega)), T(y(\omega)))) \leq F(&\psi(a(\omega)d(x(\omega), y(\omega)) + b(\omega)\max\{d(x(\omega), Tx(\omega)) + d(y(\omega), Ty(\omega))\}\\ &+ c(\omega)\max\{d(x(\omega), T(y(\omega))), d(y(\omega), T(x(\omega)))\}), \varphi(a(\omega)d(x(\omega), y(\omega))\\ &+ b(\omega)\max\{d(x(\omega), Tx(\omega)) + d(y(\omega), Ty(\omega))\}\\ &+ c(\omega)\max\{d(x(\omega), T(y(\omega))), d(y(\omega), T(x(\omega)))\}))\end{aligned} \tag{21}$$

for all $x, y \in M$, $a(\omega) + b(\omega) + 2c(\omega) < 1$, where $a(\omega), b(\omega), c(\omega) > 0$ and $\omega \in \Omega$. Then S and T have a unique common random fixed point in X..

Proof. The proof of the corollary immediately follows by putting $S = T$ in Theorem (2). This completes the proof □

Example 2.9. *Let $\Omega = [0, 1]$ ans Σ be the sigma algebra of Lebesgue's measurable subset of $[0, 1]$. Take $X = R$ with $d(x, y) = |x - y|$ for $x, y \in R$. Define random mapping T from $\Omega \times X$ to X as $T(\omega, x) = 2\omega - x$. Let $F(s, t) = s - t$ for all $s, t \in [0, \infty)$. Also define $\phi, \psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = 2t$ and $\varphi(t) = t$. Then a measurable mapping $\zeta : \Omega \to X$ defined as $\zeta(\omega) = \omega$ for all $\omega \in \Omega$,*

$$\begin{aligned}F(\psi(d(T(x(\omega)), T(y(\omega)))), \varphi(d(T(x(\omega)), T(y(\omega))))) &= \psi(d(T(x(\omega)), T(y(\omega)))) - \varphi(d(T(x(\omega)), T(y(\omega))))\\ &= \psi(\zeta(\omega)) - \varphi(\zeta(\omega))\\ &= 2\omega - \omega\\ &= \omega\end{aligned} \tag{22}$$

serve as a unique random fixed point of T.

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